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NEW PLANE AND SPHERICAL TRIGONOMETRY, SURVEYING, AND NAVIGATION

BY

G. A. WENTWORTH

AUTHOR OF A SERIES OF TEXT-BOOKS IN MATHEMATICS

TEACHERS' EDITION

GINN & COMPANY
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PREFACE

THIS edition is intended for teachers, *and for them only.* The publishers will under no circumstances sell the book except to teachers of Wentworth's Trigonometry; and every teacher must consider himself in honor bound not to leave his copy where pupils can have access to it, and not to sell his copy except to the publishers, Messrs. Ginn & Company.

It is hoped that young teachers will derive great advantage from studying the systematic arrangement of the work, and that all teachers who are pressed for time will find great relief by not being obliged to work out every problem.

G. A. WENTWORTH.

PLANE TRIGONOMETRY.

TEACHERS' EDITION.

Exercise I. Page 2.

1. Reduce the following angles to circular measure, expressing the results as fractions of π: $60°$, $45°$, $150°$, $195°$, $11° 15'$, $123° 45'$, $37° 30'$.

$$60° = \tfrac{60}{180}\pi = \tfrac{1}{3}\pi.$$
$$45° = \tfrac{45}{180}\pi = \tfrac{1}{4}\pi.$$
$$150° = \tfrac{150}{180}\pi = \tfrac{5}{6}\pi.$$
$$195° = \tfrac{195}{180}\pi = \tfrac{13}{12}\pi.$$
$$11° 15' = \frac{11\frac{1}{4}}{180}\pi = \frac{1}{16}\pi.$$
$$123° 45' = \frac{123\frac{3}{4}}{180}\pi = \frac{11}{16}\pi.$$
$$37° 30' = \frac{37\frac{1}{2}}{180}\pi = \frac{5}{24}\pi.$$

2. How many degrees are there in $\frac{2}{3}\pi$ radians? $\frac{3}{4}\pi$ radians? $\frac{5}{8}\pi$ radians? $\frac{15}{16}\pi$ radians? $\frac{7}{15}\pi$ radians?

$\frac{2}{3}\pi$ radians $= \frac{2}{3}$ of $180° = 120°$.
$\frac{3}{4}\pi$ radians $= \frac{3}{4}$ of $180° = 135°$.
$\frac{5}{8}\pi$ radians $= \frac{5}{8}$ of $180° = 112° 30'$.
$\frac{15}{16}\pi$ radians $= \frac{15}{16}$ of $180° = 168° 45'$.
$\frac{7}{15}\pi$ radians $= \frac{7}{15}$ of $180° = 84°$.

3. What decimal part of a radian is $1°$? $1'$?

$$1° = \frac{\pi}{180}\text{ radian} = \frac{3.1416}{180}\text{ radian}$$
$$= 0.017453\text{ radian.}$$
$$1' = \frac{0.017453}{60}\text{ radian}$$
$$= 0.0002909\text{ radian.}$$

4. How many seconds in a radian?

$$1\text{ radian} = 57° 17' 45''$$
$$= 206{,}265''.$$

5. Express in radians one of the interior angles of a regular octagon; of a regular dodecagon.

The sum of all the interior angles of a regular octagon is $8 \times 180° - 360° = 8\pi - 2\pi = 6\pi$. Hence, each interior angle

$$= \frac{6}{8}\pi\text{ radians} = \frac{3}{4}\pi\text{ radians.}$$

The sum of all the interior angles of a regular dodecagon is $12 \times 180° - 360° = 12\pi - 2\pi = 10\pi$. Hence, each interior angle

$$= \frac{10}{12}\pi\text{ radians} = \frac{5}{6}\pi\text{ radians.}$$

6. On the circumference of a circle of 50 feet radius an arc of 10 feet is laid off. How many degrees in the angle at the centre by this arc?

It subtends $\frac{10}{50}$ radian $= \frac{1}{5}$ radian

$$= \frac{57^\circ\ 17'\ 45''}{5} = 11^\circ\ 27'\ 33''.$$

7. The earth's equatorial radius is approximately 3963 miles. If two points on the equator are 1000 miles apart, what is their difference in longitude?

Their difference in longitude is $\frac{1000}{3963}$ radian, or

$$\frac{1000}{3963} \times 57^\circ\ 17'\ 45'' = 14^\circ\ 27'\ 28''.$$

8. If the difference in longitude of two points on the equator is 1°, what is the distance between them in miles?

By Example 3, $1^\circ = 0.017453$ radian. Hence, 1° on the equator is equal to 0.017453×3963 miles $=$ 69.166 miles.

9. What is the radius of a circle if an arc of 1 foot subtends an angle of 1° at the centre?

Since 1° of arc = 1 foot, 1 radian or $57^\circ\ 17'\ 45'' = 57\frac{17\frac{3}{4}}{60}$ feet = 57 feet 3.55 inches = the radius.

10. In how many hours is a point on the equator carried by the rotation of the earth on its axis through a distance equal to the earth's radius?

The earth turns through $360^\circ = 2\pi$ radians in 24 hours. Hence, it turns through 1 radian in $\frac{24}{2\pi}$ hours $= \frac{12}{3.1416}$ hours = 3 hours 49 minutes 11 seconds.

11. The minute hand of a clock is $3\frac{1}{2}$ feet long. How far does its extremity move in 25 minutes? [Take $\pi = \frac{22}{7}$.]

The circumference which is passed over in 60 minutes is $2\pi \times 3\frac{1}{2}$ feet. Hence, the arc passed over in 25 minutes is

$$\frac{25}{60} \text{ of } 2\pi \times 3\tfrac{1}{2} \text{ ft.}$$
$$= \frac{25}{60} \times 2 \times \frac{22}{7} \times \frac{7}{2} \text{ ft.}$$
$$= 9\tfrac{1}{6} \text{ ft.} = 9 \text{ ft. } 2 \text{ in.}$$

12. A wheel makes 15 revolutions a second. How long does it take to turn through 4 radians? [Take $\pi = \frac{22}{7}$.]

The wheel turns through 2π radians in $\frac{1}{15}$ of a second. Hence, it turns through 4 radians in $4 \times (\frac{1}{15}$ second $\div 2\pi) = \frac{7}{165}$ second.

EXERCISE II. PAGE 5.

1. What are the functions of the other acute angle B of the triangle ACB (Fig. 2)?

$$\sin B = \frac{b}{c}. \qquad \cos B = \frac{a}{c}.$$

$$\tan B = \frac{b}{a}. \qquad \sec B = \frac{c}{a}.$$

$$\cot B = \frac{a}{b}. \qquad \csc B = \frac{c}{b}.$$

2. Compare the functions of A and B, and show that

$$\begin{aligned}\sin A &= \cos B,\\ \cos A &= \sin B,\\ \tan A &= \cot B,\\ \cot A &= \tan B,\\ \sec A &= \csc B,\\ \csc A &= \sec B,\\ \text{vers } A &= \text{covers } B,\\ \text{covers } A &= \text{vers } B.\end{aligned}$$

$$\sin A = \frac{a}{c}, \qquad \cos B = \frac{a}{c}.$$

$$\cos A = \frac{b}{c}, \qquad \sin B = \frac{b}{c}.$$

$$\tan A = \frac{a}{b}, \qquad \cot B = \frac{a}{b}.$$

$$\cot A = \frac{b}{a}, \qquad \tan B = \frac{b}{a}.$$

$$\sec A = \frac{c}{b}, \qquad \csc B = \frac{c}{b}.$$

$$\csc A = \frac{c}{a}, \qquad \sec B = \frac{c}{a}.$$

$$\text{vers } A = \frac{c-b}{c}, \quad \text{covers } B = \frac{c-b}{c}.$$

$$\text{covers } A = \frac{c-a}{c}, \qquad \text{vers } B = \frac{c-a}{c}.$$

3. Find the values of the functions of A, if a, b, c, respectively, have the following values:

(i) 3, 4, 5. (iv) 9, 40, 41.
(ii) 5, 12, 13. (v) 3.9, 8, 8.9.
(iii) 8, 15, 17. (vi) 1.19, 1.20, 1.69.

(i)

$$\sin A = \frac{3}{5}. \qquad \cot A = \frac{4}{3}.$$

$$\cos A = \frac{4}{5}. \qquad \sec A = \frac{5}{4}.$$

$$\tan A = \frac{3}{4}. \qquad \csc A = \frac{5}{3}.$$

(ii)

$$\sin A = \frac{5}{13}. \qquad \cot A = \frac{12}{5}.$$

$$\cos A = \frac{12}{13}. \qquad \sec A = \frac{13}{12}.$$

$$\tan A = \frac{5}{12}. \qquad \csc A = \frac{13}{5}.$$

(iii)

$$\sin A = \frac{8}{17}. \qquad \cot A = \frac{15}{8}.$$

$$\cos A = \frac{15}{17}. \qquad \sec A = \frac{17}{15}.$$

$$\tan A = \frac{8}{15}. \qquad \csc A = \frac{17}{8}.$$

(iv)

$$\sin A = \frac{9}{41}. \qquad \cot A = \frac{40}{9}.$$

$$\cos A = \frac{40}{41}. \qquad \sec A = \frac{41}{40}.$$

$$\tan A = \frac{9}{40}. \qquad \csc A = \frac{41}{9}.$$

(v)

$$\sin A = \frac{39}{89}. \qquad \cot A = \frac{80}{39}.$$

$$\cos A = \frac{80}{89}. \qquad \sec A = \frac{89}{80}.$$

$$\tan A = \frac{39}{80}. \qquad \csc A = \frac{89}{39}.$$

(vi)

$$\sin A = \frac{119}{169}. \qquad \cot A = \frac{120}{119}.$$

$$\cos A = \frac{120}{169}. \qquad \sec A = \frac{169}{120}.$$

$$\tan A = \frac{119}{120}. \qquad \csc A = \frac{169}{119}.$$

4. **What condition must be fulfilled by the lengths of the three lines *a*, *b*, *c* (Fig. 2) in order to make them the sides of a right triangle? Is this condition fulfilled in Example 3?**

The condition is $a^2 + b^2 = c^2$.

It is; for

$3^2 + 4^2 = 5^2$. $\quad 9^2 + 40^2 = 41^2$.
$5^2 + 12^2 = 13^2$. $\quad 3.9^2 + 8^2 = 8.9^2$.
$8^2 + 15^2 = 17^2$. $\quad 1.19^2 + 1.20^2 = 1.69^2$.

5. **Find the values of the functions of *A*, if *a*, *b*, *c*, respectively, have the following values:**

(i) $2\,mn,\ m^2 - n^2,\ m^2 + n^2$.

(ii) $\dfrac{2\,xy}{x-y},\ x+y,\ \dfrac{x^2+y^2}{x-y}$.

(iii) $pqr,\ qrs,\ rsp$.

(iv) $\dfrac{mn}{pq},\ \dfrac{mv}{sq},\ \dfrac{nr}{ps}$.

(i)

$$\sin A = \frac{a}{c} = \frac{2\,mn}{m^2+n^2}.$$
$$\cos A = \frac{b}{c} = \frac{m^2-n^2}{m^2+n^2}.$$
$$\tan A = \frac{a}{b} = \frac{2\,mn}{m^2-n^2}.$$
$$\cot A = \frac{b}{a} = \frac{m^2-n^2}{2\,mn}.$$
$$\sec A = \frac{c}{b} = \frac{m^2+n^2}{m^2-n^2}.$$
$$\csc A = \frac{c}{a} = \frac{m^2+n^2}{2\,mn}.$$

(ii)

$$\sin A = \frac{2\,xy}{x-y} \times \frac{x-y}{x^2+y^2} = \frac{2\,xy}{x^2+y^2}.$$
$$\cos A = (x+y) \times \frac{x-y}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}.$$
$$\tan A = \frac{2\,xy}{x-y} \times \frac{1}{x+y} = \frac{2\,xy}{x^2-y^2}.$$
$$\cot A = (x+y) \times \frac{x-y}{2\,xy} = \frac{x^2-y^2}{2\,xy}.$$
$$\sec A = \frac{x^2+y^2}{x-y} \times \frac{1}{x+y} = \frac{x^2+y^2}{x^2-y^2}.$$
$$\csc A = \frac{x^2+y^2}{x-y} \times \frac{x-y}{2\,xy} = \frac{x^2+y^2}{2\,xy}.$$

(iii)

$$\sin A = \frac{pqr}{rsp} = \frac{q}{s}. \quad \cos A = \frac{qrs}{rsp} = \frac{q}{p}.$$
$$\tan A = \frac{pqr}{qrs} = \frac{p}{s}. \quad \cot A = \frac{qrs}{pqr} = \frac{s}{p}.$$
$$\sec A = \frac{rsp}{qrs} = \frac{p}{q}. \quad \csc A = \frac{rsp}{pqr} = \frac{s}{q}.$$

(iv)

$$\sin A = \frac{mn}{pq} \times \frac{ps}{nr} = \frac{ms}{qr}.$$
$$\cos A = \frac{mv}{sq} \times \frac{ps}{nr} = \frac{mpv}{nqr}.$$
$$\tan A = \frac{mn}{pq} \times \frac{sq}{mv} = \frac{ns}{pv}.$$
$$\cot A = \frac{mv}{sq} \times \frac{pq}{mn} = \frac{pv}{ns}.$$
$$\sec A = \frac{nr}{ps} \times \frac{sq}{mv} = \frac{nqr}{mpv}.$$
$$\csc A = \frac{nr}{ps} \times \frac{pq}{mn} = \frac{qr}{ms}.$$

6. **Prove that the values of *a*, *b*, *c* in (i) and (ii), Example 5, satisfy the condition necessary to make them the sides of a right triangle.**

(i)

$$a^2 + b^2 = c^2.$$
$$(2\,mn)^2 + (m^2 - n^2)^2 = (m^2 + n^2)^2.$$
$$4\,m^2n^2 + m^4 - 2\,m^2n^2 + n^4 = m^4 + 2\,m^2n^2 + n^4.$$
$$m^4 + 2\,m^2n^2 + n^4 = m^4 + 2\,m^2n^2 + n^4.$$

(ii)

$$\left(\frac{2xy}{x-y}\right)^2+(x+y)^2 = \left(\frac{x^2+y^2}{x-y}\right)^2.$$

$$\frac{4x^2y^2}{x^2-2xy+y^2}+x^2+2xy+y^2 = \frac{x^4+2x^2y^2+y^4}{x^2-2xy+y^2}.$$

$$4x^2y^2+x^4-2x^2y^2+y^4 = x^4+2x^2y^2+y^4.$$

$$x^4+2x^2y^2+y^4=x^4+2x^2y^2+y^4.$$

7. What equations of condition must be satisfied by the values of a, b, c in (iii) and (iv), Example 5, in order that the values may represent the sides of a right triangle?

(iii)

$$p^2q^2r^2+q^2r^2s^2=r^2s^2p^2,$$

or $$p^2q^2+q^2s^2=p^2s^2.$$

(iv)

$$\frac{m^2n^2}{p^2q^2}+\frac{m^2v^2}{s^2q^2}=\frac{n^2r^2}{p^2s^2},$$

or $$m^2n^2s^2+m^2p^2v^2=n^2q^2r^2.$$

8. Given $a^2+b^2=c^2$; find the functions of A and B when $a=24$, $b=143$.

$$\begin{aligned} c &= \sqrt{24^2+143^2} \\ &= \sqrt{21025} \\ &= 145. \end{aligned}$$

$$\sin A=\frac{24}{145}=\cos B.$$

$$\cos A=\frac{143}{145}=\sin B.$$

$$\tan A=\frac{24}{143}=\cot B.$$

$$\cot A=\frac{143}{24}=\tan B.$$

$$\sec A=\frac{145}{143}=\csc B.$$

$$\csc A=\frac{145}{24}=\sec B.$$

9. Given $a^2+b^2=c^2$; find the functions of A and B when $a=0.264$, $c=0.265$.

$$\begin{aligned} b &= \sqrt{(c+a)(c-a)} \\ &= \sqrt{0.529\times 0.001} \\ &= \sqrt{0.000529} \\ &= 0.023. \end{aligned}$$

$$\sin A=\frac{a}{c}=\frac{264}{265}=\cos B.$$

$$\cos A=\frac{b}{c}=\frac{23}{265}=\sin B.$$

$$\tan A=\frac{a}{b}=\frac{264}{23}=\cot B.$$

$$\cot A=\frac{b}{a}=\frac{23}{264}=\tan B.$$

$$\sec A=\frac{c}{b}=\frac{265}{23}=\csc B.$$

$$\csc A=\frac{c}{a}=\frac{265}{264}=\sec B.$$

10. Given $a^2+b^2=c^2$; find the functions of A and B when $b=9.5$, $c=19.3$.

$$\begin{aligned} a &= \sqrt{(c+b)(c-a)} \\ &= \sqrt{28.8\times 9.8} \\ &= \sqrt{282.24} \\ &= 16.8. \end{aligned}$$

$$\sin A=\frac{a}{c}=\frac{168}{193}=\cos B.$$

$$\cos A=\frac{b}{c}=\frac{95}{193}=\sin B.$$

$$\tan A=\frac{a}{b}=\frac{168}{95}=\cot B.$$

$$\cot A=\frac{b}{a}=\frac{95}{168}=\tan B.$$

$$\sec A = \frac{c}{b} = \frac{193}{95} = \csc B.$$

$$\csc A = \frac{c}{a} = \frac{193}{168} = \sec B.$$

11. Given $a^2 + b^2 = c^2$; find the functions of A and B when

$$a = \sqrt{p^2 + q^2},\ b = \sqrt{2\,pq}.$$

$$c = \sqrt{p^2 + q^2 + 2\,pq}$$

$$= p + q.$$

$$\sin A = \frac{a}{c} = \frac{\sqrt{p^2 + q^2}}{p + q} = \cos B.$$

$$\cos A = \frac{b}{c} = \frac{\sqrt{2\,pq}}{p + q} = \sin B.$$

$$\tan A = \frac{a}{b} = \frac{\sqrt{p^2 + q^2}}{\sqrt{2\,pq}} = \cot B.$$

$$\cot A = \frac{b}{a} = \frac{\sqrt{2\,pq}}{\sqrt{p^2 + q^2}} = \tan B.$$

$$\sec A = \frac{c}{b} = \frac{p + q}{\sqrt{2\,pq}} = \csc B.$$

$$\csc A = \frac{c}{a} = \frac{p + q}{\sqrt{p^2 + q^2}} = \sec B.$$

12. Given $a^2 + b^2 = c^2$; find the functions of A and B when

$$a = \sqrt{p^2 + pq},\ c = p + q.$$

$$b^2 = c^2 - a^2.$$

$$= q^2 + pq.$$

$$\therefore b = \sqrt{q^2 + pq}.$$

$$\sin A = \frac{a}{c} = \frac{\sqrt{p^2 + pq}}{p + q} = \cos B.$$

$$\cos A = \frac{b}{c} = \frac{\sqrt{q^2 + pq}}{p + q} = \sin B.$$

$$\tan A = \frac{a}{b} = \frac{\sqrt{p^2 + pq}}{\sqrt{q^2 + pq}}$$

$$= \sqrt{\frac{p}{q}} = \cot B.$$

$$\cot A = \frac{b}{a} = \frac{\sqrt{q^2 + pq}}{\sqrt{p^2 + pq}}$$

$$= \sqrt{\frac{q}{p}} = \tan B.$$

$$\sec A = \frac{c}{b} = \frac{p + q}{\sqrt{q^2 + pq}} = \csc B.$$

$$\csc A = \frac{c}{a} = \frac{p + q}{\sqrt{p^2 + pq}} = \sec B.$$

13. Given $a^2 + b^2 = c^2$; find the functions of A and B when

$$b = 2\sqrt{pq},\ c = p + q.$$

$$a^2 = c^2 - b^2.$$

$$a^2 = p^2 + 2\,pq + q^2 - 4\,pq$$

$$= p^2 - 2\,pq + q^2.$$

$$\therefore a = p - q.$$

$$\sin A = \frac{a}{c} = \frac{p - q}{p + q} = \cos B.$$

$$\cos A = \frac{b}{c} = \frac{2\sqrt{pq}}{p + q} = \sin B.$$

$$\tan A = \frac{a}{b} = \frac{p - q}{2\sqrt{pq}} = \cot B.$$

$$\cot A = \frac{b}{a} = \frac{2\sqrt{pq}}{p - q} = \tan B.$$

$$\sec A = \frac{c}{b} = \frac{p + q}{2\sqrt{pq}} = \csc B.$$

$$\csc A = \frac{c}{a} = \frac{p + q}{p - q} = \sec B.$$

14. Given $a^2 + b^2 = c^2$; find the functions of A when $a = 2\,b$.

$$a = 2\,b.$$

$$a^2 + b^2 = c^2.$$

$$4\,b^2 + b^2 = c^2.$$

$$5\,b^2 = c^2.$$

$$\therefore c = b\sqrt{5}.$$

$$\sin A = \frac{a}{c} = \frac{2b}{b\sqrt{5}} = \tfrac{2}{5}\sqrt{5}.$$

$$\cos A = \frac{b}{c} = \frac{b}{b\sqrt{5}} = \tfrac{1}{5}\sqrt{5}.$$

$$\tan A = \frac{a}{b} = \frac{2b}{b} = 2.$$

$$\cot A = \frac{b}{a} = \frac{1}{2}.$$

$$\sec A = \frac{c}{b} = \frac{b\sqrt{5}}{b} = \sqrt{5}.$$

$$\csc A = \frac{c}{a} = \frac{b\sqrt{5}}{2b} = \tfrac{1}{2}\sqrt{5}.$$

15. Given $a^2 + b^2 = c^2$; find the functions of A when $a = \tfrac{2}{3}c$.

$$a = \tfrac{2}{3}c.$$

$$b = \sqrt{c^2 - a^2}$$

$$= \sqrt{c^2 - \tfrac{4}{9}c^2}$$

$$= \frac{c}{3}\sqrt{5}.$$

$$\sin A = \frac{a}{c} = \frac{\tfrac{2}{3}c}{c} = \frac{2}{3}.$$

$$\cos A = \frac{b}{c} = \frac{\frac{c}{3}\sqrt{5}}{c} = \tfrac{1}{3}\sqrt{5}.$$

$$\tan A = \frac{a}{b} = \frac{\tfrac{2}{3}c}{\frac{c}{3}\sqrt{5}} = \tfrac{2}{5}\sqrt{5}.$$

$$\cot A = \frac{b}{a} = \frac{\frac{c}{3}\sqrt{5}}{\tfrac{2}{3}c} = \tfrac{1}{2}\sqrt{5}.$$

$$\sec A = \frac{c}{b} = \frac{c}{\frac{c}{3}\sqrt{5}} = \tfrac{3}{5}\sqrt{5}.$$

$$\csc A = \frac{c}{a} = \frac{c}{\tfrac{2}{3}c} = \frac{3}{2}.$$

16. Given $a^2 + b^2 = c^2$; find the functions of A when $a + b = \tfrac{5}{4}c$.

$$a + b = \tfrac{5}{4}c.$$

$$a^2 + b^2 = c^2.$$

$$a^2 + b^2 + 2ab = \tfrac{25}{16}c^2.$$

$$2ab = \tfrac{9}{16}c^2.$$

$$a^2 - 2ab + b^2 = \tfrac{7}{16}c^2.$$

$$a - b = \frac{c}{4}\sqrt{7}.$$

$$a + b = \tfrac{5}{4}c.$$

$$2b = \frac{5}{4}c - \frac{c}{4}\sqrt{7}.$$

$$b = \frac{c}{8}(5 - \sqrt{7}).$$

$$2a = \frac{5}{4}c + \frac{c}{4}\sqrt{7}.$$

$$a = \frac{c}{8}(5 + \sqrt{7}).$$

$$\sin A = \frac{a}{c} = \frac{\frac{c}{8}(5 + \sqrt{7})}{c} = \frac{5 + \sqrt{7}}{8}.$$

$$\cos A = \frac{b}{c} = \frac{\frac{c}{8}(5 - \sqrt{7})}{c} = \frac{5 - \sqrt{7}}{8}.$$

$$\tan A = \frac{a}{b} = \frac{5 + \sqrt{7}}{5 - \sqrt{7}} = \frac{16 + 5\sqrt{7}}{9}.$$

$$\cot A = \frac{b}{a} = \frac{5 - \sqrt{7}}{5 + \sqrt{7}} = \frac{16 - 5\sqrt{7}}{9}.$$

$$\sec A = \frac{c}{b} = \frac{c}{\frac{c}{8}(5 - \sqrt{7})} = \tfrac{4}{9}(5 + \sqrt{7}).$$

$$\csc A = \frac{c}{a} = \frac{c}{\frac{c}{8}(5 + \sqrt{7})} = \tfrac{4}{9}(5 - \sqrt{7}).$$

17. Given $a^2 + b^2 = c^2$; find the functions of A when $a - b = \frac{1}{4}c$.

$$
\begin{array}{rl}
a^2 - 2\,ab + b^2 &= \dfrac{c^2}{16}. \\
a^2 \qquad\quad + b^2 &= c^2. \\
\hline
2\,ab \qquad\quad &= \dfrac{15\,c^2}{16}. \\
a^2 \qquad\quad + b^2 &= c^2. \\
\hline
a^2 + 2\,ab + b^2 &= \dfrac{31\,c^2}{16}.
\end{array}
$$

$$a + b = \frac{c}{4}\sqrt{31}.$$

$$a - b = \frac{c}{4}.$$

$$2\,a = \frac{c}{4}\sqrt{31} + \frac{c}{4}.$$

$$\therefore a = \frac{c}{8}(\sqrt{31} + 1).$$

$$2\,b = \frac{c}{4}\sqrt{31} - \frac{c}{4}.$$

$$\therefore b = \frac{c}{8}(\sqrt{31} - 1).$$

$$\sin A = \frac{a}{c} = \frac{\frac{c}{8}(\sqrt{31}+1)}{c} = \frac{\sqrt{31}+1}{8}.$$

$$\cos A = \frac{b}{c} = \frac{\frac{c}{8}(\sqrt{31}-1)}{c} = \frac{\sqrt{31}-1}{8}.$$

$$\tan A = \frac{a}{b} = \frac{\sqrt{31}+1}{\sqrt{31}-1} = \frac{16+\sqrt{31}}{15}.$$

$$\cot A = \frac{b}{a} = \frac{\sqrt{31}-1}{\sqrt{31}+1} = \frac{16-\sqrt{31}}{15}.$$

$$\sec A = \frac{c}{b} = \frac{8}{\sqrt{31}-1}$$

$$= \tfrac{4}{15}(\sqrt{31} + 1).$$

$$\csc A = \frac{c}{a} = \frac{8}{\sqrt{31}+1}$$

$$= \tfrac{4}{15}(\sqrt{31} - 1).$$

18. Find a if $\sin A = \frac{3}{5}$, and $c = 20.5$.

$$\sin A = \frac{a}{c} = \frac{3}{5}.$$

$$\frac{a}{20.5} = \frac{3}{5}.$$

$$\therefore a = 12.3.$$

19. Find b if $\cos A = 0.44$, and $c = 3.5$.

$$\cos A = \frac{b}{c} = 0.44.$$

$$\frac{b}{3.5} = 0.44.$$

$$\therefore b = 1.54.$$

20. Find a if $\tan A = \frac{11}{3}$, and $b = 2\frac{5}{11}$.

$$\tan A = \frac{a}{b} = \frac{a}{2\frac{5}{11}} = \frac{11}{3}.$$

$$\therefore \frac{11\,a}{27} = \frac{11}{3}.$$

$$\therefore a = 9.$$

21. Find b if $\cot A = 4$, and $a = 17$.

$$\cot A = \frac{b}{a} = \frac{b}{17} = 4.$$

$$\therefore b = 68.$$

22. Find c if $\sec A = 2$, and $b = 20$.

$$\sec A = \frac{c}{b} = \frac{c}{20} = 2.$$

$$\therefore c = 40.$$

23. Find c if $\csc A = 6.45$, and $a = 35.6$.

$$\csc A = \frac{c}{a} = \frac{c}{35.6} = 6.45.$$

$$\therefore c = 229.62.$$

24. Construct a right triangle, given $c = 6$, $\tan A = \frac{3}{2}$.

$$\tan A = \frac{a}{b}.$$

$$\therefore a : b = 3 : 2.$$

Draw $AB = 2$.

Draw $BC \perp$ to $AB = 3$.

Join C and A.

Prolong AC to D, making $AD = 6$.

Draw $DE \perp$ to AB produced.

Rt. $\triangle AED$ is similar to rt. $\triangle ABC$.

$\therefore ADE$ is the rt. $\triangle$ required.

25. Construct a right triangle, given $a = 3.5$, $\cos A = \frac{1}{2}$.

Construct rt. $\triangle A'B'C'$ so that $b' = 1$, $c' = 2$.

Then $\cos A = \frac{1}{2}$.

Construct $\triangle ABC$ similar to $A'B'C'$, and having $a = 3.5$.

Then ABC is rt. $\triangle$ required.

26. Construct a right triangle, given $b = 2$, $\sin A = 0.6$.

Construct rt. $\triangle A'B'C'$, making $a' = 6$, $c' = 10$.

Then $\sin A' = \frac{6}{10}$.

Construct $\triangle ABC$ similar to $A'B'C'$, and having $b = 2$.

Then ABC is rt. $\triangle$ required.

27. Construct a right triangle, given $b = 4$, $\csc A = 4$.

Construct rt. $\triangle A'B'C'$, having $c' = 4$, $a' = 1$.

Then construct $\triangle ABC$ similar to $A'B'C'$, and having $b = 4$.

Then ABC is rt. $\triangle$ required.

28. In a right triangle, $c = 2.5$ miles, $\sin A = 0.6$, $\cos A = 0.8$; compute the legs.

$$\sin A = \frac{a}{c}. \qquad \cos A = \frac{b}{c}.$$

$$\therefore a = c \sin A. \qquad \therefore b = c \cos A.$$

$$\therefore a = 1.5. \qquad \therefore b = 2.$$

30. Find, by means of the table, the legs of a right triangle if $A = 20°$, $c = 1$; also, if $A = 20°$, $c = 4$.

$$A = 20°, \qquad c = 1.$$

$$\sin A = \frac{a}{c}.$$

$$\therefore a = c \sin A.$$

$$\therefore a = 0.342.$$

$$\cos A = \frac{b}{c}.$$

$$\therefore b = c \cos A.$$

$$\therefore b = 0.940.$$

$$A = 20°, \ c = 4.$$

$$\therefore a = 4 \times 0.342$$

$$= 1.368.$$

$$\therefore b = 4 \times 0.940$$

$$= 3.760.$$

31. By dividing the length of a vertical rod by the length of its horizontal shadow, the tangent of the angle of elevation of the sun at the time of observation was found to be 0.82. How high is a tower, if the length of its horizontal shadow at the same time is 174.3 yards?

$$\tan A = \frac{a}{b} = 0.82.$$

$$\therefore a = 0.82\,b.$$

$$b = 174.3 \text{ yards.}$$

$$\therefore a = 0.82 \text{ of } 174.3 \text{ yards}$$

$$= 142.926 \text{ yards.}$$

19. Show by construction that $2\sin A > \sin 2A$.

Construct $\angle BOC$ and $\angle COA$ each equal to the given $\angle A$.

Then $AB = 2\sin A$, and AD, the $\perp$ let fall from A to OB, $= \sin 2A$. But $AB > AD$.

Hence, $2\sin A > \sin 2A$.

20. Given two angles A and B, $A + B$ being less than 90°; show that $\sin(A + B) < \sin A + \sin B$.

Construct $HOK = \angle A$, and $COH = \angle B$.

Then $\sin(A + B) = CP$, $\sin A = HK$, $\sin B = CD$.

Now $\quad CP < CD + DE,$

and $\quad HK > DE.$

$$\therefore CP < CD + HK.$$

$$\therefore \sin(A + B) < \sin A + \sin B.$$

21. Given $\sin x$ in a unit circle; find the length of a line corresponding in position to $\sin x$ in a circle whose radius is r.

$1 : r = \sin x :$ the required line.

$\therefore$ length of line required $= r \sin x$.

22. In a right triangle, given the hypotenuse c, and also $\sin A = m$, $\cos A = n$; find the legs.

$$\sin A = \frac{a}{c} = m.$$

$$\therefore a = cm.$$

$$\cos A = \frac{b}{c} = n.$$

$$\therefore b = cn.$$

Exercise IV. Page 12.

1. Express as functions of the complementary angle:

sin 30°.
cos 45°.
tan 89°.
cot 15°.
csc 18° 10′.
cos 37° 24′.
cot 82° 19′.
csc 54° 46′.

$\sin 30° = \cos(90° - 30°) = \cos 60°.$
$\cos 45° = \sin(90° - 45°) = \sin 45°.$
$\tan 89° = \cot(90° - 89°) = \cot 1°.$
$\cot 15° = \tan(90° - 15°) = \tan 75°.$

$$\csc 18° 10' = \sec(90° - 18° 10') = \sec 71° 50'.$$

$$\cos 37° 24' = \sin(90° - 37° 24') = \sin 52° 36'.$$

$$\cot 82° 19 = \tan(90° - 82° 19') = \tan 7° 41'.$$

$$\csc 54° 46' = \sec(90° - 54° 46') = \sec 35° 14'.$$

2. Express as functions of an angle less than 45°:

sin 60°.
cos 75°.
tan 57°.
cot 84°.
csc 69° 2′.
cos 85° 39′.
cot 89° 59′.
csc 45° 1′.

$\sin 60° = \cos(90° - 60°) = \cos 30°.$
$\cos 75° = \sin(90° - 75°) = \sin 15°.$
$\tan 57° = \cot(90° - 57°) = \cot 33°.$
$\cot 84° = \tan(90° - 84°) = \tan 6°.$

$$\csc 69° 2' = \sec(90° - 69° 2') = \sec 20° 58'.$$

$$\cos 85° 39' = \sin(90° - 85° 39') = \sin 4° 21'.$$

$$\cot 89° 59' = \tan(90° - 89° 59') = \tan 0° 1'.$$

$$\csc 45° 1' = \sec(90° - 45° 1') = \sec 44° 59'.$$

3. Given $\tan 30^\circ = \frac{1}{3}\sqrt{3}$; find $\cot 60^\circ$.

$$\begin{aligned} \tan 30^\circ &= \cot(90^\circ - 30^\circ) \\ &= \cot 60^\circ. \\ \therefore \cot 60^\circ &= \tfrac{1}{3}\sqrt{3}. \end{aligned}$$

4. Given $\tan A = \cot A$; find A.

$$\begin{aligned} \tan A &= \cot(90^\circ - A), \\ 90^\circ - A &= A, \\ 2A &= 90^\circ. \\ \therefore A &= 45^\circ. \end{aligned}$$

5. Given $\cos A = \sin 2A$; find A.

$$\begin{aligned} \cos A &= \sin(90^\circ - A), \\ 90^\circ - A &= 2A, \\ 3A &= 90^\circ. \\ \therefore A &= 30^\circ. \end{aligned}$$

6. Given $\sin A = \cos 2A$; find A.

$$\begin{aligned} \sin A &= \cos(90^\circ - A), \\ 90^\circ - A &= 2A, \\ 3A &= 90^\circ. \\ \therefore A &= 30^\circ. \end{aligned}$$

7. Given $\cos A = \sin(45^\circ - \frac{1}{2}A)$; find A.

$$\begin{aligned} \cos A &= \sin(90^\circ - A), \\ 90^\circ - A &= 45^\circ - \tfrac{1}{2}A, \\ 180^\circ - 2A &= 90^\circ - A. \\ \therefore A &= 90^\circ. \end{aligned}$$

8. Given $\cot \frac{1}{2}A = \tan A$; find A.

$$\begin{aligned} \tan A &= \cot(90^\circ - A), \\ \tfrac{1}{2}A &= 90^\circ - A, \\ A &= 180^\circ - 2A, \\ 3A &= 180^\circ. \\ \therefore A &= 60^\circ. \end{aligned}$$

9. Given $\tan(45^\circ + A) = \cot A$; find A.

$$\begin{aligned} \cot A &= \tan(90^\circ - A), \\ \tan(90^\circ - A) &= \tan(45^\circ + A), \\ 90^\circ - A &= 45^\circ + A, \\ 2A &= 45^\circ. \\ \therefore A &= 22^\circ\, 30'. \end{aligned}$$

10. Find A if $\sin A = \cos 4A$.

$$\begin{aligned} \sin A &= \cos(90^\circ - A), \\ 90^\circ - A &= 4A, \\ 5A &= 90^\circ. \\ \therefore A &= 18^\circ. \end{aligned}$$

11. Find A if $\cot A = \tan 8A$.

$$\begin{aligned} \cot A &= \tan(90^\circ - A), \\ 8A &= 90^\circ - A, \\ 9A &= 90^\circ. \\ \therefore A &= 10^\circ. \end{aligned}$$

12. Find A if $\cot A = \tan nA$.

$$\begin{aligned} \cot A &= \tan(90^\circ - A), \\ 90^\circ - A &= nA, \\ 90^\circ &= A(n+1). \\ \therefore A &= \frac{90^\circ}{n+1}. \end{aligned}$$

EXERCISE V. PAGE 14.

1. Prove Formulas [1], [2], [3], using for the functions the line values in the unit circle given in Sect. III, p. 8.

[1] $\sin^2 A + \cos^2 A = 1.$

[2] $\tan A = \dfrac{\sin A}{\cos A}.$

[3] $$\begin{aligned} \sin A \times \csc A &= 1, \\ \cos A \times \sec A &= 1, \\ \tan A \times \cot A &= 1. \end{aligned}$$

[1] $$\begin{aligned} MP &= \sin A, \\ OM &= \cos A, \\ \overline{MP}^2 + \overline{OM}^2 &= \overline{OP}^2; \end{aligned}$$

but $$\overline{OP}^2 = 1.$$

4. What condition must be fulfilled by the lengths of the three lines a, b, c (Fig. 2) in order to make them the sides of a right triangle? Is this condition fulfilled in Example 3?

The condition is $a^2 + b^2 = c^2$.

It is; for

$$3^2 + 4^2 = 5^2. \qquad 9^2 + 40^2 = 41^2.$$
$$5^2 + 12^2 = 13^2. \qquad 3.9^2 + 8^2 = 8.9^2.$$
$$8^2 + 15^2 = 17^2. \qquad 1.19^2 + 1.20^2 = 1.69^2.$$

5. Find the values of the functions of A, if a, b, c, respectively, have the following values:

(i) $2mn,\ m^2 - n^2,\ m^2 + n^2.$

(ii) $\dfrac{2xy}{x-y},\ x + y,\ \dfrac{x^2+y^2}{x-y}.$

(iii) $pqr,\ qrs,\ rsp.$

(iv) $\dfrac{mn}{pq},\ \dfrac{mv}{sq},\ \dfrac{nr}{ps}.$

(i)

$$\sin A = \frac{a}{c} = \frac{2mn}{m^2+n^2}.$$
$$\cos A = \frac{b}{c} = \frac{m^2-n^2}{m^2+n^2}.$$
$$\tan A = \frac{a}{b} = \frac{2mn}{m^2-n^2}.$$
$$\cot A = \frac{b}{a} = \frac{m^2-n^2}{2mn}.$$
$$\sec A = \frac{c}{b} = \frac{m^2+n^2}{m^2-n^2}.$$
$$\csc A = \frac{c}{a} = \frac{m^2+n^2}{2mn}.$$

(ii)

$$\sin A = \frac{2xy}{x-y} \times \frac{x-y}{x^2+y^2} = \frac{2xy}{x^2+y^2}.$$
$$\cos A = (x+y) \times \frac{x-y}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}.$$
$$\tan A = \frac{2xy}{x-y} \times \frac{1}{x+y} = \frac{2xy}{x^2-y^2}.$$
$$\cot A = (x+y) \times \frac{x-y}{2xy} = \frac{x^2-y^2}{2xy}.$$
$$\sec A = \frac{x^2+y^2}{x-y} \times \frac{1}{x+y} = \frac{x^2+y^2}{x^2-y^2}.$$
$$\csc A = \frac{x^2+y^2}{x-y} \times \frac{x-y}{2xy} = \frac{x^2+y^2}{2xy}.$$

(iii)

$$\sin A = \frac{pqr}{rsp} = \frac{q}{s}. \quad \cos A = \frac{qrs}{rsp} = \frac{q}{p}.$$
$$\tan A = \frac{pqr}{qrs} = \frac{p}{s}. \quad \cot A = \frac{qrs}{pqr} = \frac{s}{p}.$$
$$\sec A = \frac{rsp}{qrs} = \frac{p}{q}. \quad \csc A = \frac{rsp}{pqr} = \frac{s}{q}.$$

(iv)

$$\sin A = \frac{mn}{pq} \times \frac{ps}{nr} = \frac{ms}{qr}.$$
$$\cos A = \frac{mv}{sq} \times \frac{ps}{nr} = \frac{mpv}{nqr}.$$
$$\tan A = \frac{mn}{pq} \times \frac{sq}{mv} = \frac{ns}{pv}.$$
$$\cot A = \frac{mv}{sq} \times \frac{pq}{mn} = \frac{pv}{ns}.$$
$$\sec A = \frac{nr}{ps} \times \frac{sq}{mv} = \frac{nqr}{mpv}.$$
$$\csc A = \frac{nr}{ps} \times \frac{pq}{mn} = \frac{qr}{ms}.$$

6. Prove that the values of a, b, c in (i) and (ii), Example 5, satisfy the condition necessary to make them the sides of a right triangle.

(i)

$$a^2 + b^2 = c^2.$$
$$(2mn)^2 + (m^2 - n^2)^2 = (m^2 + n^2)^2.$$
$$4m^2n^2 + m^4 - 2m^2n^2 + n^4 = m^4 + 2m^2n^2 + n^4.$$
$$m^4 + 2m^2n^2 + n^4 = m^4 + 2m^2n^2 + n^4.$$

(ii)

$$\left(\frac{2xy}{x-y}\right)^2 + (x+y)^2 = \left(\frac{x^2+y^2}{x-y}\right)^2.$$

$$\frac{4x^2y^2}{x^2-2xy+y^2} + x^2 + 2xy + y^2 = \frac{x^4+2x^2y^2+y^4}{x^2-2xy+y^2}.$$

$$4x^2y^2 + x^4 - 2x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4.$$

$$x^4 + 2x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4.$$

7. What equations of condition must be satisfied by the values of a, b, c in (iii) and (iv), Example 5, in order that the values may represent the sides of a right triangle?

(iii)

$$p^2q^2r^2 + q^2r^2s^2 = r^2s^2p^2,$$

or $$p^2q^2 + q^2s^2 = p^2s^2.$$

(iv)

$$\frac{m^2n^2}{p^2q^2} + \frac{m^2v^2}{s^2q^2} = \frac{n^2r^2}{p^2s^2},$$

or $m^2n^2s^2 + m^2p^2v^2 = n^2q^2r^2.$

8. Given $a^2 + b^2 = c^2$; find the functions of A and B when $a = 24$, $b = 143$.

$$\begin{aligned} c &= \sqrt{24^2 + 143^2} \\ &= \sqrt{21025} \\ &= 145. \end{aligned}$$

$$\sin A = \frac{24}{145} = \cos B.$$

$$\cos A = \frac{143}{145} = \sin B.$$

$$\tan A = \frac{24}{143} = \cot B.$$

$$\cot A = \frac{143}{24} = \tan B.$$

$$\sec A = \frac{145}{143} = \csc B.$$

$$\csc A = \frac{145}{24} = \sec B.$$

9. Given $a^2 + b^2 = c^2$; find the functions of A and B when

$$a = 0.264, \ c = 0.265.$$

$$\begin{aligned} b &= \sqrt{(c+a)(c-a)} \\ &= \sqrt{0.529 \times 0.001} \\ &= \sqrt{0.000529} \\ &= 0.023. \end{aligned}$$

$$\sin A = \frac{a}{c} = \frac{264}{265} = \cos B.$$

$$\cos A = \frac{b}{c} = \frac{23}{265} = \sin B.$$

$$\tan A = \frac{a}{b} = \frac{264}{23} = \cot B.$$

$$\cot A = \frac{b}{a} = \frac{23}{264} = \tan B.$$

$$\sec A = \frac{c}{b} = \frac{265}{23} = \csc B.$$

$$\csc A = \frac{c}{a} = \frac{265}{264} = \sec B.$$

10. Given $a^2 + b^2 = c^2$; find the functions of A and B when

$$b = 9.5, \ c = 19.3.$$

$$\begin{aligned} a &= \sqrt{(c+b)(c-a)} \\ &= \sqrt{28.8 \times 9.8} \\ &= \sqrt{282.24} \\ &= 16.8. \end{aligned}$$

$$\sin A = \frac{a}{c} = \frac{168}{193} = \cos B.$$

$$\cos A = \frac{b}{c} = \frac{95}{193} = \sin B.$$

$$\tan A = \frac{a}{b} = \frac{168}{95} = \cot B.$$

$$\cot A = \frac{b}{a} = \frac{95}{168} = \tan B.$$

$$\sec A = \frac{c}{b} = \frac{193}{95} = \csc B.$$

$$\csc A = \frac{c}{a} = \frac{193}{168} = \sec B.$$

11. Given $a^2 + b^2 = c^2$; find the functions of A and B when

$$a = \sqrt{p^2 + q^2},\ b = \sqrt{2\,pq}.$$

$$c = \sqrt{p^2 + q^2 + 2\,pq}$$

$$= p + q.$$

$$\sin A = \frac{a}{c} = \frac{\sqrt{p^2 + q^2}}{p + q} = \cos B.$$

$$\cos A = \frac{b}{c} = \frac{\sqrt{2\,pq}}{p + q} = \sin B.$$

$$\tan A = \frac{a}{b} = \frac{\sqrt{p^2 + q^2}}{\sqrt{2\,pq}} = \cot B.$$

$$\cot A = \frac{b}{a} = \frac{\sqrt{2\,pq}}{\sqrt{p^2 + q^2}} = \tan B.$$

$$\sec A = \frac{c}{b} = \frac{p + q}{\sqrt{2\,pq}} = \csc B.$$

$$\csc A = \frac{c}{a} = \frac{p + q}{\sqrt{p^2 + q^2}} = \sec B.$$

12. Given $a^2 + b^2 = c^2$; find the functions of A and B when

$$a = \sqrt{p^2 + pq},\ c = p + q.$$

$$b^2 = c^2 - a^2$$

$$= q^2 + pq.$$

$$\therefore b = \sqrt{q^2 + pq}.$$

$$\sin A = \frac{a}{c} = \frac{\sqrt{p^2 + pq}}{p + q} = \cos B.$$

$$\cos A = \frac{b}{c} = \frac{\sqrt{q^2 + pq}}{p + q} = \sin B.$$

$$\tan A = \frac{a}{b} = \frac{\sqrt{p^2 + pq}}{\sqrt{q^2 + pq}}$$

$$= \sqrt{\frac{p}{q}} = \cot B.$$

$$\cot A = \frac{b}{a} = \frac{\sqrt{q^2 + pq}}{\sqrt{p^2 + pq}}$$

$$= \sqrt{\frac{q}{p}} = \tan B.$$

$$\sec A = \frac{c}{b} = \frac{p + q}{\sqrt{q^2 + pq}} = \csc B.$$

$$\csc A = \frac{c}{a} = \frac{p + q}{\sqrt{p^2 + pq}} = \sec B.$$

13. Given $a^2 + b^2 = c^2$; find the functions of A and B when

$$b = 2\sqrt{pq},\ c = p + q.$$

$$a^2 = c^2 - b^2.$$

$$a^2 = p^2 + 2\,pq + q^2 - 4\,pq$$

$$= p^2 - 2\,pq + q^2.$$

$$\therefore a = p - q.$$

$$\sin A = \frac{a}{c} = \frac{p - q}{p + q} = \cos B.$$

$$\cos A = \frac{b}{c} = \frac{2\sqrt{pq}}{p + q} = \sin B.$$

$$\tan A = \frac{a}{b} = \frac{p - q}{2\sqrt{pq}} = \cot B.$$

$$\cot A = \frac{b}{a} = \frac{2\sqrt{pq}}{p - q} = \tan B.$$

$$\sec A = \frac{c}{b} = \frac{p + q}{2\sqrt{pq}} = \csc B.$$

$$\csc A = \frac{c}{a} = \frac{p + q}{p - q} = \sec B.$$

14. Given $a^2 + b^2 = c^2$; find the functions of A when $a = 2\,b$.

$$a = 2\,b.$$

$$a^2 + b^2 = c^2.$$

$$4\,b^2 + b^2 = c^2.$$

$$5\,b^2 = c^2.$$

$$\therefore c = b\sqrt{5}.$$

$$\sin A = \frac{a}{c} = \frac{2b}{b\sqrt{5}} = \tfrac{2}{5}\sqrt{5}.$$

$$\cos A = \frac{b}{c} = \frac{b}{b\sqrt{5}} = \tfrac{1}{5}\sqrt{5}.$$

$$\tan A = \frac{a}{b} = \frac{2b}{b} = 2.$$

$$\cot A = \frac{b}{a} = \frac{1}{2}.$$

$$\sec A = \frac{c}{b} = \frac{b\sqrt{5}}{b} = \sqrt{5}.$$

$$\csc A = \frac{c}{a} = \frac{b\sqrt{5}}{2b} = \tfrac{1}{2}\sqrt{5}.$$

15. Given $a^2 + b^2 = c^2$; find the functions of A when $a = \frac{2}{3}c$.

$$\begin{aligned} a &= \tfrac{2}{3}c. \\ b &= \sqrt{c^2 - a^2} \\ &= \sqrt{c^2 - \tfrac{4}{9}c^2} \\ &= \frac{c}{3}\sqrt{5}. \end{aligned}$$

$$\sin A = \frac{a}{c} = \frac{\frac{2}{3}c}{c} = \frac{2}{3}.$$

$$\cos A = \frac{b}{c} = \frac{\frac{c}{3}\sqrt{5}}{c} = \tfrac{1}{3}\sqrt{5}.$$

$$\tan A = \frac{a}{b} = \frac{\frac{2}{3}c}{\frac{c}{3}\sqrt{5}} = \tfrac{2}{5}\sqrt{5}.$$

$$\cot A = \frac{b}{a} = \frac{\frac{c}{3}\sqrt{5}}{\frac{2}{3}c} = \tfrac{1}{2}\sqrt{5}.$$

$$\sec A = \frac{c}{b} = \frac{c}{\frac{c}{3}\sqrt{5}} = \tfrac{3}{5}\sqrt{5}.$$

$$\csc A = \frac{c}{a} = \frac{c}{\frac{2}{3}c} = \frac{3}{2}.$$

16. Given $a^2 + b^2 = c^2$; find the functions of A when $a + b = \frac{5}{4}c$.

$$\begin{aligned} a + b &= \tfrac{5}{4}c. \\ a^2 + b^2 &= c^2. \\ a^2 + b^2 + 2ab &= \tfrac{25}{16}c^2. \\ 2ab &= \tfrac{9}{16}c^2. \\ a^2 - 2ab + b^2 &= \tfrac{7}{16}c^2. \\ a - b &= \frac{c}{4}\sqrt{7}. \\ a + b &= \tfrac{5}{4}c. \\ 2b &= \frac{5}{4}c - \frac{c}{4}\sqrt{7}. \\ b &= \frac{c}{8}(5 - \sqrt{7}). \\ 2a &= \frac{5}{4}c + \frac{c}{4}\sqrt{7}. \\ a &= \frac{c}{8}(5 + \sqrt{7}). \end{aligned}$$

$$\sin A = \frac{a}{c} = \frac{\frac{c}{8}(5 + \sqrt{7})}{c} = \frac{5 + \sqrt{7}}{8}.$$

$$\cos A = \frac{b}{c} = \frac{\frac{c}{8}(5 - \sqrt{7})}{c} = \frac{5 - \sqrt{7}}{8}.$$

$$\tan A = \frac{a}{b} = \frac{5 + \sqrt{7}}{5 - \sqrt{7}} = \frac{16 + 5\sqrt{7}}{9}.$$

$$\cot A = \frac{b}{a} = \frac{5 - \sqrt{7}}{5 + \sqrt{7}} = \frac{16 - 5\sqrt{7}}{9}.$$

$$\sec A = \frac{c}{b} = \frac{c}{\frac{c}{8}(5 - \sqrt{7})} = \tfrac{4}{9}(5 + \sqrt{7}).$$

$$\csc A = \frac{c}{a} = \frac{c}{\frac{c}{8}(5 + \sqrt{7})} = \tfrac{4}{9}(5 - \sqrt{7}).$$

From [1], $$1 - \sin^2 A = \cos^2 A.$$

$$\therefore \csc A - \sin A = \frac{\cos^2 A}{\sin A}.$$

Also, from [2], $$\frac{\cos A}{\sin A} = \cot A.$$

$$\therefore \csc A - \sin A = \cos A \cot A.$$

22. Prove that $\dfrac{\cos A}{1 - \sin A} = \dfrac{1 + \sin A}{\cos A}$.

Clearing of fractions, this becomes,

$$\cos^2 A = 1 - \sin^2 A,$$

which is correct by [1].

$$\therefore \frac{\cos A}{1 - \sin A} = \frac{1 + \sin A}{\cos A}.$$

Exercise VI. Page 16.

1. Find the values of the other functions when $\sin A = \frac{12}{13}$.

$$\sin^2 A + \cos^2 A = 1,$$

$$\cos^2 A = 1 - \left(\frac{12}{13}\right)^2,$$

$$\cos A = \sqrt{1 - \left(\frac{12}{13}\right)^2}$$

$$= \sqrt{\frac{25}{169}}.$$

$$\therefore \cos A = \frac{5}{13}.$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{12}{5}.$$

$\cot A$ is reciprocal of $\tan A$.

$$\therefore \cot A = \frac{5}{12}.$$

$\sec A$ is reciprocal of $\cos A$.

$$\therefore \sec A = \frac{13}{5}.$$

$\csc A$ is reciprocal of $\sin A$.

$$\therefore \csc A = \frac{13}{12}.$$

2. Find the values of the other functions when $\sin A = 0.8$.

$$\sin^2 A + \cos^2 A = 1,$$

$$\cos^2 A = 1 - (0.8)^2,$$

$$\cos A = \sqrt{1 - 0.64}.$$

$$\therefore \cos A = 0.6.$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{0.8}{0.6}.$$

$$\therefore \tan A = 1.3333.$$

$$\cot A = \frac{0.6}{0.8}.$$

$$\therefore \cot A = 0.75.$$

$$\sec A = \frac{1}{0.6}.$$

$$\therefore \sec A = 1.6667.$$

$$\csc A = \frac{1}{0.8}.$$

$$\therefore \csc A = 1.25.$$

3. Find the values of the other functions when $\cos A = \frac{60}{61}$.

$$\sin^2 A + \cos^2 A = 1,$$

$$\sin A = \sqrt{1 - \frac{3600}{3721}} = \sqrt{\frac{121}{3721}} = \frac{11}{61}.$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{11}{60}.$$

$$\cot A = \frac{1}{\tan A} = \frac{60}{11}.$$

$$\sec A = \frac{1}{\cos A} = \frac{61}{60}.$$

$$\csc A = \frac{1}{\sin A} = \frac{61}{11}.$$

4. Find the values of the other functions when $\cos A = 0.28$.

$$\begin{aligned} \sin^2 A + \cos^2 A &= 1, \\ \sin A &= \sqrt{1 - (0.28)^2} \\ &= \sqrt{0.9216} = 0.96. \end{aligned}$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{0.96}{0.28} = 3.4286.$$

$$\cot A = \frac{1}{\tan A} = \frac{1}{3.4286} = 0.2917.$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{0.28} = 3.5714.$$

$$\csc A = \frac{1}{\sin A} = \frac{1}{0.96} = 1.0417.$$

5. Find the values of the other functions when $\tan A = \frac{4}{3}$.

$$\tan A = \frac{4}{3}.$$

$$\therefore \cot A = \frac{3}{4}.$$

$$\tan A = \frac{\sin A}{\cos A},$$

$$\frac{4}{3} = \frac{\sin A}{\sqrt{1 - \sin^2 A}}.$$

$$\begin{aligned} 3 \sin A &= 4\sqrt{1 - \sin^2 A}, \\ 9 \sin^2 A &= 16 - 16 \sin^2 A, \\ 5 \sin A &= 4. \\ \therefore \sin A &= \frac{4}{5}. \end{aligned}$$

$$\cos A = \frac{\sin A}{\tan A} = \frac{3}{5}.$$

$$\sec A = \frac{1}{\cos A} = \frac{5}{3}.$$

$$\csc A = \frac{1}{\sin A} = \frac{5}{4}.$$

6. Find the values of the other functions when $\cot A = 1$.

$$\begin{aligned} \cot A &= 1. \\ \therefore \tan A &= 1. \\ \tan A &= \frac{\sin A}{\cos A}, \end{aligned}$$

$$1 = \frac{\sin A}{\sqrt{1 - \sin^2 A}},$$

$$\begin{aligned} \sin A &= \sqrt{1 - \sin^2 A}, \\ \sin^2 A &= 1 - \sin^2 A, \\ 2 \sin^2 A &= 1, \\ \sin^2 A &= \frac{1}{2}. \\ \therefore \sin A &= \sqrt{\tfrac{1}{2}} = \tfrac{1}{2}\sqrt{2}. \end{aligned}$$

$$\cos A = \frac{\sin A}{\tan A} = \tfrac{1}{2}\sqrt{2}.$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{\frac{1}{2}\sqrt{2}} = \sqrt{2}.$$

$$\csc A = \frac{1}{\sin A} = \frac{1}{\frac{1}{2}\sqrt{2}} = \sqrt{2}.$$

7. Find the values of the other functions when $\cot A = 0.5$.

$$\tan A = \frac{1}{\cot A} = \frac{1}{0.5} = 2.$$

$$\tan A = \frac{\sin A}{\cos A} = 2.$$

$$\begin{aligned} 2 \cos A &= \sin A. \\ 4 \cos^2 A - \sin^2 A &= 0. \\ \cos^2 A + \sin^2 A &= 1. \\ \text{Add,} \qquad 5 \cos^2 A &= 1. \end{aligned}$$

$$\cos A = \sqrt{\frac{1}{5}} = 0.45.$$

$$\begin{array}{r} 4\cos^2 A + 4\sin^2 A = 4 \\ 4\cos^2 A - \quad \sin^2 A = 0 \\ \hline 5\sin^2 A = 4 \end{array}$$

$$\sin A = \sqrt{\frac{4}{5}} = 0.90.$$

$$\sec A = \frac{1}{\cos A} = 2.22.$$

$$\csc A = \frac{1}{\sin A} = 1.11.$$

8. Find the values of the other functions when $\sec A = 2$.

$$\cos A = \frac{1}{\sec A} = \frac{1}{2}.$$

$$\sin A = \sqrt{1 - \cos^2 A}$$

$$= \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}}.$$

$$\therefore \sin A = \tfrac{1}{2}\sqrt{3}.$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}} = \sqrt{3}.$$

$$\cot A = \frac{1}{\tan A} = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3}.$$

$$\csc A = \frac{1}{\sin A} = \tfrac{2}{3}\sqrt{3}.$$

9. Find the values of the other functions when $\csc A = \sqrt{2}$.

$$\sin A = \frac{1}{\sqrt{2}} = \tfrac{1}{2}\sqrt{2}.$$

$$\cos A = \sqrt{1 - (\tfrac{1}{2}\sqrt{2})^2} = \sqrt{1 - \tfrac{1}{2}}$$

$$= \sqrt{\tfrac{1}{2}} = \tfrac{1}{2}\sqrt{2}.$$

$$\tan A = \frac{\frac{1}{2}\sqrt{2}}{\frac{1}{2}\sqrt{2}} = 1.$$

$$\cot A = \frac{1}{1} = 1.$$

$$\sec A = \frac{1}{\frac{1}{2}\sqrt{2}} = \sqrt{2}.$$

10. Find the values of the other functions when $\sin A = m$.

$$\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - m^2}.$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{m}{\sqrt{1 - m^2}}$$

$$= \frac{m\sqrt{1 - m^2}}{1 - m^2}.$$

$$\cot A = \frac{1}{\tan A} = \frac{1}{m}\sqrt{1 - m^2}.$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1 - m^2}}.$$

$$\csc A = \frac{1}{\sin A} = \frac{1}{m}.$$

11. Find the values of the other functions when $\sin A = \dfrac{2m}{1 + m^2}$.

$$\cos A = \sqrt{1 - \sin^2 A}.$$

$$\therefore \cos A = \sqrt{1 - \frac{4m^2}{1 + 2m^2 + m^4}}$$

$$= \sqrt{\frac{1 - 2m^2 + m^4}{1 + 2m^2 + m^4}}$$

$$= \frac{1 - m^2}{1 + m^2}.$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{2m}{1 - m^2}.$$

$$\cot A = \frac{1}{\tan A} = \frac{1 - m^2}{2m}.$$

$$\sec A = \frac{1}{\cos A} = \frac{1 + m^2}{1 - m^2}.$$

$$\csc A = \frac{1}{\sin A} = \frac{1 + m^2}{2m}.$$

12. Find the values of the other functions when $\cos A = \dfrac{2mn}{m^2 + n^2}$.

$$\sin A = \sqrt{1 - \cos^2 A}.$$

$$= \sqrt{1 - \frac{4m^2n^2}{m^4 + 2m^2n^2 + n^4}}$$

$$= \sqrt{\frac{m^4 - 2m^2n^2 + n^4}{m^4 + 2m^2n^2 + n^4}}$$

$$= \frac{m^2 - n^2}{m^2 + n^2}.$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{m^2 - n^2}{2mn}.$$

$$\cot A = \frac{1}{\tan A} = \frac{2mn}{m^2 - n^2}.$$

$$\sec A = \frac{1}{\cos A} = \frac{m^2 + n^2}{2mn}.$$

$$\csc A = \frac{1}{\sin A} = \frac{m^2 + n^2}{m^2 - n^2}.$$

13. Given $\tan 45° = 1$; find the other functions of 45°.

$$\frac{\sin 45°}{\cos 45°} = \tan 45°.$$

$$\frac{\sin 45°}{\cos 45°} = 1. \qquad (1)$$

$$\sin^2 45° + \cos^2 45° = 1. \qquad (2)$$

By (1), $\sin 45° = \cos 45°$.

By (2), $\cos^2 45° + \cos^2 45° = 1.$

$$2\cos^2 45° = 1,$$

$$\cos^2 45° = \frac{1}{2},$$

$$\cos 45° = \sqrt{\frac{1}{2}} = \tfrac{1}{2}\sqrt{2}.$$

$$\sin 45° = \tfrac{1}{2}\sqrt{2}.$$

$$\cot 45° = \frac{1}{\tan 45°} = \frac{1}{1} = 1.$$

$$\sec 45° = \frac{1}{\frac{1}{2}\sqrt{2}} = \sqrt{2}.$$

$$\csc 45° = \frac{1}{\frac{1}{2}\sqrt{2}} = \sqrt{2}.$$

14. Given $\sin 30° = \frac{1}{2}$; find the other functions of 30°.

$$\sin^2 30° + \cos^2 30° = 1.$$

$$\cos 30° = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \tfrac{1}{2}\sqrt{3}.$$

$$\tan 30° = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} = \tfrac{1}{3}\sqrt{3}.$$

$$\cot 30° = \frac{1}{\frac{1}{3}\sqrt{3}} = \sqrt{3}.$$

$$\sec 30° = \frac{1}{\frac{1}{2}\sqrt{3}} = \tfrac{2}{3}\sqrt{3}.$$

$$\csc 30° = \frac{1}{\frac{1}{2}} = 2.$$

15. Given $\csc 60° = \frac{2}{3}\sqrt{3}$; find the other function of 60°.

$$\sin 60° = \frac{1}{\csc 60°}$$

$$= \frac{1}{\frac{2}{3}\sqrt{3}} = \tfrac{1}{2}\sqrt{3}.$$

$$\cos 60° = \sqrt{1 - \sin^2 60°}$$

$$= \sqrt{1 - (\tfrac{1}{2}\sqrt{3})^2}$$

$$= \sqrt{1 - \frac{3}{4}} = \frac{1}{2}.$$

$$\tan 60° = \frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}} = \sqrt{3}.$$

$$\cot 60° = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3}.$$

$$\sec 60° = \frac{1}{\frac{1}{2}} = 2.$$

16. Given $\tan 15° = 2 - \sqrt{3}$; find the other functions of 15°.

$$\tan 15° = \frac{\sin 15°}{\cos 15°} = 2 - \sqrt{3}.$$

$$\sin^2 15° + \cos^2 15° = 1.$$

$$\sin 15° = (2 - \sqrt{3})\cos 15°.$$

$$[(2 - \sqrt{3})\cos 15°]^2 + \cos^2 15° = 1,$$

$$(4 - 4\sqrt{3} + 3)\cos^2 15° + \cos^2 15° = 1,$$

$$(8 - 4\sqrt{3})\cos^2 15° = 1.$$

$$\cos^2 15° = \frac{1}{4(2 - \sqrt{3})} = \frac{2 + \sqrt{3}}{4},$$

$$\cos 15^\circ = \sqrt{\frac{2+\sqrt{3}}{4}} = \tfrac{1}{2}\sqrt{2+\sqrt{3}}.$$

$$\sin^2 15^\circ = 1 - \cos^2 15^\circ.$$

$$\sin^2 15^\circ = 1 - \frac{2+\sqrt{3}}{4} = \frac{2-\sqrt{3}}{4},$$

$$\sin 15^\circ = \tfrac{1}{2}\sqrt{2-\sqrt{3}}.$$

$$\cot 15^\circ = \frac{1}{\tan 15^\circ} = \frac{1}{2-\sqrt{3}}$$

$$= 2 + \sqrt{3}.$$

$$\sec 15^\circ = \frac{1}{\tfrac{1}{2}\sqrt{2+\sqrt{3}}}$$

$$= 2(2-\sqrt{3})\sqrt{2+\sqrt{3}}.$$

$$\csc 15^\circ = \frac{1}{\tfrac{1}{2}\sqrt{2-\sqrt{3}}}$$

$$= 2(2+\sqrt{3})\sqrt{2-\sqrt{3}}.$$

17. Given $\cot 22^\circ 30' = \sqrt{2} + 1$; find the other functions of $22^\circ 30'$.

$$\tan 22\tfrac{1}{2}^\circ = \frac{1}{\cot 22\tfrac{1}{2}^\circ} = \frac{1}{\sqrt{2}+1}$$

$$= \sqrt{2} - 1.$$

$$\frac{\sin 22\tfrac{1}{2}^\circ}{\cos 22\tfrac{1}{2}^\circ} = \tan 22\tfrac{1}{2}^\circ, \qquad (1)$$

$$\cos^2 22\tfrac{1}{2}^\circ + \sin^2 22\tfrac{1}{2}^\circ = 1. \qquad (2)$$

From (1),

$$\cos 22\tfrac{1}{2}^\circ \tan 22\tfrac{1}{2}^\circ = \sin 22\tfrac{1}{2}^\circ.$$

Square,

$$\cos^2 22\tfrac{1}{2}^\circ \tan^2 22\tfrac{1}{2}^\circ = \sin^2 22\tfrac{1}{2}^\circ.$$

From (2),

$$\underline{\cos^2 22\tfrac{1}{2}^\circ = -\sin^2 22\tfrac{1}{2}^\circ + 1}$$

Add,

$$\cos^2 22\tfrac{1}{2}^\circ \tan^2 22\tfrac{1}{2}^\circ + \cos^2 22\tfrac{1}{2}^\circ = 1.$$

$$\cos^2 22\tfrac{1}{2}^\circ (\tan^2 22\tfrac{1}{2}^\circ + 1) = 1,$$

$$\cos^2 22\tfrac{1}{2}^\circ (4 - 2\sqrt{2}) = 1,$$

$$\cos 22\tfrac{1}{2}^\circ \sqrt{4 - 2\sqrt{2}} = 1.$$

$$\therefore \cos 22\tfrac{1}{2}^\circ = \frac{1}{\sqrt{4-2\sqrt{2}}}$$

$$= \sqrt{\frac{4+2\sqrt{2}}{8}}$$

$$= \tfrac{1}{2}\sqrt{2+\sqrt{2}}.$$

$$\sin 22\tfrac{1}{2}^\circ = \sqrt{1 - \frac{2+\sqrt{2}}{4}}$$

$$= \sqrt{\frac{4-2-\sqrt{2}}{4}}$$

$$= \sqrt{\frac{2-\sqrt{2}}{4}}$$

$$= \tfrac{1}{2}\sqrt{2-\sqrt{2}}.$$

$$\sec 22\tfrac{1}{2}^\circ = \frac{1}{\tfrac{1}{2}\sqrt{2+\sqrt{2}}}$$

$$= (2-\sqrt{2})\sqrt{2+\sqrt{2}}.$$

$$\csc 22\tfrac{1}{2}^\circ = \frac{1}{\tfrac{1}{2}\sqrt{2-\sqrt{2}}}$$

$$= (2+\sqrt{2})\sqrt{2-\sqrt{2}}.$$

18. Given $\sin 0^\circ = 0$; find the other functions of 0°.

$$\cos 0^\circ = \sqrt{1 - \sin^2 0^\circ}$$

$$= \sqrt{1-0}$$

$$= 1.$$

$$\tan 0^\circ = \frac{\sin 0^\circ}{\cos 0^\circ} = \frac{0}{1} = 0.$$

$$\cot 0^\circ = \frac{1}{\tan 0^\circ} = \frac{1}{0} = \infty.$$

$$\sec 0^\circ = \frac{1}{\cos 0^\circ} = \frac{1}{1} = 1.$$

$$\csc 0^\circ = \frac{1}{\sin 0^\circ} = \frac{1}{0} = \infty.$$

19. Given $\sin 90^\circ = 1$; find the other functions of 90°.

$$\sin 99^\circ = 1.$$

$$\cos 90^\circ = \sqrt{1 - \sin^2 90^\circ} = 0.$$

$$\tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \frac{1}{0} = \infty.$$

$$\cot 90^\circ = \frac{1}{\tan 90^\circ} = \frac{1}{\infty} = 0.$$

$$\sec 90^\circ = \frac{1}{\cos 90^\circ} = \frac{1}{0} = \infty.$$

$$\csc 90^\circ = \frac{1}{\sin 90^\circ} = \frac{1}{1} = 1.$$

20. Given $\tan 90^\circ = \infty$; find the other functions of 90°.

$$\tan 90^\circ = \infty.$$

$$\cot 90^\circ = \frac{1}{\tan 90^\circ} = \frac{1}{\infty} = 0.$$

$$\frac{\cos 90^\circ}{\sin 90^\circ} = \cot 90^\circ = 0.$$

$$\therefore \cos 90^\circ = 0.$$

$$\sin^2 90^\circ + \cos^2 90^\circ = 1.$$

$$\therefore \sin^2 90^\circ = 1,$$

$$\sin 90^\circ = 1.$$

$$\sec 90^\circ = \frac{1}{0} = \infty.$$

$$\csc 90^\circ = 1.$$

21. Express the values of all the other functions in terms of $\sin A$.

$$\cos A = \sqrt{1 - \sin^2 A}.$$

$$\tan A = \frac{\sin A}{\sqrt{1 - \sin^2 A}}.$$

$$\cot A = \frac{\sqrt{1 - \sin^2 A}}{\sin A}.$$

$$\sec A = \frac{1}{\sqrt{1 - \sin^2 A}}.$$

$$\csc A = \frac{1}{\sin A}.$$

22. Express the values of all the other functions in terms of $\cos A$.

$$\sin A = \sqrt{1 - \cos^2 A}.$$

$$\tan A = \frac{\sqrt{1 - \cos^2 A}}{\cos A}.$$

$$\cot A = \frac{\cos A}{\sqrt{1 - \cos^2 A}}.$$

$$\sec A = \frac{1}{\cos A}.$$

$$\csc A = \frac{1}{\sqrt{1 - \cos^2 A}}.$$

23. Express the values of all the other functions in terms of $\tan A$.

$$\cot A = \frac{1}{\tan A}.$$

$$\frac{a}{b} = \tan A.$$

$$a = b \tan A.$$

$$a^2 = b^2 \tan^2 A.$$

$$\begin{array}{l} a^2 - b^2 \tan^2 A = 0 \\ \underline{a^2 + b^2 \qquad\qquad = 1} \\ b^2(1 + \tan^2 A) = 1 \end{array}$$

$$b^2 = \frac{1}{1 + \tan^2 A} = \cos^2 A.$$

$$\cos A = \sqrt{\frac{1}{1 + \tan^2 A}}.$$

$$\sin A = \sqrt{1 - \cos^2 A}$$

$$= \sqrt{1 - \frac{1}{1 + \tan^2 A}}$$

$$= \frac{\tan A}{\sqrt{1 + \tan^2 A}}.$$

$$\sec A = \frac{1}{\cos A} = \sqrt{1 + \tan^2 A}.$$

$$\csc A = \frac{1}{\sin A} = \frac{\sqrt{1 + \tan^2 A}}{\tan A}.$$

24. Express the values of all the other functions in terms of $\cot A$.

$$\frac{1}{\cot A} = \tan A.$$

$$\frac{\sin A}{\cos A} = \tan A.$$

Let $x = \sin A,\ y = \cos A.$

$$\frac{x}{y} = \frac{1}{\cot A}.$$

$$x \cot A = y,$$

$$x^2 \cot^2 A = y^2.$$

$$\begin{array}{r}x^2\cot^2 A - y^2 = 0\\ x^2 \qquad\quad + y^2 = 1\\ \hline x^2(1+\cot^2 A) = 1\end{array}$$

$$x^2 = \frac{1}{1+\cot^2 A}.$$

$$\sin A = \frac{1}{\sqrt{1+\cot^2 A}}.$$

$$\cos A = \sqrt{1-\sin^2 A}$$

$$= \sqrt{1 - \frac{1}{1+\cot^2 A}}$$

$$= \sqrt{\frac{1+\cot^2 A - 1}{1+\cot^2 A}}$$

$$= \frac{\cot A}{\sqrt{1+\cot^2 A}}.$$

$$\sec A = \frac{1}{\cos A} = \frac{\sqrt{1+\cot^2 A}}{\cot A}.$$

$$\csc A = \frac{1}{\sin A} = \sqrt{1+\cot^2 A}.$$

25. Given $2 \sin A = \cos A$; find $\sin A$ and $\cos A$.

$$\sin^2 A + \cos^2 A = 1.$$

$$\sin^2 A + 4\sin^2 A = 1.$$

$$5 \sin^2 A = 1,$$

$$\sin^2 A = \frac{1}{5},$$

$$\sin A = \sqrt{\frac{1}{5}} = \tfrac{1}{5}\sqrt{5}.$$

$$\therefore \cos A = \tfrac{2}{5}\sqrt{5}.$$

26. Given $4 \sin A = \tan A$; find $\sin A$ and $\tan A$.

$$\tan A = \frac{\sin A}{\cos A}.$$

But $\tan A = 4 \sin A.$

$$\therefore \sin A = \frac{\sin A}{\cos A}.$$

$$4 \sin A \times \cos A = \sin A.$$

$$\therefore \cos A = \frac{\sin A}{4 \sin A} = \frac{1}{4}.$$

$$\sin^2 A + \cos^2 A = 1.$$

$$\therefore \sin A = \sqrt{1 - \frac{1}{16}} = \sqrt{\frac{15}{16}}$$

$$= \tfrac{1}{4}\sqrt{15}.$$

$$\tan A = \frac{\sin A}{\cos A}$$

$$= \frac{\tfrac{1}{4}\sqrt{15}}{\tfrac{1}{4}} = \sqrt{15}.$$

27. If $\sin A : \cos A = 9 : 40$, find $\sin A$ and $\cos A$.

$$40 \sin A = 9 \cos A.$$

Square,

$$1600 \sin^2 A = 81 \cos^2 A.$$

$$1600 \sin^2 A - 81 \cos^2 A = 0.$$

But $\qquad \sin^2 A + \cos^2 A = 1.$

Multiply by 81 and add,

$$1681 \sin^2 A = 81.$$

$$\therefore 41 \sin A = 9.$$

$$\sin A = \frac{9}{41}.$$

$$\sin^2 A + \cos^2 A = 1.$$

$$\cos A = \sqrt{1 - \sin^2 A}.$$

$$\therefore \cos A = \sqrt{1 - \left(\frac{9}{41}\right)^2} = \frac{40}{41}.$$

28. Transform the quantity

$$\tan^2 A + \cot^2 A - \sin^2 A - \cos^2 A$$

into a form containing only $\cos A$.

$$\tan^2 A = \frac{\sin^2 A}{\cos^2 A} = \frac{1-\cos^2 A}{\cos^2 A}.$$

$$\cot^2 A = \frac{\cos^2 A}{\sin^2 A} = \frac{\cos^2 A}{1-\cos^2 A}.$$

$$\frac{1-\cos^2 A}{\cos^2 A}+\frac{\cos^2 A}{1-\cos^2 A}$$

$$-1+\cos^2 A-\cos^2 A$$

$$=\frac{1-2\cos^2 A+2\cos^4 A-\cos^2 A+\cos^4 A}{\cos^2 A-\cos^4 A}$$

$$=\frac{1-3\cos^2 A+3\cos^4 A}{\cos^2 A-\cos^4 A}.$$

29. Prove that

$\sin A+\cos A=(1+\tan A)\cos A.$

$$\frac{\sin A}{\cos A}=\tan A.$$

$$\sin A=\tan A\cos A.$$

$$\sin A+\cos A=\tan A\cos A+\cos A$$

$$=(1+\tan A)\cos A.$$

30. Prove that

$\tan A+\cot A=\sec A\times\csc A.$

$$\tan A=\frac{\sin A}{\cos A}.$$

$$\cot A=\frac{\cos A}{\sin A}.$$

$$\tan A+\cot A=\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}$$

$$=\frac{\sin^2 A+\cos^2 A}{\cos A\sin A}.$$

But $\sin^2 A+\cos^2 A=1.$

$$\therefore \tan A+\cot A=\frac{1}{\cos A\sin A}$$

$$=\sec A\times\csc A.$$

EXERCISE VII. PAGE 18.

1. Solve the equation

$$2\cos x=\sec x.$$

$$2\cos x=\frac{1}{\cos x}.$$

$$\therefore 2\cos^2 x=1.$$

$$\cos x=\sqrt{\tfrac{1}{2}}.$$

$$\therefore x=45^\circ.$$

2. Solve the equation

$$4\sin x=\csc x.$$

$$4\sin x=\frac{1}{\sin x}.$$

$$\therefore 4\sin^2 x=1.$$

$$\sin x=\tfrac{1}{2}.$$

$$\therefore x=30^\circ.$$

3. Solve the equation

$$\tan x=2\sin x.$$

$$\frac{\sin x}{\cos x}=2\sin x.$$

$$\sin x=2\sin x\cos x.$$

$$(1-2\cos x)\sin x=0.$$

$$\therefore \sin x=0. \qquad (1)$$

$$\therefore x=0.$$

$$1-2\cos x=0. \qquad (2)$$

$$\cos x=\tfrac{1}{2}.$$

$$\therefore x=60^\circ.$$

$$\therefore x=0^\circ \text{ or } 60^\circ.$$

4. Solve the equation

$$\sec x=\sqrt{2}\tan x.$$

$$\frac{1}{\cos x}=\sqrt{2}\,\frac{\sin x}{\cos x}.$$

$$1=\sqrt{2}\sin x.$$

$$\sin x=\frac{1}{\sqrt{2}}.$$

$$\therefore x=45^\circ.$$

5. Solve the equation

$$\sin^2 x=3\cos^2 x$$

$$\frac{\sin^2 x}{\cos^2 x}=3.$$

$$\tan^2 x=3.$$

$$\tan x=\sqrt{3}.$$

$$\therefore x=60^\circ.$$

6. Solve the equation

$$\begin{aligned} 2\sin^2 x + \cos^2 x &= \tfrac{3}{2}. \\ \sin^2 x + (\sin^2 x + \cos^2 x) &= \tfrac{3}{2}. \\ \sin^2 x + 1 &= \tfrac{3}{2}. \\ \sin^2 x &= \tfrac{1}{2}. \\ \sin x &= \frac{1}{\sqrt{2}}. \\ \therefore x &= 45^\circ. \end{aligned}$$

7. Solve the equation

$$\begin{aligned} 3\tan^2 x - \sec^2 x &= 1. \\ 3\tan^2 x - (\tan^2 x + 1) &= 1. \\ 2\tan^2 x - 1 &= 1. \\ 2\tan^2 x &= 2. \\ \tan x &= 1. \\ \therefore x &= 45^\circ. \end{aligned}$$

8. Solve the equation

$$\begin{aligned} \tan x + \cot x &= 2. \\ \tan x + \frac{1}{\tan x} &= 2. \\ \tan^2 x - 2\tan x + 1 &= 0. \\ \tan x - 1 &= 0. \\ \therefore \tan x &= 1. \\ \therefore x &= 45^\circ. \end{aligned}$$

9. Solve the equation

$$\begin{aligned} \sin^2 x - \cos x &= \tfrac{1}{4}. \\ (1 - \cos^2 x) - \cos x &= \tfrac{1}{4}. \\ \cos^2 x + \cos x &= \tfrac{3}{4}. \\ \cos^2 x + \cos x + \tfrac{1}{4} &= 1. \\ \cos x + \tfrac{1}{2} &= 1. \\ \cos x &= \tfrac{1}{2}. \\ \therefore x &= 60^\circ. \end{aligned}$$

10. Solve the equation

$$\begin{aligned} \tan^2 x - \sec x &= 1. \\ (\sec^2 x - 1) - \sec x &= 1. \\ \sec^2 x - \sec x &= 2. \\ \sec^2 x - \sec x + \tfrac{1}{4} &= \tfrac{9}{4}. \\ \sec x - \tfrac{1}{2} &= \pm\tfrac{3}{2}. \\ \sec x &= 2. \\ \therefore x &= 60^\circ. \end{aligned}$$

11. Solve the equation

$$\begin{aligned} \sin x + \sqrt{3}\cos x &= 2. \\ \sqrt{3}\cos x &= 2 - \sin x. \\ 3\cos^2 x &= (2 - \sin x)^2. \\ 3(1 - \sin^2 x) &= 4 - 4\sin x + \sin^2 x. \\ 4\sin^2 x - 4\sin x + 1 &= 0. \\ 2\sin x - 1 &= 0. \\ \sin x &= \tfrac{1}{2}. \\ \therefore x &= 30^\circ. \end{aligned}$$

12. Solve the equation

$$\begin{aligned} \tan^2 x + \csc^2 x &= 3. \\ \tan^2 x + (1 + \cot^2 x) &= 3. \\ \tan^2 x - 2 + \cot^2 x &= 0. \\ \tan x - \cot x &= 0. \\ \tan x - \frac{1}{\tan x} &= 0. \\ \tan^2 x &= 1. \\ \tan x &= 1. \\ \therefore x &= 45^\circ. \end{aligned}$$

13. Solve the equation

$$\begin{aligned} 2\cos x + \sec x &= 3. \\ 2\cos x + \frac{1}{\cos x} &= 3. \\ 2\cos^2 x + 1 &= 3\cos x. \\ 2\cos^2 x - 3\cos x + 1 &= 0. \\ (2\cos x - 1)(\cos x - 1) &= 0. \\ \cos x &= 1 \text{ or } \tfrac{1}{2}. \\ \therefore x &= 0^\circ \text{ or } 60^\circ. \end{aligned}$$

14. Solve the equation

$$\begin{aligned} \cos^2 x - \sin^2 x &= \sin x. \\ (1 - \sin^2 x) - \sin^2 x &= \sin x. \\ 2\sin^2 x + \sin x - 1 &= 0. \\ (2\sin x - 1)(\sin x + 1) &= 0. \\ \sin x &= -1 \text{ or } \tfrac{1}{2}. \\ \therefore x &= 30^\circ. \end{aligned}$$

15. Solve the equation

$$2\sin x + \cot x = 1 + 2\cos x.$$

$$2\sin x + \frac{\cos x}{\sin x} = 1 + 2\cos x.$$

$$2\sin^2 x + \cos x$$
$$= \sin x + 2\cos x \sin x.$$

$$2\sin^2 x - \sin x$$
$$= 2\cos x \sin x - \cos x.$$

$$\sin x(2\sin x - 1)$$
$$= \cos x(2\sin x - 1).$$

$$(\sin x - \cos x)(2\sin x - 1) = 0.$$

$$\therefore \sin x = \cos x.$$
$$\tan x = 1.$$
$$\therefore x = 45^\circ.$$
$$\sin x = \tfrac{1}{2}.$$
$$\therefore x = 30^\circ.$$

Hence $x = 30^\circ$ or 45°.

16. Solve the equation

$$\sin^2 x + \tan^2 x = 3\cos^2 x.$$

$$\sin^2 x + \frac{\sin^2 x}{\cos^2 x} = 3\cos^2 x.$$

$$1 - \cos^2 x + \frac{1 - \cos^2 x}{\cos^2 x} = 3\cos^2 x.$$

$$-4\cos^2 x + \frac{1}{\cos^2 x} = 0.$$

$$4\cos^4 x = 1.$$
$$\cos x = \sqrt{\tfrac{1}{2}}.$$
$$\therefore x = 45^\circ.$$

17. Solve the equation

$$\tan x + 2\cot x = \tfrac{5}{2}\csc x.$$

$$\frac{\sin x}{\cos x} + 2\frac{\cos x}{\sin x} = \frac{5}{2\sin x}.$$

$$\sin^2 x + 2\cos^2 x = \tfrac{5}{2}\cos x.$$
$$1 - \cos^2 x + 2\cos^2 x = \tfrac{5}{2}\cos x.$$
$$\cos^2 x - \tfrac{5}{2}\cos x + 1 = 0.$$
$$(\cos x - 2)(\cos x - \tfrac{1}{2}) = 0.$$
$$\cos x = 2 \text{ or } \tfrac{1}{2}.$$
$$\therefore x = 60^\circ.$$

EXERCISE VIII. PAGE 24.

1. In Case II give another way of finding c, after b has been found.

$$\cos A = \frac{b}{c}.$$
$$b = c\cos A.$$
$$\therefore c = \frac{b}{\cos A}.$$

2. In Case III give another way of finding c, after a has been found.

$$\sin A = \frac{a}{c}.$$
$$c\sin A = a.$$
$$\therefore c = \frac{a}{\sin A}.$$

3. In Case IV give another way of finding b, after the angles have been found.

$$\cos A = \frac{b}{c}.$$
$$\therefore b = c\cos A.$$

4. In Case V give another way of finding c, after the angles have been found.

$$\sin A = \frac{a}{c}.$$
$$c\sin A = a.$$
$$\therefore c = \frac{a}{\sin A}.$$

5. Given B and c; find A, a, b.

$$A = 90^\circ - B.$$
$$\cos B = \frac{a}{c}.$$
$$\therefore a = c\cos B.$$
$$\sin B = \frac{b}{c}.$$
$$\therefore b = c\sin B.$$

6. Given B and b; find A, a, c.

$$A = 90^\circ - B.$$
$$\cot B = \frac{a}{b}.$$
$$\therefore a = b \cot B.$$
$$\sin B = \frac{b}{c}.$$
$$\therefore c = \frac{b}{\sin B}.$$

7. Given B and a; find A, b, c.

$$A = 90^\circ - B.$$
$$\cot B = \frac{a}{b}.$$
$$\therefore b = \frac{a}{\cot B} = a \tan B.$$
$$\cos B = \frac{a}{c}.$$
$$\therefore c = \frac{a}{\cos B}.$$

8. Given b and c; find A, B, a.

$$\cos A = \frac{b}{c}.$$
$$B = 90^\circ - A.$$
$$\begin{aligned} a &= \sqrt{c^2 - b^2} \\ &= \sqrt{(c+b)(c-b)}. \end{aligned}$$

9. Given $a = 3$, $b = 4$; required $A = 36^\circ\ 52'$, $B = 53^\circ\ 8'$, $c = 5$.

$$\tan A = \frac{a}{b} = \frac{3}{4} = 0.7500.$$
$$\begin{aligned} \therefore A &= 36^\circ\ 52'. \\ B &= 90^\circ - A \\ &= 53^\circ\ 8'. \\ c &= \sqrt{a^2 + b^2} \\ &= 5. \end{aligned}$$

10. Given $a = 7$, $c = 13$; required $A = 32^\circ 35'$, $B = 57^\circ 25'$, $b = 10.954$.

$$\sin A = \frac{a}{c} = \frac{7}{13} = 0.5385.$$
$$\begin{aligned} \therefore A &= 32^\circ\ 35'. \\ B &= 90^\circ - A \\ &= 57^\circ\ 25'. \\ b &= \sqrt{(c-a)(c+a)} \\ &= \sqrt{120} \\ &= 10.954. \end{aligned}$$

11. Given $a = 5.3$, $A = 12^\circ 17'$; required $B = 77^\circ\ 43'$, $b = 24.342$, $c = 24.918$.

$$\begin{aligned} B &= 90^\circ - A \\ &= 77^\circ\ 43'. \end{aligned}$$
$$\frac{b}{a} = \cot A.$$
$$\begin{aligned} \therefore b &= a \cot A \\ &= 5.3 \times 4.5928 \\ &= 24.342. \end{aligned}$$
$$\frac{a}{c} = \sin A.$$
$$\begin{aligned} \therefore c &= \frac{a}{\sin A} \\ &= \frac{5.3}{0.2127} \\ &= 24.918. \end{aligned}$$

12. Given $a = 10.4$, $B = 43^\circ\ 18'$; required $A = 46^\circ 42'$, $b = 9.800$, $c = 14.290$.

$$\begin{aligned} A &= 90^\circ - B \\ &= 46^\circ\ 42'. \end{aligned}$$
$$\frac{b}{a} = \tan B.$$
$$\begin{aligned} \therefore b &= a \tan B \\ &= 10.4 \times 0.9424 \\ &= 9.800. \end{aligned}$$
$$\frac{a}{c} = \cos B.$$
$$\begin{aligned} \therefore c &= \frac{a}{\cos B} \\ &= \frac{10.4}{0.7278} \\ &= 14.290. \end{aligned}$$

13. Given $c = 26$, $A = 37° 42'$; required $B = 52° 18'$, $a = 15.900$, $b = 20.572$

$$\begin{aligned} B &= 90° - A \\ &= 52° 18'. \\ \frac{a}{c} &= \sin A. \\ \therefore a &= c \sin A \\ &= 26 \times 0.6115 \\ &= 15.900. \\ \frac{b}{c} &= \cos A. \\ \therefore b &= c \cos A \\ &= 26 \times 0.7912 \\ &= 20.572. \end{aligned}$$

14. Given $c = 140$, $B = 24° 12'$; required $A = 65° 48'$, $a = 127.694$, $b = 57.386$.

$$\begin{aligned} A &= 90° - B \\ &= 65° 48'. \\ \frac{a}{c} &= \cos B. \\ \therefore a &= c \cos B \\ &= 140 \times 0.9121 \\ &= 127.694. \\ \frac{b}{c} &= \sin B. \\ \therefore b &= c \sin B \\ &= 140 \times 0.4099 \\ &= 57.386. \end{aligned}$$

15. Given $b = 19$, $c = 23$; required $A = 34° 18'$, $B = 55° 42'$, $a = 12.961$.

$$\cos A = \frac{b}{c} = \frac{19}{23} = 0.8261.$$

$$\begin{aligned} \therefore A &= 34° 18'. \\ \therefore B &= 90° - A \\ &= 55° 42'. \\ a &= \sqrt{(c - b)(c + b)} \\ &= \sqrt{168} \\ &= 12.961. \end{aligned}$$

16. Given $b = 98$, $c = 135.2$; required $A = 43° 33'$, $B = 46° 27'$, $a = 93.139$.

$$\begin{aligned} \cos A &= \frac{b}{c} = \frac{98}{135.2} \\ &= 0.7248. \\ \therefore A &= 43° 33'. \\ \therefore B &= 90° - A \\ &= 46° 27'. \\ a &= \sqrt{(c - b)(c + b)} \\ &= \sqrt{8675.04} \\ &= 93.139. \end{aligned}$$

17. Given $b = 42.4$, $A = 32° 14'$; required $B = 57° 46'$, $a = 26.733$, $c = 50.124$.

$$\begin{aligned} B &= 90° - A \\ &= 57° 46'. \\ \frac{a}{b} &= \tan A. \\ \therefore a &= b \tan A \\ &= 42.4 \times 0.6305 \\ &= 26.733. \\ \frac{b}{c} &= \cos A. \\ c &= \frac{b}{\cos A} \\ &= \frac{42.4}{0.8459} \\ &= 50.124. \end{aligned}$$

18. Given $b = 200$, $B = 46° 11'$; required $A = 43° 49'$, $a = 191.900$, $c = 277.160$.

$$\begin{aligned} A &= 90° - B \\ &= 43° 49'. \\ \frac{a}{b} &= \cot B. \\ \therefore a &= b \cot B \\ &= 200 \times 0.9595 \\ &= 191.900. \\ \frac{b}{c} &= \sin B. \end{aligned}$$

$$\therefore c = \frac{b}{\sin B}$$
$$= \frac{200}{0.7216}$$
$$= 277.160.$$

19. Given $a = 95$, $b = 37$; required $A = 68° 43'$, $B = 21° 17'$, $c = 101.951$.

$$\tan A = \frac{a}{b} = \frac{95}{37} = 2.5676.$$
$$\therefore A = 68° 43'.$$
$$\therefore B = 90° - A$$
$$= 21° 17'.$$
$$c = \sqrt{a^2 + b^2}$$
$$= \sqrt{10394}$$
$$= 101.951.$$

20. Given $a = 6$, $c = 103$; required $A = 3° 21'$, $B = 86° 39'$, $b = 102.825$.

$$\sin A = \frac{a}{c} = \frac{6}{103} = 0.0583.$$
$$\therefore A = 3° 21'.$$
$$\therefore B = 90° - A$$
$$= 86° 39'.$$
$$b = \sqrt{c^2 - a^2}$$
$$= \sqrt{10573}$$
$$= 102.825.$$

21. Given $a = 3.12$, $B = 5° 8'$; required $A = 84° 52'$, $b = 0.280$, $c = 3.133$.

$$A = 90° - B$$
$$= 84° 52'.$$
$$\frac{b}{a} = \tan B.$$
$$\therefore b = a \tan B$$
$$= 3.12 \times 0.0898$$
$$= 0.280.$$
$$\frac{a}{c} = \cos B.$$
$$\therefore c = \frac{a}{\cos B}$$
$$= \frac{3.12}{0.9960}$$
$$= 3.133.$$

22. Given $a = 17$, $c = 18$; required $A = 70° 48'$, $B = 19° 12'$, $b = 5.916$.

$$\tan \tfrac{1}{2} B = \sqrt{\frac{c - a}{c + a}}$$
$$= \sqrt{\frac{1}{35}}$$
$$= 0.1690.$$
$$\therefore \tfrac{1}{2} B = 9° 36'.$$
$$\therefore B = 19° 12'.$$
$$\therefore A = 90° - B$$
$$= 70° 48'.$$
$$b = \sqrt{(c - a)(c + a)}$$
$$= \sqrt{35}$$
$$= 5.916.$$

23. Given $c = 57$, $A = 38° 29'$; required $B = 51° 31'$, $a = 35.471$, $b = 44.620$.

$$B = 90° - A$$
$$= 51° 31'.$$
$$\frac{a}{c} = \sin A.$$
$$\therefore a = c \sin A$$
$$= 57 \times 0.6223$$
$$= 35.471.$$
$$\frac{b}{c} = \cos A.$$
$$\therefore b = c \cos A$$
$$= 57 \times 0.7828$$
$$= 44.620.$$

24. Given $a + c = 18$, $b = 12$; required $A = 22° 37'$, $B = 67° 23'$, $a = 5$, $c = 13$.

$$c^2 - a^2 = b^2.$$
$$(c + a)(c - a) = b^2.$$
$$18(c - a) = 144.$$
$$c - a = 8.$$
$$\therefore c = 13.$$
$$\therefore a = 5.$$
$$\sin A = \frac{a}{c} = \frac{5}{13} = 0.3846.$$
$$\therefore A = 22^\circ\ 37'.$$
$$\therefore B = 90^\circ - A$$
$$= 67^\circ\ 23'.$$

25. Given $a + b = 9$, $c = 8$; required $A = 82^\circ\ 18'$, $B = 7^\circ\ 42'$, $a = 7.928$, $b = 1.072$.

$$a^2 + b^2 = c^2 = 64$$
$$\underline{a^2 + 2\,ab + b^2 = 9^2 = 81}$$
$$2\,ab = 17$$
$$a^2 - 2\,ab + b^2 = 64 - 17 = 47.$$
$$\therefore a - b = \sqrt{47} = 6.856.$$

But $a + b = 9.$

$$\therefore a = 7.928$$

and $b = 1.072.$

$$\tan \tfrac{1}{2} B = \sqrt{\frac{c - a}{c + a}}$$
$$= \sqrt{\frac{0.072}{15.928}}$$
$$= 0.0672.$$
$$\therefore \tfrac{1}{2} B = 3^\circ\ 51'.$$
$$\therefore B = 7^\circ\ 42'.$$
$$\therefore A = 90^\circ - B = 82^\circ\ 18'.$$

Exercise IX. Page 28.

1. Given $a = 6$, $c = 12$; required $A = 30^\circ$, $B = 60^\circ$, $b = 10.392$.

$$\sin A = \frac{a}{c} = \frac{1}{2}.$$
$$\therefore A = 30^\circ.$$
$$B = (90^\circ - A) = 60^\circ.$$
$$\cos A = \frac{b}{c}.$$
$$\therefore b = c \cos A.$$
$$\log \cos A = 9.93753$$
$$\underline{\log 12 = 1.07918}$$
$$\log b = 1.01671$$
$$b = 10.392.$$

2. Given $A = 60^\circ$, $b = 4$; required $B = 30^\circ$, $c = 8$, $a = 6.9282$.

$$B = 90^\circ - A = 30^\circ.$$
$$\cos A = \frac{b}{c}.$$
$$\therefore c = \frac{b}{\cos A}.$$
$$\log b = 0.60206$$
$$\underline{\text{colog} \cos A = 0.30103}$$
$$\log c = 0.90309$$
$$c = 8.$$
$$c^2 = a^2 + b^2.$$
$$\therefore c^2 - b^2 = a^2 = 48.$$
$$\log 48 = \log a^2 = 1.68124.$$
$$\therefore \log a = 0.84062.$$
$$a = 6.9282.$$

3. Given $A = 30^\circ$, $a = 3$; required $B = 60$, $c = 6$, $b = 5.1961$.

$$B = (90^\circ - A) = 60^\circ.$$
$$\sin A = \frac{a}{c}.$$
$$\therefore c = \frac{a}{\sin A}.$$
$$\log a = 0.47712$$
$$\underline{\text{colog} \sin A = 0.30103}$$
$$\log c = 0.77815$$
$$c = 6.$$
$$c^2 = a^2 + b^2.$$

$$\begin{aligned}\therefore c^2 - a^2 &= b^2 = 27.\\ \log 27 &= \log b^2 = 1.43136.\\ \therefore \log b &= 0.71568.\\ b &= 5.1961.\end{aligned}$$

4. Given $a = 4$, $b = 4$; required $A = B = 45°$, $c = 5.6568$.

Since a and b each $= 4$, the △ is isosceles and $\angle A = \angle B$.

$$\begin{aligned}\therefore A &= \tfrac{1}{2} \text{ of } 90° = 45°,\\ \text{and } B &= \tfrac{1}{2} \text{ of } 90° = 45°.\\ c^2 &= a^2 + b^2 = 32°.\\ \log 32 &= \log c^2 = 1.50515.\\ \therefore \log c &= 0.75257.\\ c &= 5.6568.\end{aligned}$$

5. Given $a = 2$, $c = 2.82843$; required $A = B = 45°$, $b = 2$.

$$\begin{aligned}b &= \sqrt{c^2 - a^2}\\ &= \sqrt{(c + a)(c - a)}.\\ \log b^2 &= \log(c + a) + \log(c - a).\\ \log(c + a) &= 0.68381\\ \log(c - a) &= \underline{9.91826 - 10}\\ \log b^2 &= 0.60207\\ \log b &= 0.30103.\\ b &= 2.\end{aligned}$$

∴ the △ is an isosceles rt △.

$\therefore A = B = 45°.$

6. Given $c = 627$, $A = 23° 30'$; required $B = 66° 30'$, $a = 250.02$, $b = 575.0$.

$$\begin{aligned}B &= (90° - A) = 66° 30'.\\ a &= c \sin A.\\ \log a &= \log c + \log \sin A.\\ \log c &= 2.79727\\ \log \sin A &= \underline{9.60070}\\ \log a &= 2.39797\\ a &= 250.02.\\ b &= c \cos A.\\ \log b &= \log c + \log \cos A.\\ \log c &= 2.79727\\ \log \cos A &= \underline{9.96240}\\ \log b &= 2.75967\\ b &= 575.\end{aligned}$$

7. Given $c = 2280$, $A = 28° 5'$; required $B = 61° 55'$, $a = 1073.3$, $b = 2011.5$.

$$\begin{aligned}B &= (90° - A) = 61° 55'.\\ a &= c \sin A.\\ \log a &= \log c + \log \sin A.\\ \log c &= 3.35793\\ \log \sin A &= \underline{9.67280}\\ \log a &= 3.03073\\ a &= 1073.3.\\ b &= c \cos A.\\ \log b &= \log c + \log \cos A.\\ \log c &= 3.35793\\ \log \cos A &= \underline{9.94560}\\ \log b &= 3.30353\\ b &= 2011.5.\end{aligned}$$

8. Given $c = 72.15$, $A = 39° 34'$; required $B = 50° 26'$, $a = 45.958$, $b = 55.620$.

$$\begin{aligned}B &= (90° - A) = 50° 26'.\\ a &= c \sin A.\\ \log a &= \log c + \log \sin A.\\ \log c &= 1.85824\\ \log \sin A &= \underline{9.80412}\\ \log a &= 1.66236\\ a &= 45.958.\\ b &= c \cos A.\\ \log b &= \log c + \log \cos A.\\ \log c &= 1.85824\\ \log \cos A &= \underline{9.88699}\\ \log b &= 1.74523\\ b &= 55.620.\end{aligned}$$

9. Given $c = 1$, $A = 36°$; required $B = 54°$, $a = 0.58779$, $b = 0.80902$.

$$B = (90^\circ - A) = 54^\circ.$$

$$\sin A = \frac{a}{c}.$$

$$\therefore a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\begin{aligned} \log c &= 0.00000 \\ \log \sin A &= \underline{9.76922} \\ \log a &= 9.76922 - 10 \end{aligned}$$

$$a = 0.58779.$$

$$\cos A = \frac{b}{c}.$$

$$\therefore b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\begin{aligned} \log c &= 0.00000 \\ \log \cos A &= \underline{9.90796} \\ \log b &= 9.90796 - 10 \end{aligned}$$

$$b = 0.80902.$$

10. Given $c = 200$, $B = 21^\circ\, 47'$; required $A = 68^\circ\, 13'$, $a = 185.72$, $b = 74.22$.

$$A = (90^\circ - B) = 68^\circ\, 13'.$$

$$\sin A = \frac{a}{c}.$$

$$\therefore a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\begin{aligned} \log c &= 2.30103 \\ \log \sin A &= \underline{9.96783} \\ \log a &= 2.26886 \end{aligned}$$

$$a = 185.72.$$

$$\cos a = \frac{b}{c}.$$

$$\therefore b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\begin{aligned} \log c &= 2.30103 \\ \log \cos A &= \underline{9.56949} \\ \log b &= 1.87052 \end{aligned}$$

$$b = 74.220.$$

11. Given $c = 93.4$, $B = 76^\circ\, 25'$; required $A = 13^\circ\, 35'$, $a = 21.936$, $b = 90.788$.

$$A = (90^\circ - B) = 13^\circ\, 35'.$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\begin{aligned} \log c &= 1.97035 \\ \log \sin A &= \underline{9.37081} \\ \log a &= 1.34116 \end{aligned}$$

$$a = 21.936.$$

$$b = a \cot A.$$

$$\log b = \log a + \log \cot A.$$

$$\begin{aligned} \log a &= 1.34116 \\ \log \cot A &= \underline{0.61687} \\ \log b &= 1.95803 \end{aligned}$$

$$b = 90.788.$$

12. Given $a = 637$, $A = 4^\circ\, 35'$; required $B = 85^\circ\, 25'$, $b = 7946$, $c = 7971.5$.

$$B = (90^\circ - A) = 85^\circ\, 25'.$$

$$b = a \cot A.$$

$$\log b = \log a + \log \cot A.$$

$$\begin{aligned} \log a &= 2.80414 \\ \log \cot A &= \underline{1.09601} \\ \log b &= 3.90015 \end{aligned}$$

$$b = 7946.$$

$$\log c = \log a + \operatorname{colog} \sin A.$$

$$\begin{aligned} \log a &= 2.80414 \\ \operatorname{colog} \sin A &= \underline{1.09740} \\ \log c &= 3.90154 \end{aligned}$$

$$c = 7971.5.$$

13. Given $a = 48.532$, $A = 36^\circ 44'$; required $B = 53^\circ\, 16'$, $B = 65.031$, $c = 81.144$.

$$\begin{aligned} B &= 90^\circ - A \\ &= 90^\circ - 36^\circ\, 44' \\ &= 53^\circ\, 16'. \end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$\therefore c = \frac{a}{\sin A}.$$

$$\log c = \log a + \text{colog} \sin A.$$

$$\begin{aligned} \log a &= 1.68603 \\ \text{colog} \sin A &= \underline{0.22323} \\ \log c &= 1.90926 \\ c &= 81.144. \end{aligned}$$

$$\cos A = \frac{b}{c}.$$

$$\therefore b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\begin{aligned} \log c &= 1.90926 \\ \log \cos A &= \underline{9.90386} \\ \log b &= 1.81312 \\ b &= 65.031. \end{aligned}$$

14. Given $a = 0.0008$, $A = 86°$; required $B = 4°$, $b = 0.0000559$, $c = 0.000802$.

$$\begin{aligned} B &= 90° - A \\ &= 90° - 86° \\ &= 4°. \end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$\therefore c = \frac{a}{\sin A}.$$

$$\log c = \log a + \text{colog} \sin A.$$

$$\begin{aligned} \log a &= 6.90309 - 10 \\ \text{colog} \sin A &= \underline{0.00106 \qquad\quad} \\ \log c &= 6.90415 - 10 \\ c &= 0.000802. \end{aligned}$$

$$\cos A = \frac{b}{c}.$$

$$\therefore b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\begin{aligned} \log c &= 6.90415 - 10 \\ \log \cos A &= \underline{8.84358 - 10} \\ \log b &= 5.74773 - 10 \\ b &= 0.0000559. \end{aligned}$$

15. Given $b = 50.937$, $B = 43° 48'$; required $A = 46° 12'$, $a = 53.116$, $c = 73.59$.

$$A = (90° - B) = 46° 12'.$$

$$\tan A = \frac{a}{b}.$$

$$\therefore a = b \tan A.$$

$$\begin{aligned} \log b &= 1.70703 \\ \log \tan A &= \underline{0.01820} \\ \log a &= 1.72523 \\ a &= 53.116. \end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$\therefore c = \frac{a}{\sin A}.$$

$$\begin{aligned} \log a &= 1.72523 \\ \text{colog} \sin A &= \underline{0.14161} \\ \log c &= 1.86684 \\ c &= 73.593. \end{aligned}$$

16. Given $b = 2$, $B = 3° 38'$; required $A = 86° 22'$, $a = 31.496$, $c = 31.559$.

$$A = (90° - B) = 86° 22'.$$

$$\tan A = \frac{a}{b}.$$

$$\therefore a = b \tan A.$$

$$\begin{aligned} \log b &= 0.30103 \\ \log \tan A &= \underline{1.19723} \\ \log a &= 1.49826 \\ a &= 31.496. \end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$\therefore c = \frac{a}{\sin A}.$$

$$\begin{aligned} \log a &= 1.49826 \\ \text{colog} \sin A &= \underline{0.00087} \\ \log c &= 1.49913 \\ c &= 31.559. \end{aligned}$$

17. Given $a = 992$, $B = 76° 19'$; required $A = 13° 41'$, $b = 4074.5$, $c = 4193.5$.

$$\begin{aligned} A &= 90° - 76° 19' \\ &= 13° 41'. \end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$\log c = \log a + \operatorname{colog} \sin A.$$

$$\begin{aligned} \log a &= 2.99651 \\ \operatorname{colog} \sin A &= \underline{0.62607} \\ \log c &= 3.62258 \end{aligned}$$

$$c = 4193.5.$$

$$\sin B = \frac{b}{c}.$$

$$\therefore b = c \sin B.$$

$$\log b = \log c + \log \sin B.$$

$$\begin{aligned} \log c &= 3.62258 \\ \log \sin B &= \underline{9.98750} \\ \log b &= 3.61008 \end{aligned}$$

$$b = 4074.5.$$

18. Given $a = 73$, $B = 68° 52'$; required $A = 21° 8'$, $b = 188.86$, $c = 202.47$.

$$A = (90° - B) = 21° 8'.$$

$$\sin A = \frac{a}{c}.$$

$$\therefore c = \frac{a}{\sin A}.$$

$$\log c = \log a + \operatorname{colog} \sin A.$$

$$\begin{aligned} \log A &= 1.86332 \\ \operatorname{colog} \sin A &= \underline{0.44305} \\ \log c &= 2.30637 \end{aligned}$$

$$c = 202.47.$$

$$\sin B = \frac{b}{c}.$$

$$\therefore b = c \sin B.$$

$$\log b = \log c + \log \sin B.$$

$$\begin{aligned} \log c &= 2.30637 \\ \log \sin B &= \underline{9.96976} \\ \log b &= 2.27613 \end{aligned}$$

$$b = 188.86.$$

19. Given $a = 2.189$, $B = 45° 25'$; required $A = 44° 35'$, $b = 2.2211$, $c = 3.1185$.

$$\begin{aligned} A &= 90° - 45° 25' \\ &= 44° 35'. \end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$\therefore c = \frac{a}{\sin A}.$$

$$\log c = \log a + \operatorname{colog} \sin A.$$

$$\begin{aligned} \log a &= 0.34025 \\ \operatorname{colog} \sin A &= \underline{0.15370} \\ \log c &= 0.49395 \end{aligned}$$

$$c = 3.1185.$$

$$\cos A = \frac{b}{c}.$$

$$\therefore b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\begin{aligned} \log c &= 0.49395 \\ \log \cos A &= \underline{9.85262} \\ \log b &= 0.34657 \end{aligned}$$

$$b = 2.2211.$$

20. Given $b = 4$, $A = 37° 56'$; required $B = 52° 4'$, $a = 3.1176$, $c = 5.0714$.

$$B = (90° - A) = 52° 4'.$$

$$\cos A = \frac{b}{c}.$$

$$\therefore c = \frac{b}{\cos A}.$$

$$\log c = \log b + \operatorname{colog} \cos A.$$

$$\begin{aligned} \log b &= 0.60206 \\ \operatorname{colog} \cos A &= \underline{0.10307} \\ \log c &= 0.70513 \end{aligned}$$

$$c = 5.0714.$$

$$\tan A = \frac{a}{b}.$$

$$\therefore a = b \tan A.$$

$$\log a = \log b + \log \tan A.$$

$$\begin{aligned} \log b &= 0.60206 \\ \log \tan A &= \underline{9.89177} \\ \log a &= 0.49383 \end{aligned}$$

$$a = 3.1176.$$

21. Given $c = 8590$, $a = 4476$; required $A = 31° 24'$, $B = 58° 36'$, $b = 7332.8$.

$$\sin A = \frac{a}{c}.$$

$$\log \sin A = \log a + \operatorname{colog} c.$$

$$\begin{aligned} \log a &= 3.65089 \\ \operatorname{colog} c &= \underline{6.06601 - 10} \\ \log \sin A &= 9.71690 - 10 \end{aligned}$$

$$A = 31° 24'.$$

$$\therefore B = 58° 36.$$

$$\cot A = \frac{b}{a}.$$

$$b = a \cot a.$$

$$\log b = \log a + \log \cot A.$$

$$\begin{aligned} \log a &= 3.65089 \\ \log \cot A &= \underline{0.21438} \\ \log b &= 3.86527 \end{aligned}$$

$$b = 7332.8.$$

22. Given $c = 86.53$, $a = 71.78$; required $A = 56° 3'$, $B = 33° 57'$, $b = 48.324$.

$$\sin A = \frac{a}{c}.$$

$$\log \sin A = \log a + \operatorname{colog} c.$$

$$\begin{aligned} \log a &= 1.85600 \\ \operatorname{colog} c &= \underline{8.06283 - 10} \\ \log \sin A &= 9.91883 - 10 \end{aligned}$$

$$A = 56° 3'.$$

$$\therefore B = 33° 57'.$$

$$b = \frac{a}{\cot A}.$$

$$\log b = \log a + \log \cot A.$$

$$\begin{aligned} \log a &= 1.85600 \\ \log \cot A &= \underline{9.82817} \\ \log b &= 1.68417 \end{aligned}$$

$$b = 48.324.$$

23. Given $c = 9.35$, $a = 8.49$; required $A = 65° 14'$, $B = 24° 46'$, $b = 3.917$.

$$\sin A = \frac{a}{c}.$$

$$\begin{aligned} \operatorname{colog} c &= 9.02919 - 10 \\ \log a &= \underline{0.92891} \\ \log \sin A &= 9.95810 - 10 \end{aligned}$$

$$A = 65° 14'.$$

$$\therefore B = 24° 46'.$$

$$\cos A = \frac{b}{c}.$$

$$\therefore b = c \cos a.$$

$$\begin{aligned} \log c &= 0.97081 \\ \log \cos A &= \underline{9.62214} \\ \log b &= 0.59295 \end{aligned}$$

$$b = 3.917.$$

24. Given $c = 2194$, $b = 1312.7$; required $A = 53° 15'$, $B = 36° 45'$, $a = 1758$.

$$\cos A = \frac{b}{c}.$$

$$\begin{aligned} \log B &= 3.11816 \\ \operatorname{colog} c &= \underline{6.65876 - 10} \\ \log \cos A &= 9.77692 - 10 \end{aligned}$$

$$A = 53° 15'.$$

$$\begin{aligned} \therefore B &= (90° - A) \\ &= 36° 45'. \end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$\therefore a = c \sin A.$$

$$\begin{aligned}\log c &= 3.34124\\ \log\sin A &= \underline{9.90377}\\ \log A &= 3.24501\end{aligned}$$

$$a = 1758.$$

25. Given $c = 30.69$, $b = 18.256$; required $A = 53° 30'$, $B = 36° 30'$, $a = 24.67$.

$$\cos A = \frac{b}{c}.$$

$$\log\cos A = \log B + \text{colog}\, c.$$

$$\begin{aligned}\log b &= 1.26140\\ \text{colog}\, c &= \underline{8.51300 - 10}\\ \log\cos A &= 9.77440 - 10\end{aligned}$$

$$\begin{aligned}A &= 53° 30'.\\ \therefore B &= 36° 30'.\end{aligned}$$

$$\tan A = \frac{a}{b}.$$

$$a = b\tan A.$$

$$\log a = \log\tan A + \log b.$$

$$\begin{aligned}\log\tan A &= 0.13079\\ \log b &= \underline{1.26140}\\ \log a &= 1.39219\end{aligned}$$

$$a = 24.671.$$

26. Given $a = 38.313$, $b = 19.522$; required $A = 63°$, $B = 27°$, $c = 43$.

$$\tan A = \frac{a}{b}.$$

$$\log\tan A = \log a + \text{colog}\, b.$$

$$\begin{aligned}\log a &= \ \ 1.58335\\ \text{colog}\, b &= \underline{\ \ 8.70948 - 10}\\ \log\tan A &= 10.29283 - 10\end{aligned}$$

$$\begin{aligned}A &= 63°.\\ \therefore B &= 27°.\end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$\therefore c = \frac{a}{\sin A}.$$

$$\log c = \log a + \text{colog}\sin A.$$

$$\begin{aligned}\log a &= 1.58335\\ \text{colog}\sin A &= \underline{0.05012}\\ \log c &= 1.63347\end{aligned}$$

$$c = 43.$$

27. Given $a = 1.2291$, $b = 14.950$; required $A = 4° 42'$, $B = 85° 18'$, $c = 15$.

$$\tan A = \frac{a}{b}.$$

$$\begin{aligned}\log a &= 0.08959\\ \text{colog}\, b &= \underline{8.82536 - 10}\\ \log\tan A &= 8.91495 - 10\end{aligned}$$

$$\begin{aligned}A &= 4° 42'.\\ \therefore B &= 85° 18'.\end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$\therefore c = \frac{a}{\sin A}.$$

$$\begin{aligned}\log a &= 0.08959\\ \text{colog}\sin A &= \underline{1.08651}\\ \log c &= 1.17610\end{aligned}$$

$$c = 15.$$

28. Given $a = 415.38$, $b = 62.080$ required $A = 81° 30'$, $B = 8° 30'$ $c = 420$.

$$\tan A = \frac{a}{b}.$$

$$\begin{aligned}\log a &= \ \ 2.61845\\ \text{colog}\, b &= \underline{\ \ 8.20705 - 10}\\ \log\tan A &= 10.82550 - 10\end{aligned}$$

$$\begin{aligned}A &= 81° 30'.\\ \therefore B &= 8° 30'.\end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$\therefore c = \frac{a}{\sin A}.$$

$$\begin{aligned}\log a &= 2.61845\\ \text{colog}\sin A &= \underline{0.00480}\\ \log c &= 2.62325\end{aligned}$$

$$c = 420.$$

29. Given $a = 13.690$, $b = 16.926$; required $A = 38° 58'$, $B = 51° 2'$, $c = 21.769$.

$$\tan A = \frac{a}{b}.$$

$$\log \tan A = \log a + \operatorname{colog} b.$$

$$\begin{aligned} \log a &= 1.13640 \\ \operatorname{colog} b &= \underline{8.77144 - 10} \\ \log \tan A &= 9.90784 - 10 \end{aligned}$$

$$A = 38° 58'.$$

$$\therefore B = 51° 2'.$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a + \operatorname{colog} \sin A.$$

$$\begin{aligned} \log a &= 1.13640 \\ \operatorname{colog} \sin A &= \underline{0.20144} \\ \log c &= 1.33784 \end{aligned}$$

$$c = 21.769.$$

30. Given $c = 91.92$, $a = 2.19$; required $A = 1° 22'$, $B = 88° 38'$, $b = 91.894$.

$$\sin A = \frac{a}{c}.$$

$$\log \sin A = \log a + \operatorname{colog} c.$$

$$\begin{aligned} \log a &= 0.34044 \\ \operatorname{colog} c &= \underline{8.03659 - 10} \\ \log \sin A &= 8.37703 - 10 \end{aligned}$$

$$A = 1° 22'.$$

$$\therefore B = 88° 38'.$$

$$\cos A = \frac{b}{c}.$$

$$\therefore b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\begin{aligned} \log c &= 1.96341 \\ \log \cos A &= \underline{9.99988} \\ \log b &= 1.96329 \end{aligned}$$

$$b = 91.894.$$

31. Compute the unknown parts and also the area, having given $a = 5$, $b = 6$.

$$\tan A = \frac{a}{b}.$$

$$\log \tan A = \log a + \operatorname{colog} b.$$

$$\begin{aligned} \log a &= 0.69897 \\ \operatorname{colog} b &= \underline{9.22185 - 10} \\ \log \tan A &= 9.92082 - 10 \end{aligned}$$

$$A = 39° 48'.$$

$$\therefore B = 50° 12'.$$

$$\sin A = \frac{a}{c}.$$

$$\therefore c = \frac{a}{\sin A}.$$

$$\log c = \log a + \operatorname{colog} \sin A.$$

$$\begin{aligned} \log a &= 0.69897 \\ \operatorname{colog} \sin A &= \underline{0.19375} \\ \log c &= 0.89272 \end{aligned}$$

$$c = 7.8112.$$

$$\therefore F = \frac{ab}{2} = \frac{30}{2} = 15.$$

32. Compute the unknown parts and also the area, having given $a = 0.615$, $c = 70$.

$$\sin A = \frac{a}{c}.$$

$$\log \sin A = \log a + \operatorname{colog} c.$$

$$\begin{aligned} \log a &= 9.78888 - 10 \\ \operatorname{colog} c &= \underline{8.15490 - 10} \\ \log \sin A &= 7.94378 - 10 \end{aligned}$$

$$A = 30' 12''.$$

$$\therefore B = 89° 29' 48''.$$

$$b = \sqrt{(c + a)(c - a)}.$$

$$\log b = \frac{\log (c + a) + \log (c - a)}{2}.$$

$$\begin{aligned} \log (c + a) &= 1.84890 \\ \log (c - a) &= \underline{1.84126} \\ & 3.69016 \end{aligned}$$

$$\begin{aligned}\log b &= 1.84508.\\ b &= 69.997.\\ F &= \tfrac{1}{2}ab.\end{aligned}$$

$$\begin{aligned}\log a &= 9.78888-10\\ \log b &= 1.84508\\ \operatorname{colog} 2 &= \underline{9.69897-10}\\ \log F &= 1.33293\\ F &= 21.525.\end{aligned}$$

33. Compute the unknown parts and also the area, having given $b=\sqrt[3]{2}$, $c=\sqrt{3}$.

$$\begin{aligned}\sqrt[3]{2} &= 1.25991.\\ \sqrt{3} &= 1.73205.\\ \cos A &= \frac{b}{c}.\\ \log\cos A &= \log b+\operatorname{colog} c.\end{aligned}$$

$$\begin{aligned}\log b &= 0.10034\\ \operatorname{colog} c &= \underline{9.76144-10}\\ \log\cos A &= 9.86178-10\\ A &= 43^\circ\,20'.\\ \therefore B &= 46^\circ\,40'.\\ \sin A &= \frac{a}{c}.\\ \therefore a &= c\sin A.\\ \log a &= \log c+\log\sin A.\end{aligned}$$

$$\begin{aligned}\log c &= 0.23856\\ \log\sin A &= \underline{9.83648}\\ \log a &= 0.07504\\ a &= 1.1886.\\ F &= \tfrac{1}{2}ab.\end{aligned}$$

$$\begin{aligned}\log a &= 0.07504\\ \log b &= 0.10034\\ \operatorname{colog} 2 &= \underline{9.69897-10}\\ \log F &= 9.87435-10\\ F &= 0.74876.\end{aligned}$$

34. Compute the unknown parts and also the area, having given $a=7$, $A=18^\circ\,14'$.

$$\begin{aligned}B &= 71^\circ\,46'.\\ \sin A &= \frac{a}{c}.\\ \therefore c &= \frac{a}{\sin A}.\\ \log c &= \log a+\operatorname{colog}\sin A.\end{aligned}$$

$$\begin{aligned}\log a &= 0.84510\\ \operatorname{colog}\sin A &= \underline{0.50461}\\ \log c &= 1.34971\\ c &= 22.372.\\ \tan A &= \frac{a}{b}.\\ \therefore b &= \frac{a}{\tan A}.\\ \log b &= \log a+\operatorname{colog}\tan A.\end{aligned}$$

$$\begin{aligned}\log a &= 0.84510\\ \operatorname{colog}\tan A &= \underline{0.48224-10}\\ \log b &= 1.32734\\ b &= 21.249.\\ F &= \tfrac{1}{2}ab.\end{aligned}$$

$$\begin{aligned}\log a &= 0.84510\\ \log b &= 1.32734\\ \operatorname{colog} 2 &= \underline{9.69897-10}\\ \log F &= 1.87141\\ F &= 74.372.\end{aligned}$$

35. Compute the unknown parts and also the area, having given $b=12$, $A=29^\circ\,8'$.

$$\begin{aligned}A &= 29^\circ\,8'.\\ \therefore B &= 60^\circ\,52'\\ \cos A &= \frac{b}{c}.\\ \therefore c &= \frac{b}{\cos A}.\\ \log c &= \log b+\operatorname{colog}\cos A.\end{aligned}$$

$$\begin{aligned}\log b &= 1.07918\\ \operatorname{colog}\cos A &= \underline{0.05874}\\ \log c &= 1.13792\\ c &= 13.738.\end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$\therefore a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\begin{aligned} \log c &= 1.13792 \\ \log \sin A &= \underline{9.68739} \\ \log a &= 0.82531 \end{aligned}$$

$$a = 6.6882.$$

$$F = \tfrac{1}{2} ab.$$

$$\log F = \log a + \log b + \operatorname{colog} 2.$$

$$\begin{aligned} \log a &= 0.82531 \\ \log b &= 1.07918 \\ \operatorname{colog} 2 &= \underline{9.69897 - 10} \\ \log F &= 1.60346 \end{aligned}$$

$$F = 40.129.$$

36. Compute the unknown parts and also the area, having given $c = 68$, $A = 69° 54'$.

$$A = 69° 54'.$$

$$\therefore B = 20° 6'.$$

$$\sin A = \frac{a}{c}.$$

$$\therefore a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\begin{aligned} \log c &= 1.83251 \\ \log \sin A &= \underline{9.97271} \\ \log a &= 1.80522 \end{aligned}$$

$$a = 63.859.$$

$$\cos A = \frac{b}{c}.$$

$$\therefore b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\begin{aligned} \log c &= 1.83251 \\ \log \cos A &= \underline{9.53613} \\ \log b &= 1.36864 \end{aligned}$$

$$b = 23.369.$$

$$ab.$$

$$\begin{aligned} \log a &= 1.80522 \\ \log b &= 1.36864 \\ \operatorname{colog} 2 &= \underline{9.69897 - 10} \\ \log F &= 2.87283 \end{aligned}$$

$$F = 746.15.$$

37. Compute the unknown parts and also the area, having given $c = 27$, $B = 44° 4'$.

$$A = 45° 56'.$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\begin{aligned} \log c &= 1.43136 \\ \log \sin A &= \underline{9.85645} \\ \log a &= 1.28781 \end{aligned}$$

$$a = 19.40.$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\begin{aligned} \log c &= 1.43136 \\ \log \cos A &= \underline{9.84229} \\ \log b &= 1.27365 \end{aligned}$$

$$b = 18.778.$$

$$F = \tfrac{1}{2} ab.$$

$$\begin{aligned} \log a &= 1.28781 \\ \log b &= 1.27365 \\ \operatorname{colog} 2 &= \underline{9.69897 - 10} \\ \log F &= 2.26043 \end{aligned}$$

$$F = 182.15.$$

38. Compute the unknown parts and also the area, having given $a = 47$, $B = 48° 49'$.

$$A = 41° 11'.$$

$$b = a \cot A.$$

$$\log b = \log a + \log \cot A.$$

$$\begin{aligned} \log a &= 1.67210 \\ \log \cot A &= \underline{10.05803} \\ \log b &= 1.73013 \end{aligned}$$

$$b = 53.719.$$

$$c = \frac{a}{\sin A}.$$

$$\begin{aligned}
\log c &= \log a + \operatorname{colog} \sin A.\\
\log a &= 1.67210\\
\operatorname{colog} \sin a &= \underline{0.18146}\\
\log c &= 1.85356\\
c &= 71.377.\\
F &= \tfrac{1}{2} ab.\\
\log a &= 1.67210\\
\log b &= 1.73013\\
\operatorname{colog} 2 &= \underline{9.69897 - 10}\\
\log F &= 3.10120\\
F &= 1262.4.
\end{aligned}$$

39. Compute the unknown parts and also the area, having given $b = 9$, $B = 34° 44'$.

$$\begin{aligned}
A &= 55° 16'.\\
a &= b \tan A\\
\log a &= \log b + \log \tan A.\\
\log b &= 0.95424\\
\log \tan A &= \underline{10.15908}\\
\log a &= 1.11332\\
a &= 12.981.
\end{aligned}$$

$$c = \frac{a}{\sin A}.$$

$$\begin{aligned}
\log c &= \log a + \operatorname{colog} \sin A.\\
\log a &= 1.11332\\
\operatorname{colog} \sin A &= \underline{0.08523}\\
\log c &= 1.19855\\
c &= 15.796.\\
F &= \tfrac{1}{2} ab.\\
\log a &= 1.11332\\
\log b &= 0.95424\\
\operatorname{colog} 2 &= \underline{9.69897 - 10}\\
\log F &= 1.76653\\
F &= 58.416.
\end{aligned}$$

40. Compute the unknown parts and also the area, having given $c = 8.462$, $B = 86° 4'$.

$$\begin{aligned}
A &= 3° 56'.\\
a &= c \sin A.\\
\log a &= \log c + \log \sin A.\\
\log c &= 0.92747\\
\log \sin A &= \underline{8.83630 - 10}\\
\log a &= 9.76377 - 10\\
a &= 0.58046.\\
b &= c \cos A.\\
\log b &= \log c + \log \cos A.\\
\log c &= 0.92747\\
\log \cos A &\doteq \underline{9.99898}\\
\log b &= 0.92645\\
b &= 8.442.\\
F &= \tfrac{1}{2} ab.\\
\log a &= 9.76377 - 10\\
\log b &= 0.92645\\
\operatorname{colog} 2 &= \underline{9.69897 - 10}\\
\log F &= 0.38919\\
F &= 2.4501.
\end{aligned}$$

41. Find the value of F in terms of c and A.

$$\begin{aligned}
F &= \tfrac{1}{2} ab.\\
\sin A &= \frac{a}{c}.\\
\therefore a &= c \sin A.\\
\cos A &= \frac{b}{c}.\\
\therefore b &= c \cos A.\\
F &= \tfrac{1}{2} ab\\
&= \tfrac{1}{2} c^2 \sin A \cos A.
\end{aligned}$$

42. Find the value of F in terms of a and A.

$$\begin{aligned}
F &= \tfrac{1}{2} ab.\\
\cot A &= \frac{b}{a}.\\
\therefore b &= a \cot A.\\
F &= \tfrac{1}{2} ab\\
&= \tfrac{1}{2} a^2 \cot A.
\end{aligned}$$

43. Find the value of F in terms of b and A.

$$F = \tfrac{1}{2}ab.$$

$$\tan A = \frac{a}{b}.$$

$$\therefore a = b \tan A.$$

$$\begin{aligned} F &= \tfrac{1}{2}ab \\ &= \tfrac{1}{2}b^2 \tan A. \end{aligned}$$

44. Find the value of F in terms of a and c.

$$\begin{aligned} F &= \tfrac{1}{2}ab. \\ c^2 &= a^2 + b^2. \\ b^2 &= c^2 + a^2. \\ \therefore b &= \sqrt{c^2 - a^2}. \\ F &= \tfrac{1}{2}a\sqrt{c^2 - a^2}. \end{aligned}$$

45. Given $F = 58$, $a = 10$; solve the triangle.

$$F = \tfrac{1}{2}ab.$$

$$b = \frac{2F}{a}.$$

$$\log b = \log 2F + \operatorname{colog} a.$$

$$\begin{aligned} \log 2F &= 2.06446 \\ \operatorname{colog} a &= \underline{9.00000 - 10} \\ \log b &= 1.06446 \\ b &= 11.6. \end{aligned}$$

$$\tan A = \frac{a}{b}.$$

$$\log \tan A = \log a + \operatorname{colog} b.$$

$$\begin{aligned} \log a &= 1.00000 \\ \operatorname{colog} b &= \underline{8.93554 - 10} \\ \log \tan A &= 9.93554 - 10 \\ A &= 40° \, 45' \, 48''. \\ \therefore B &= 49° \, 14' \, 12''. \end{aligned}$$

$$c = \frac{a}{\sin A}.$$

$$a + \operatorname{colog} \sin A.$$

$$\begin{aligned} \log a &= 1.00000 \\ \operatorname{colog} \sin A &= \underline{0.18513} \\ \log c &= 1.18513 \\ c &= 15.315. \end{aligned}$$

46. Given $F = 18$, $b = 5$; solve the triangle.

$$F = \tfrac{1}{2}ab.$$

$$a = \frac{2F}{b}.$$

$$\log a = \log 2F + \operatorname{colog} b.$$

$$\begin{aligned} \log 2F &= 1.55630 \\ \operatorname{colog} b &= \underline{9.30103 - 10} \\ \log a &= 0.85733 \\ a &= 7.2. \end{aligned}$$

$$\tan A = \frac{a}{b}.$$

$$\log \tan A = \log a + \operatorname{colog} b.$$

$$\begin{aligned} \log a &= 0.85733 \\ \operatorname{colog} b &= \underline{9.30103 - 10} \\ \log \tan A &= 10.15836 - 10 \\ A &= 55° \, 13' \, 20''. \\ \therefore B &= 34° \, 46' \, 40''. \end{aligned}$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a + \operatorname{colog} \sin A.$$

$$\begin{aligned} \log a &= 0.85733 \\ \operatorname{colog} \sin A &= \underline{0.08546} \\ \log c &= 0.94279 \\ c &= 8.7658. \end{aligned}$$

47. Given $F = 12$, $A = 29°$; solve the triangle.

$$\begin{aligned} B &= 61°. \\ F &= \tfrac{1}{2}ab = 12. \\ ab &= 24. \end{aligned}$$

$$a = \frac{24}{b}.$$

$$\tan A = \frac{a}{b}.$$

$$\tan 29^\circ = \frac{24}{b^2}.$$

$$b^2 = \frac{24}{\tan 29^\circ}.$$

$$\log b = \tfrac{1}{2}(\log 24 + \text{colog}\tan 29^\circ).$$

$$\begin{aligned} \log 24 &= 1.38021 \\ \text{colog}\tan 29^\circ &= \underline{0.25625} \\ 2&\overline{)1.63646} \\ \log b &= 0.81823 \\ b &= 6.58. \end{aligned}$$

$$\tan 29^\circ = \frac{a}{b}.$$

$$\begin{aligned} \therefore a &= b\tan 29^\circ. \\ \log a &= \log b + \log\tan 29^\circ. \\ \log b &= 0.81823 \\ \log\tan 29^\circ &= \underline{9.74375} \\ \log a &= 0.56198 \\ a &= 3.6474. \end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$\therefore c = \frac{a}{\sin 29^\circ}.$$

$$\begin{aligned} \log c &= \log a + \text{colog}\sin 29^\circ. \\ \log a &= 0.56198 \\ \text{colog}\sin 29^\circ &= \underline{0.31443} \\ \log c &= 0.87641 \\ c &= 7.5233. \end{aligned}$$

48. Given $F = 100$, $c = 22$; solve the triangle.

$$\begin{aligned} F &= \tfrac{1}{2}ab = 100. \\ ab &= 200. \end{aligned}$$

$$a = \frac{200}{b}.$$

$$a^2 = \frac{40000}{b^2}.$$

$$a^2 + b^2 = c^2 = 484.$$

Substitute,

$$\frac{40000}{b^2} + b^2 = 484.$$

$$\therefore 40{,}000 + b^4 = 484\,b^2.$$

$$\begin{aligned} b^4 - 484\,b^2 &= -40{,}000. \\ b^4 - (\,) + (242)^2 &= 18{,}564. \\ \log\sqrt{18564} &= \tfrac{1}{2}(4.26867) \\ &= 2.13434. \\ \text{But}\quad 2.13434 &= \log 136.25. \\ \therefore b^2 - 242 &= 136.25. \\ b^2 &= 378.25. \\ \log b &= \tfrac{1}{2}\log 378.25 \\ &= 1.28889. \\ b &= 19.449. \end{aligned}$$

$$\cos A = \frac{b}{c}.$$

$$\begin{aligned} \log\cos A &= \log b + \text{colog}\,c. \\ \log b &= 1.28889 \\ \text{colog}\,c &= \underline{8.65758} \\ \log\cos A &= 9.94647 \\ A &= 27^\circ\,52'. \\ \therefore B &= 62^\circ\,8'. \end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$\begin{aligned} \therefore a &= c\sin A. \\ \log a &= \log c + \log\sin A. \\ \log c &= 1.34242 \\ \log\sin A &= \underline{9.66970} \\ \log a &= 1.01212 \\ a &= 10.283. \end{aligned}$$

49. Find the angles of a right triangle if the hypotenuse is equal to three times one of the legs.

Let $c =$ hypotenuse,

and $c =$ three times a, one of the legs,

$$\sin A = \frac{a}{c} = \frac{1}{3}.$$

$$\begin{aligned} \log\sin A &= \log a + \text{colog}\,c. \\ \log a &= 0.00000 \\ \text{colog}\,c &= \underline{9.52288 - 10} \\ \log\sin A &= 9.52288 \\ A &= 19^\circ\,28'\,17''. \\ \therefore B &= 70^\circ\,31'\,43''. \end{aligned}$$

50. Find the legs of a right triangle if the hypotenuse is 6, and one angle is twice the other.

Let $c = \text{hypotenuse} = 6$,

and $B = 2A$.

Then $B = 60°$, and $A = 30°$.

$$\sin A = \frac{a}{c}.$$

$$\therefore a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\begin{aligned} \log c &= 0.77815 \\ \log \sin A &= \underline{9.69897} \\ \log a &= 0.47712 \end{aligned}$$

$$a = 3.$$

$$\sin B = \frac{b}{c}.$$

$$\therefore b = c \sin B.$$

$$\log b = \log c + \log \sin B.$$

$$\begin{aligned} \log c &= 0.77815 \\ \log \sin B &= \underline{9.93753} \\ \log b &= 0.71568 \end{aligned}$$

$$b = 5.1961.$$

51. In a right triangle given c, and $A = nB$; find a and b.

$$B = 90° - A = 90° - nB.$$

$$B(n+1) = 90°.$$

$$\therefore B = \frac{90°}{n+1}.$$

$$\cos B = \frac{a}{c}.$$

$$\cos \frac{90°}{n+1} = \frac{a}{c}.$$

$$\therefore a = c \cos \frac{90°}{n+1}.$$

$$\sin B = \frac{b}{c}.$$

$$\sin \frac{90°}{n+1} = \frac{b}{c}.$$

$$\therefore b = c \sin \frac{90°}{n+1}.$$

52. In a right triangle the difference between the hypotenuse and the greater leg is equal to the difference between the two legs. Find the angles.

$$c - a = a - b.$$

$$2a - b = c. \qquad (1)$$

$$a^2 + b^2 = c^2. \qquad (2)$$

Square (1),

$$\begin{aligned} 4a^2 - 4ab + b^2 &= c^2 \\ \underline{a^2 \qquad\quad + b^2} &\underline{= c^2} \\ 3a^2 - 4ab \qquad\quad &= 0 \end{aligned}$$

$$3a^2 = 4ab.$$

$$3a = 4b.$$

$$\therefore a = \frac{4b}{3}.$$

$$\tan A = \frac{a}{b} = \frac{4}{3}.$$

$$\log \tan A = \log 4 + \operatorname{colog} 3.$$

$$\begin{aligned} \log 4 &= 0.60206 \\ \operatorname{colog} 3 &= \underline{9.52288 - 10} \\ \log \tan A &= 10.12494 \end{aligned}$$

$$A = 53° \, 7' \, 48''.$$

$$\therefore B = 36° \, 52' \, 12''.$$

53. At a horizontal distance of 120 feet from the foot of a steeple, the angle of elevation of the top was found to be 60° 30′. Find the height of the steeple.

$$\tan A = \frac{a}{b}.$$

$$\therefore a = b \tan A.$$

$$\log a = \log b + \log \tan A.$$

$$\begin{aligned} \log b &= 2.07918 \\ \log \tan A &= \underline{10.24736} \\ \log a &= 2.32654 \end{aligned}$$

$$a = 212.1.$$

Height of steeple is 212.1 feet.

54. From the top of a rock that rises vertically 326 feet out of the water, the angle of depression of a boat was found to be 24°. Find the distance of the boat from the foot of the rock.

$$\cot A = \frac{b}{a}.$$

$$\therefore b = a \cot A.$$

$$\log b = \log a + \log \cot A.$$

$$\begin{aligned} \log a &= 2.51322 \\ \log \cot A &= \underline{10.35142} \\ \log b &= 2.86464 \\ b &= 732.22. \end{aligned}$$

Distance is 732.22 feet.

55. How far is a monument, in a level plain, from the eye, if the height of the monument is 200 feet and the angle of elevation of the top 3° 30′?

$$\cot A = \frac{b}{a}.$$

$$\therefore b = a \cot A.$$

$$\log b = \log a + \log \cot A.$$

$$\begin{aligned} \log a &= 2.30103 \\ \log \cot A &= \underline{1.21351} \\ \log b &= 3.51454 \\ b &= 3270. \end{aligned}$$

Monument is 3270 feet from the eye.

56. A distance AB is measured 96 feet along the bank of a river from a point A opposite a tree C on the other bank. The angle ABC is 21° 14′. Find the breadth of the river.

$$\tan B = AC \div AB.$$

$$\therefore AC = AB \times \tan B.$$

$$\log AC = \log AB + \log \tan B.$$

$$\begin{aligned} \log AB &= 1.98227 \\ \log \tan B &= \underline{9.58944} \\ \log AC &= 1.57171 \\ AC &= 37.3. \end{aligned}$$

Breadth of river is 37.3 feet.

57. What is the angle of elevation of an inclined plane if it rises 1 foot in a horizontal distance of 40 feet?

$$\tan A = \frac{a}{b}.$$

$$\log \tan A = \log a + \operatorname{colog} b.$$

$$\begin{aligned} \log a &= 0.00000 \\ \operatorname{colog} b &= \underline{8.39794 - 10} \\ \log \tan A &= 8.39794 \\ A &= 1^\circ\, 25'\, 56''. \end{aligned}$$

58. Find the angle of elevation of the sun when a tower 120 feet high casts a horizontal shadow 70 feet long.

$$\tan A = \frac{a}{b}.$$

$$\tan A = \frac{120}{70}.$$

$$\log \tan A = \log 120 + \operatorname{colog} 70.$$

$$\begin{aligned} \log 120 &= 2.07918 \\ \operatorname{colog} 70 &= \underline{8.15490 - 10} \\ \log \tan A &= 10.23408 \\ A &= 59^\circ\, 44'\, 35''. \end{aligned}$$

59. How high is a tree that casts a horizontal shadow 80 feet in length when the angle of elevation of the sun is 50°?

$$\tan A = \frac{a}{b}.$$

$$\therefore a = b \tan A.$$

$$\log a = \log b + \log \tan A.$$

$$\begin{aligned}\log b &= 1.90309\\ \log\tan A &= \underline{10.07619}\\ \log a &= 1.97928\end{aligned}$$

$$a = 95.34.$$

Tree is 95.34 feet high.

60. A ship is sailing due northeast at a rate of 10 miles an hour. Find the rate at which she is moving due north, and also due east.

Let AB be the direction of the vessel, and equal 1 hour's progress $= 10$ miles.

$AC =$ distance due east passed over in 1 hour.

As the direction of the ship is northeast,

$$\begin{aligned}A &= 45^\circ.\\ b &= c\cos A.\\ \log b &= \log c + \log\cos A.\end{aligned}$$

$$\begin{aligned}\log c &= 1.00000\\ \log\cos A &= \underline{9.84949}\\ \log b &= 0.84949\end{aligned}$$

$$b = 7.0712.$$

$$AP = AC.$$

Ship moves 7.0712 miles due north, and also due east.

61. In front of a window 20 feet high is a flower-bed 6 feet wide. How long is a ladder that will just reach from the edge of the bed to the window?

$$\tan A = \frac{a}{b}.$$

$$\log\tan A = \log 20 + \operatorname{colog} 6.$$

$$\begin{aligned}\log 20 &= 1.30103\\ \operatorname{colog} 6 &= \underline{9.22185 - 10}\\ \tan A &= 10.52288\end{aligned}$$

$$A = 73^\circ\ 18'.$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log 20 + \operatorname{colog}\sin A.$$

$$\begin{aligned}\log a &= 1.30103\\ \operatorname{colog}\sin A &= \underline{0.01871}\\ \log c &= 1.31974\end{aligned}$$

$$c = 20.88.$$

Length of ladder is 20.88 feet.

62. A ladder 40 feet long may be so placed that it will reach a window 33 feet high on one side of the street, and by turning it over without moving its foot it will reach a window 21 feet high on the other side. Find the breadth of the street.

$$\cos B = \frac{33}{40}.$$

$$\begin{aligned}\log 33 &= 1.51851\\ \operatorname{colog} 40 &= \underline{8.39794 - 10}\\ \log\cos B &= 9.91645 - 10\end{aligned}$$

$$B = 34^\circ\ 24'\ 45''.$$

$$\tan B = \frac{b}{33}.$$

$$\therefore b = 33\tan B.$$

$$\begin{aligned}\log 33 &= 1.51851\\ \log\tan B &= \underline{9.83571}\\ \log b &= 1.35422\end{aligned}$$

$$b = 22.606.$$

$$\cos B' = \frac{21}{40}.$$

$$\begin{aligned}\log 21 &= 1.32222\\ \operatorname{colog} 40 &= \underline{8.39794 - 10}\\ \log\cos B' &= 9.72016\end{aligned}$$

$$B' = 58^\circ\ 19'\ 54''.$$

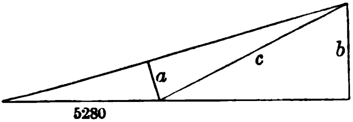
$$\tan B' = \frac{b'}{21}.$$

$$b' = 21\tan B'.$$

$$\begin{aligned} \log 21 &= 1.32222 \\ \log \tan B' &= \underline{0.20982} \\ \log b' &= 1.53204 \\ b' &= 34.044. \end{aligned}$$

Width of street
$= 32.606$ feet $+ 34.044$ feet
$= 56.65$ feet.

63. From the top of a hill the angles of depression of two successive milestones, on a straight level road leading to the hill, are observed to be 5° and 15°. Find the height of the hill.



$$\sin 5° = \frac{a}{5280}.$$

$$\therefore a = 5280 \sin 5°.$$

$$\begin{aligned} \log 5280 &= 3.72263 \\ \log \sin 5° &= \underline{8.94030 - 10} \\ \log a &= 2.66293 \end{aligned}$$

$$\sin 10° = \frac{a}{c}.$$

$$\therefore c = \frac{a}{\sin 10°}.$$

$$\begin{aligned} \log a &= 2.66293 \\ \operatorname{colog} \sin 10° &= \underline{0.76033} \\ \log c &= 3.42326 \end{aligned}$$

$$\cos 75° = \frac{b}{c}.$$

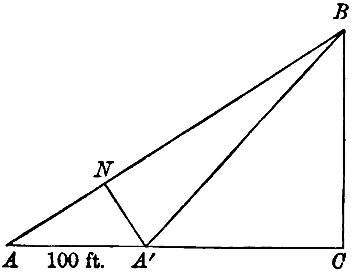
$$\therefore b = c \cos 75°.$$

$$\begin{aligned} \log c &= 3.42326 \\ \log \cos 75° &= \underline{9.41300 - 10} \\ \log b &= 2.83626 \end{aligned}$$

$$b = 685.9$$

Height of hill is 685.9 feet.

64. A fort stands on a horizontal plain. The angle of elevation at a certain point on the plain is 30°, and at a point 100 feet nearer the fort it is 45°. How high is the fort?



Let B represent the fort, AC the horizontal plain, BC a $\perp$ from fort to plain.

$BAC =$ angle made by line from eye of observer $= 30°$.

$BA'C = 45° =$ angle of elevation 100 feet nearer.

From A' draw $A'N \perp$ to AB.

In rt. $\triangle AA'N$,

$$\angle NAA' = 30°,$$

and $\angle NA'A = 60°$.

$$\therefore NA' = 50 \text{ feet}.$$

$$\begin{aligned} \therefore AN &= \sqrt{(100)^2 - (50)^2} \\ &= \sqrt{7500} = 50\sqrt{3} \\ &= 86.602. \end{aligned}$$

In rt. $\triangle BNA'$,

$$\frac{BN}{NA'} = \cot NBA' = \cot 15°.$$

and $BN = NA' \cot 15°$.

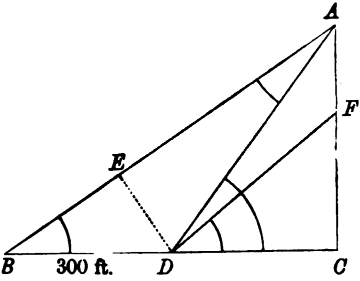
$$\begin{aligned} \log NA' &= 1.69897 \\ \log \cot 15° &= \underline{0.57195} \\ \log BN &= 2.27092 \end{aligned}$$

$$\begin{aligned} BN &= 186.60 \\ AN &= \underline{\ 86.60} \\ AB &= 273.20 \end{aligned}$$

In rt. $\triangle ABC$,

$$\begin{aligned} \angle BAC &= 30^\circ, \\ \text{and} \quad \angle ABC &= 60^\circ. \\ \therefore BC &= \tfrac{1}{2} AB \\ &= \tfrac{1}{2} \times 273.20 \text{ feet} \\ &= 136.60 \text{ feet.} \end{aligned}$$

65. From a certain point on the ground the angles of elevation of the belfry of a church and of the top of a steeple were found to be 40° and 51° respectively. From a point 300 feet farther off, on a horizontal line, the angle of elevation of the top of the steeple is found to be 33° 45′. Find the distance from the belfry to the top of the steeple.



Draw $DE \perp$ to AB from D.

In $\triangle BED$,

$$\frac{ED}{BD} = \sin 33^\circ 45'.$$

$$\therefore ED = 300 \times \sin 33^\circ 45'.$$

$$\begin{aligned} \log 300 &= 2.47712 \\ \log \sin 33^\circ 45' &= \underline{9.74474} \\ \log ED &= 2.22186 \end{aligned}$$

$\angle EAD = 180^\circ - 33^\circ 45' - (180^\circ - 51^\circ) = 17^\circ 15'$.

In $\triangle ADE$,

$$\frac{ED}{AD} = \sin 17^\circ 15'.$$

$$\therefore AD = \frac{ED}{\sin 17^\circ 15'}.$$

$$\begin{aligned} \log ED &= 2.22186 \\ \text{colog} \sin 17^\circ 15' &= \underline{0.52791} \\ \log AD &= 2.74977 \end{aligned}$$

In $\triangle ADC$,

$$\frac{DC}{AD} = \cos 51^\circ.$$

$$\therefore DC = AD \cos 51^\circ.$$

$$\begin{aligned} \log AD &= 2.74977 \\ \log \cos 51^\circ &= \underline{9.79887} \\ \log DC &= 2.54864 \end{aligned}$$

In $\triangle ADC$,

$$\frac{AC}{DC} = \tan 51^\circ.$$

$$\therefore AC = DC \tan 51^\circ.$$

$$\begin{aligned} \log DC &= 2.54864 \\ \log \tan 51^\circ &= \underline{10.09163 - 10} \\ \log AC &= 2.64027 \\ AC &= 436.79. \end{aligned}$$

In $\triangle FDC$,

$$\frac{FC}{DC} = \tan 40^\circ.$$

$$\therefore FC = DC \tan 40^\circ.$$

$$\begin{aligned} \log DC &= 2.54864 \\ \log \tan 40^\circ &= \underline{9.92381} \\ \log FC &= 2.47245 \\ FC &= 296.79. \end{aligned}$$

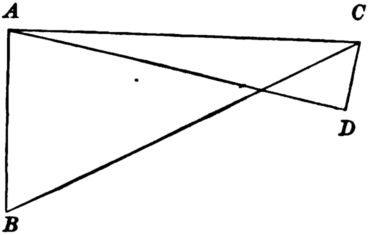
Distance from belfry to top of steeple is 436.79 feet − 296.79 feet = 140 feet.

66. The angle of elevation of the top C of an inaccessible fort observed from a point A is 12°. At a point B, 219 feet from A and on a line AB perpendicular to AC, the angle ABC is 61° 45′. Find the height of the fort.

In rt. $\triangle\, CAB$, $\frac{AB}{AC} = \cot ABC.$

$$\therefore AC = \frac{AB}{\cot ABC}.$$

$$\log AC = \log AB + \operatorname{colog} \cot ABC.$$



$$\log AB = 2.34044$$
$$\operatorname{colog} \cot ABC = \underline{0.26977}$$
$$\log AC = 2.61021$$

In rt. $\triangle\, ADC$,

$$\frac{CD}{AC} = \sin CAD.$$

$$\therefore CD = AC \sin CAD.$$

$$\log CD = \log AC + \log \sin CAD.$$

$$\log AC = 2.61021$$
$$\log \sin CAD = \underline{9.31788 - 10}$$
$$\log CD = 1.92809$$
$$CD = 84.74.$$

Height of fort is 84.74 feet.

EXERCISE X. PAGE 33.

1. In an isosceles triangle, given a and A; find C, c, h.

$$C = 180^\circ - 2A$$
$$= 2(90^\circ - A).$$
$$\frac{\frac{1}{2}c}{a} = \cos A.$$
$$c = 2a \cos A.$$
$$\frac{h}{a} = \sin A.$$
$$h = a \sin A.$$

2. In an isosceles triangle, given a and C; find A, c, h.

$$C + 2A = 180^\circ.$$
$$A = 90^\circ - \tfrac{1}{2}C.$$
$$\frac{\frac{1}{2}c}{a} = \cos A.$$
$$c = 2a \cos A.$$
$$\frac{h}{a} = \sin A.$$
$$h = a \sin A.$$

3. In an isosceles triangle, given c and A; find C, a, h.

$$C = 180^\circ - 2A$$
$$= 2(90^\circ - A).$$
$$\frac{\frac{1}{2}c}{a} = \cos A.$$
$$2a = \frac{c}{\cos A}.$$
$$a = \frac{c}{2\cos A}.$$
$$\frac{h}{a} = \sin A.$$
$$h = a \sin A.$$

4. In an isosceles triangle, given c and C; find A, a, h.

$$A = 90^\circ - \tfrac{1}{2}C.$$
$$\frac{\frac{1}{2}c}{a} = \cos A.$$
$$a = \frac{c}{2\cos A}.$$
$$\frac{h}{a} = \sin A.$$
$$h = a \sin A.$$

5. In an isosceles triangle, given h and A; find C, a, c.

$$C = 2(90^\circ - A).$$
$$\sin A = \frac{h}{a}.$$

$$\therefore a = \frac{h}{\sin A}.$$

$$\cos A = \frac{\frac{1}{2}c}{a} = \frac{c}{2a}.$$

$$\therefore c = 2a\cos A.$$

6. In an isosceles triangle, given h and C; find A, a, c.

$$A = 90^\circ - \tfrac{1}{2}C.$$

$$\sin A = \frac{h}{a}.$$

$$\therefore a = \frac{h}{\sin A}.$$

$$\cos A = \frac{\frac{1}{2}c}{a} = \frac{c}{2a}.$$

$$\therefore c = 2a\cos A.$$

7. In an isosceles triangle, given a and h; find A, C, c.

$$\sin A = h \div a.$$

$$C = 2(90^\circ - A).$$

$$\cos A = \frac{\frac{1}{2}c}{a} = \frac{c}{2a}.$$

$$\therefore c = 2a\cos A.$$

8. In an isosceles triangle, given c and h; find A, C, a.

$$\tan A = \frac{h}{\frac{1}{2}c}.$$

$$C = 2(90^\circ - A).$$

$$\sin A = \frac{h}{a}.$$

$$a = \frac{h}{\sin A}.$$

9. In an isosceles triangle, given $a = 14.3$, $c = 11$; find A, C, h.

$$\cos A = \frac{\frac{1}{2}c}{a}.$$

$$\log\cos A = \log\tfrac{1}{2}c + \operatorname{colog} a.$$

$$\begin{aligned} \log\tfrac{1}{2}c &= 0.74036 \\ \operatorname{colog} a &= \underline{8.84466 - 10} \\ \log\cos A &= 9.58502 - 10 \end{aligned}$$

$$A = 67^\circ\, 22'\, 50''.$$

$$\begin{aligned} \therefore C &= 2(90^\circ - A) \\ &= 45^\circ\, 14'\, 20''. \end{aligned}$$

$$\sin A = \frac{h}{a}.$$

$$\therefore h = a\sin A.$$

$$\log h = \log a + \log\sin A.$$

$$\begin{aligned} \log a &= 1.15534 \\ \log\sin A &= \underline{9.96524} \\ \log h &= 1.12058 \end{aligned}$$

$$h = 13.2.$$

10. In an isosceles triangle, given $a = 0.295$, $A = 68^\circ\, 10'$; find c, h, F.

$$\sin A = \frac{h}{a}.$$

$$\therefore h = a\sin A.$$

$$\log h = \log a + \log\sin A.$$

$$\begin{aligned} \log a &= 9.46982 - 10 \\ \log\sin A &= \underline{9.96767 - 10} \\ \log h &= 9.43749 - 10 \end{aligned}$$

$$h = 0.27384.$$

$$\cos A = \frac{\frac{1}{2}c}{a}.$$

$$\therefore \tfrac{1}{2}c = a\cos A.$$

$$\log\tfrac{1}{2}c = \log a + \log\cos A.$$

$$\begin{aligned} \log a &= 9.46982 - 10 \\ \log\cos A &= \underline{9.57044 - 10} \\ \log\tfrac{1}{2}c &= 9.04026 - 10 \end{aligned}$$

$$\tfrac{1}{2}c = 0.109713.$$

$$\therefore c = 0.21943.$$

$$F = \tfrac{1}{2}ch.$$

$$2F = ch.$$

$$\log 2F = \log c + \log h.$$

$$\begin{aligned} \log c &= 9.34130 - 10 \\ \log h &= \underline{9.43749 - 10} \\ \log 2F &= 8.77879 - 10 \end{aligned}$$

$$2F = 0.060089.$$

$$F = 0.03004.$$

11. In an isosceles triangle, given $c = 2.352$, $C = 69° 49'$; find a, h, F.

$$\tfrac{1}{2}C = 34° 54' 30''.$$

$$\sin\tfrac{1}{2}C = \frac{\frac{1}{2}c}{a}.$$

$$\therefore a = \frac{\frac{1}{2}c}{\sin\frac{1}{2}C}.$$

$$\log a = \log\tfrac{1}{2}c + \operatorname{colog}\sin\tfrac{1}{2}C.$$

$$\begin{aligned}\log\tfrac{1}{2}c &= 0.07041\\ \operatorname{colog}\sin\tfrac{1}{2}C &= \underline{0.24240}\\ \log a &= 0.31281\end{aligned}$$

$$a = 2.055.$$

$$\cos\tfrac{1}{2}C = \frac{h}{a}.$$

$$\therefore h = a\cos\tfrac{1}{2}C.$$

$$\log h = \log a + \log\cos\tfrac{1}{2}C.$$

$$\begin{aligned}\log a &= 0.31281\\ \log\cos\tfrac{1}{2}C &= \underline{9.91385}\\ \log h &= 0.22666\end{aligned}$$

$$h = 1.6852.$$

$$F = \tfrac{1}{2}ch.$$

$$2F = ch.$$

$$\log 2F = \log c + \log h.$$

$$\begin{aligned}\log c &= 0.37144\\ \log h &= \underline{0.22666}\\ \log 2F &= 0.59810\end{aligned}$$

$$2F = 3.9637.$$

$$F = 1.9819.$$

12. In an isosceles triangle, given $h = 7.4847$, $A = 76° 14'$; find a, c, F.

$$\sin A = \frac{h}{a}.$$

$$\therefore a = \frac{h}{\sin A}.$$

$$\log a = \log h + \operatorname{colog}\sin A.$$

$$\begin{aligned}\log h &= 0.87417\\ \operatorname{colog}\sin A &= \underline{0.01266}\\ \log a &= 0.88683\end{aligned}$$

$$a = 7.706.$$

$$\tan A = \frac{h}{\frac{1}{2}c}.$$

$$\therefore \tfrac{1}{2}c = \frac{h}{\tan A}.$$

$$\log\tfrac{1}{2}c = \log h + \operatorname{colog}\tan A.$$

$$\begin{aligned}\log h &= 0.87417\\ \operatorname{colog}\tan A &= \underline{9.38918 - 10}\\ \log\tfrac{1}{2}c &= 0.26335\end{aligned}$$

$$\tfrac{1}{2}c = 1.8338.$$

$$c = 3.6676.$$

$$F = \tfrac{1}{2}ch.$$

$$\log F = \log\tfrac{1}{2}c + \log h.$$

$$\begin{aligned}\log\tfrac{1}{2}c &= 0.26335\\ \log h &= \underline{0.87417}\\ \log F &= 1.13752\end{aligned}$$

$$F = 13.725.$$

13. In an isosceles triangle, given $a = 6.71$, $h = 6.6$; find A, C, c.

$$\sin A = \frac{h}{a}.$$

$$\log\sin A = \log h + \operatorname{colog} a.$$

$$\begin{aligned}\log h &= 0.81954\\ \operatorname{colog} a &= \underline{9.17328 - 10}\\ \log\sin A &= 9.99282 - 10\end{aligned}$$

$$A = 79° 36' 30''.$$

$$\therefore C = 20° 47'.$$

$$\cos A = \frac{\frac{1}{2}c}{a}.$$

$$\therefore \tfrac{1}{2}c = a\cos A.$$

$$\log\tfrac{1}{2}c = \log a + \log\cos A.$$

$$\begin{aligned}\log a &= 0.82672\\ \log\cos A &= \underline{9.25617 - 10}\\ \log\tfrac{1}{2}c &= 0.08289\end{aligned}$$

$$\tfrac{1}{2}c = 1.2103.$$

$$c = 2.4206.$$

14. In an isosceles triangle, given $c = 9$, $h = 20$; find A, C, a.

$$\tan\tfrac{1}{2}C = \frac{\frac{1}{2}c}{h}.$$

$\log\tan\frac{1}{2}C = \log\frac{1}{2}c + \text{colog}\, h.$

$\log\frac{1}{2}c = 0.65321$

$\text{colog}\, h = \underline{8.69897 - 10}$

$\log\tan\frac{1}{2}C = 9.35218$

$\frac{1}{2}C = 12^\circ\ 40'\ 49''.$

$C = 25^\circ\ 21'\ 38''.$

$2A = 180^\circ - C.$

$\therefore A = 77^\circ\ 19'\ 11''.$

$\sin A = \frac{h}{a}.$

$\therefore a = \frac{h}{\sin A}.$

$\log a = \log h + \text{colog}\sin A.$

$\log h = 1.30103$

$\text{colog}\sin A = \underline{0.01072}$

$\log a = 1.31175$

$a = 20.5.$

15. In an isosceles triangle, given $c = 147$, $F = 2572.5$; find A, C, a, h.

$F = \frac{1}{2}ch.$

$\therefore h = \frac{2F}{c}.$

$\log h = \log 2F + \text{colog}\, c.$

$\log 2F = 3.71139$

$\text{colog}\, c = \underline{7.83268 - 10}$

$\log h = 1.54407$

$h = 35.$

$\tan A = \frac{h}{\frac{1}{2}c}.$

$\log\tan A = \log h + \text{colog}\,\frac{1}{2}c.$

$\log \text{h} = 1.54407$

$\text{colog}\,\frac{1}{2}c = \underline{8.13371 - 10}$

$\log\tan A = 9.67778 - 10$

$A = 25^\circ\ 27'\ 47''.$

$\therefore C = 2(90^\circ - A)$

$= 129^\circ\ 4'\ 26''.$

$a = \frac{h}{\sin A}.$

$\log a = \log h + \text{colog}\sin A.$

$\log h = 1.54407$

$\text{colog}\sin A = \underline{0.36661}$

$\log a = 1.91068$

$a = 81.41.$

16. In an isosceles triangle, given $h = 16.8$, $F = 43.68$; find A, C, a, c.

$F = \frac{1}{2}ch.$

$\frac{1}{2}c = \frac{F}{h}.$

$\log\frac{1}{2}c = \log F + \text{colog}\, h.$

$\log F = 1.64028$

$\text{colog}\, h = \underline{8.77469 - 10}$

$\log\frac{1}{2}c = 0.41497$

$\frac{1}{2}c = 2.60.$

$c = 5.2.$

$\tan A = \frac{h}{\frac{1}{2}C}.$

$\log\tan A = \log h + \text{colog}\,\frac{1}{2}c.$

$\log h = 1.22531$

$\text{colog}\,\frac{1}{2}c = \underline{9.58503 - 10}$

$\log\tan A = 10.81034 - 10$

$A = 81^\circ\ 12'\ 9''.$

$\frac{1}{2}C = 8^\circ\ 47'\ 51''.$

$C = 17^\circ\ 35'\ 42''.$

$\cos A = \frac{\frac{1}{2}c}{a}.$

$\log a = \log\frac{1}{2}c + \text{colog}\cos A.$

$\log\frac{1}{2}c = 0.41497$

$\text{colog}\cos A = \underline{0.81547}$

$\log a = 1.23044$

$a = 17.$

17. In an isosceles triangle, find the value of F in terms of a and c.

$F = \frac{1}{2}ch.$

$h = \sqrt{a^2 - \frac{c^2}{4}}$

$= \sqrt{\frac{4a^2 - c^2}{4}}$

$$= \tfrac{1}{2}\sqrt{4a^2 - c^2}.$$

$$F = \tfrac{1}{2}c\left(\tfrac{1}{2}\sqrt{4a^2 - c^2}\right)$$

$$= \tfrac{1}{4}c\sqrt{4a^2 - c^2}.$$

18. In an isosceles triangle, find the value of F in terms of a and C.

$$F = \tfrac{1}{2}ch.$$
$$\tfrac{1}{2}c = a \sin \tfrac{1}{2}C.$$
$$h = a \cos \tfrac{1}{2}C.$$
$$F = a \sin \tfrac{1}{2}C \times a \cos \tfrac{1}{2}C$$
$$= a^2 \sin \tfrac{1}{2}C \cos \tfrac{1}{2}C.$$

19. In an isosceles triangle, find the value of F in terms of a and A.

$$F = \tfrac{1}{2}ch.$$
$$\tfrac{1}{2}c = a \cos A.$$
$$h = a \sin A.$$
$$F = a \cos A \times a \sin A$$
$$= a^2 \sin A \cos A.$$

20. In an isosceles triangle, find the value of F in terms of h and C.

$$F = \tfrac{1}{2}ch.$$
$$\tfrac{1}{2}c = h \tan \tfrac{1}{2}C.$$
$$F = h(h \tan \tfrac{1}{2}C)$$
$$= h^2 \tan \tfrac{1}{2}C.$$

21. A barn is 40 × 80 feet, the pitch of the roof is 45°; find the length of the rafters and the area of the whole roof.

$$40 \div 2 = 20 = \tfrac{1}{2}c.$$
$$\cos A = \frac{\tfrac{1}{2}c}{a} = \frac{20}{a}.$$
$$20 = a \cos A.$$
$$\therefore a = \frac{20}{\cos A}.$$
$$\log a = \log 20 + \operatorname{colog} \cos A.$$

$$\begin{aligned} \log 20 &= 1.30103 \\ \operatorname{colog} \cos A &= \underline{0.15051} \\ \log a &= 1.45154 \end{aligned}$$

$$a = 28.284.$$
$$2 \times 28.284 \times 80 = 4525.44.$$

Length of rafters is 28.284 feet; area of roof is 4525.44 square feet.

22. In a unit circle, what is the length of the chord corresponding to the angle 45° at the centre?

$$\sin \tfrac{1}{2}C = \frac{\tfrac{1}{2}c}{a}.$$
$$\log \tfrac{1}{2}c = \log a + \log \sin \tfrac{1}{2}C.$$

$$\begin{aligned} \log a &= 0.00000 \\ \log \sin \tfrac{1}{2}C &= \underline{9.58284 - 10} \\ \log \tfrac{1}{2}c &= 9.58284 - 10 \end{aligned}$$

$$\tfrac{1}{2}c = 0.38268.$$
$$c = 0.76536.$$

23. If the radius of a circle is 30, and the length of a chord is 44, find the angle subtended at the centre.

$$\sin \tfrac{1}{2}C = \frac{\tfrac{1}{2}c}{a}.$$
$$\log \sin \tfrac{1}{2}C = \log \tfrac{1}{2}c + \operatorname{colog} a.$$

$$\begin{aligned} \log \tfrac{1}{2}c &= 1.34242 \\ \operatorname{colog} a &= \underline{8.52288 - 10} \\ \log \sin \tfrac{1}{2}C &= 9.86530 - 10 \end{aligned}$$

$$\tfrac{1}{2}C = 47^\circ\ 10'.$$
$$C = 94^\circ\ 20'.$$

24. Find the radius of a circle if a chord whose length is 5 subtends at the centre an angle of 133°.

$$\sin \tfrac{1}{2}C = \frac{\tfrac{1}{2}c}{a}.$$
$$\log a = \log \tfrac{1}{2}c + \operatorname{colog} \sin \tfrac{1}{2}C.$$

$$\begin{aligned} \log \tfrac{1}{2}c &= 0.39794 \\ \operatorname{colog} \sin \tfrac{1}{2}C &= \underline{0.03760} \\ \log a &= 0.43554 \end{aligned}$$

$$a = 2.7261.$$

25. What is the angle at the centre of a circle if the corresponding chord is equal to $\frac{3}{8}$ of the radius?

Let $a = 3$, then $c = 2$, and $\frac{1}{2}c = 1$.

$$\sin\tfrac{1}{2}C = \frac{1}{3}.$$

$$\begin{aligned}
\log\sin\tfrac{1}{2}C &= \log 1 + \operatorname{colog} 3.\\
\log 1 &= 0.00000\\
\operatorname{colog} 3 &= \underline{9.52288 - 10}\\
\log\sin\tfrac{1}{2}C &= 9.52288 - 10\\
\tfrac{1}{2}C &= 19^\circ\ 28'\ 17''.\\
C &= 38^\circ\ 56'\ 33''.
\end{aligned}$$

26. Find the area of a circular sector if the radius of the circle is 12, and the angle of the sector is 30°.

$$\text{Area } \odot = \pi R^2.$$

$$\text{Area sector} = \frac{30\,\pi R^2}{360}.$$

$$\begin{aligned}
\log \text{area sector}\\
&= \log 30 + \operatorname{colog} 360\\
&\qquad + \log\pi + 2\log R.\\
\log 30 &= 1.47712\\
\operatorname{colog} 360 &= 7.44370 - 10\\
\log\pi &= 0.49715\\
2\log R &= \underline{2.15836\qquad}\\
\log \text{area} &= 1.57633\\
\text{Area} &= 37.699.
\end{aligned}$$

Exercise XI. Page 35.

1. Find the remaining parts of a regular polygon, given $n = 10$, $c = 1$.

$$\begin{aligned}
\tfrac{1}{2}C &= \frac{180^\circ}{10} = 18^\circ.\\
\tfrac{1}{2}c &= 0.5.\\
A &= 72^\circ.\\
h &= \tfrac{1}{2}c\tan A.\\
\log h &= \log\tfrac{1}{2}c + \log\tan A.\\
\log\tfrac{1}{2}c &= 9.69897 - 10\\
\log\tan A &= \underline{10.48822 - 10}\\
\log h &= 0.18719\\
h &= 1.5388.\\
\log r &= \log\tfrac{1}{2}c + \operatorname{colog}\cos A.\\
\log\tfrac{1}{2}c &= 9.69897 - 10\\
\operatorname{colog}\cos A &= \underline{0.51002\qquad}\\
\log r &= 0.20899\\
r &= 1.618.\\
F &= \tfrac{1}{2}hp.\\
\log h &= 0.18719\\
\log p &= \underline{1.00000}\\
\log 2F &= 1.18719\\
2F &= 15.388.\\
F &= 7.694.
\end{aligned}$$

2. Find the remaining parts of a regular polygon, given $n = 18$, $r = 1$.

$$\begin{aligned}
\tfrac{1}{2}C &= 10^\circ.\\
A &= 80^\circ.\\
h &= r\sin A.\\
\log r &= 0.00000\\
\log\sin A &= \underline{9.99335 - 10}\\
\log h &= 9.99335 - 10\\
h &= 0.9848.\\
\tfrac{1}{2}c &= r\cos A.\\
\log r &= 0.00000\\
\log\cos A &= \underline{9.23967 - 10}\\
\log\tfrac{1}{2}c &= 9.23967 - 10\\
\tfrac{1}{2}c &= 0.17365.\\
p &= 6.2514.\\
F &= \tfrac{1}{2}hp.\\
\log h &= 9.99335 - 10\\
\log p &= \underline{0.79598\qquad}\\
\log 2F &= 0.78933\\
2F &= 6.1564.\\
F &= 3.0782.
\end{aligned}$$

3. Find the remaining parts of a regular polygon, given $n = 20$, $r = 20$.

$$
\begin{aligned}
\tfrac{1}{2}C &= 9°.\\
A &= 81°.\\
h &= r\sin A.\\
\log r &= 1.30103\\
\log\sin A &= \underline{9.99462-10}\\
\log h &= 1.29565\\
h &= 19.754.\\
\tfrac{1}{2}c &= r\cos A.\\
\log r &= 1.30103\\
\log\cos A &= \underline{9.19433-10}\\
\log\tfrac{1}{2}c &= 0.49536\\
\tfrac{1}{2}c &= 3.1286.\\
c &= 6.257.\\
p &= 125.14.\\
F &= \tfrac{1}{2}hp.\\
\log h &= 1.29565\\
\log p &= \underline{2.09740}\\
\log 2F &= 3.39305\\
2F &= 2472.\\
F &= 1236.
\end{aligned}
$$

4. Find the remaining parts of a regular polygon, given $n = 8$, $h = 1$.

$$
\begin{aligned}
\tfrac{1}{2}C &= 22°\ 30'.\\
\tan\tfrac{1}{2}C &= \frac{\tfrac{1}{2}c}{h}.\\
\log\tfrac{1}{2}c &= \log h + \log\tan\tfrac{1}{2}C.\\
\log h &= 0.00000\\
\log\tan\tfrac{1}{2}C &= \underline{9.61722-10}\\
\log\tfrac{1}{2}c &= 9.61722-10\\
\tfrac{1}{2}c &= 0.41421.\\
c &= 0.82842.\\
\cos\tfrac{1}{2}C &= \frac{h}{r}.\\
\log r &= \log h + \operatorname{colog}\cos\tfrac{1}{2}C.\\
\log h &= 0.00000\\
\operatorname{colog}\cos\tfrac{1}{2}C &= \underline{0.03438}\\
\log r &= 0.03438\\
r &= 1.0824.\\
F &= \tfrac{1}{2}hp\\
&= 3.3137.
\end{aligned}
$$

5. Find the remaining parts of a regular polygon, given $n = 11$, $F = 20$.

$$
\begin{aligned}
2F &= ph.\\
40 &= ph.\\
c &= \frac{p}{11}.\\
h &= \frac{40}{p}.\\
\tfrac{1}{2}C &= 16°\ 22'.\\
\tan\tfrac{1}{2}C &= \frac{\tfrac{1}{2}c}{h}.
\end{aligned}
$$

Substitute values of h and c,

$$
\begin{aligned}
\tan\tfrac{1}{2}C &= \frac{p}{22} \div \frac{40}{p} = \frac{p^2}{880}.\\
\log p &= \tfrac{1}{2}(\log 880 + \log\tan\tfrac{1}{2}C).\\
\log 880 &= 2.94448\\
\log\tan\tfrac{1}{2}C &= \underline{9.46788-10}\\
&\quad 2\,\overline{)2.41236}\\
\log p &= 1.20618\\
p &= 16.076.\\
c &= 1.4615.\\
\sin\tfrac{1}{2}C &= \frac{\tfrac{1}{2}c}{r}.\\
\log r &= \log\tfrac{1}{2}c + \operatorname{colog}\sin\tfrac{1}{2}C.\\
\log\tfrac{1}{2}c &= 9.86377-10\\
\operatorname{colog}\sin\tfrac{1}{2}C &= \underline{0.55008}\\
\log r &= 0.41385\\
r &= 2.5933.\\
\cos\tfrac{1}{2}C &= \frac{h}{r}.\\
\log h &= \log r + \log\cos\tfrac{1}{2}C.\\
\log r &= 0.41385\\
\log\cos\tfrac{1}{2}C &= \underline{9.98204}\\
\log h &= 0.39589\\
h &= 2.4882.
\end{aligned}
$$

6. Find the remaining parts of a regular polygon, given $n = 7$, $F = 7$

$$14 = ph.$$

$$h = \frac{14}{p}.$$

$$c = \frac{p}{7}.$$

$$\tfrac{1}{2}C = 25^\circ\, 43'.$$

$$\tan \tfrac{1}{2}C = \frac{\frac{1}{2}c}{h}.$$

$$\tan \tfrac{1}{2}C = \frac{p}{14} \div \frac{14}{p} = \frac{p^2}{196}.$$

$$\log p = \tfrac{1}{2}(\log 196 + \log \tan \tfrac{1}{2}C).$$

$$\begin{aligned} \log 196 &= 2.29226 \\ \log \tan \tfrac{1}{2}C &= \underline{9.68271 - 10} \\ &2\overline{)1.97497} \\ \log p &= 0.98749 \\ p &= 9.716. \\ \tfrac{1}{2}c &= 0.694. \end{aligned}$$

$$\tan \tfrac{1}{2}C = \frac{\frac{1}{2}c}{h}.$$

$$\log h = \log \tfrac{1}{2}c + \operatorname{colog} \tan \tfrac{1}{2}C.$$

$$\begin{aligned} \log \tfrac{1}{2}c &= 9.84136 - 10 \\ \operatorname{colog} \tan \tfrac{1}{2}C &= \underline{0.31729} \\ \log h &= 0.15865 \\ h &= 1.441. \end{aligned}$$

$$\sin \tfrac{1}{2}C = \frac{\frac{1}{2}c}{r}.$$

$$\log r = \log \tfrac{1}{2}c + \operatorname{colog} \sin \tfrac{1}{2}C.$$

$$\begin{aligned} \log \tfrac{1}{2}c &= 9.84136 - 10 \\ \operatorname{colog} \sin \tfrac{1}{2}C &= \underline{0.36259} \\ \log r &= 0.20395 \\ r &= 1.5994. \end{aligned}$$

7. Find the side of a regular decagon inscribed in a unit circle.

$$\tfrac{1}{2}C = 18^\circ.$$

$$\sin \tfrac{1}{2}C = \frac{\frac{1}{2}c}{r}.$$

$$\log c = \log 2 + \log \sin \tfrac{1}{2}C.$$

$$\begin{aligned} \log 2 &= 0.30103 \\ \log \sin \tfrac{1}{2}C &= \underline{9.48998} \\ \log c &= 9.79101 - 10 \\ c &= 0.61803. \end{aligned}$$

8. Find the side of a regular decagon circumscribed about a unit circle.

$$\tfrac{1}{2}C = 18^\circ.$$

$$\tan \tfrac{1}{2}C = \frac{\frac{1}{2}c}{h}.$$

$$\log \tfrac{1}{2}c = \log h + \log \tan \tfrac{1}{2}C.$$

$$\begin{aligned} \log h &= 0.00000 \\ \log \tan \tfrac{1}{2}C &= \underline{9.51178} \\ \log \tfrac{1}{2}c &= 9.51178 - 10 \\ \tfrac{1}{2}c &= 0.32492. \\ c &= 0.64984. \end{aligned}$$

9. If the side of an inscribed regular hexagon is equal to 1, find the side of an inscribed regular dodecagon.

Let O be the centre of the circle. BC a side of the hexagon, and BA a side of the dodecagon. Also let OD be $\perp$ to BA.

Then $OB = BC = 1.$

$$\angle BOD = 15^\circ.$$

In rt. $\triangle\, ODB$,

$$\sin BOD = \tfrac{1}{2}AB \div OB.$$

$$\therefore AB = 2\,OB \times \sin BOD.$$

$$\log AB = \log 2\,OB + \log \sin BOD.$$

$$\begin{aligned} \log 2\,OB &= 0.30103 \\ \log \sin 15^\circ &= \underline{9.41300} \\ \log AB &= 9.71403 - 10 \\ AB &= 0.51764. \end{aligned}$$

10. Given n and c, and let b denote the side of the inscribed regular polygon having $2n$ sides; find b in terms of n and c.

Let O be the centre of the circle, BC the side of the polygon having n sides, BA the side of the polygon having $2n$ sides. Then OA is $\perp$ to BC at its middle point D.

$$\angle BOA = \frac{360^\circ}{2n} = \frac{180^\circ}{n}.$$

$$\angle OBC = 90^\circ - \frac{180^\circ}{n}.$$

The $\triangle BOA$ is isosceles.

$$\therefore \angle OBA = \frac{1}{2}\left(180^\circ - \frac{180^\circ}{n}\right)$$
$$= 90^\circ - \frac{90^\circ}{n}.$$

$$\angle ABC = \angle OBA - \angle OBC$$
$$= \left(90^\circ - \frac{90^\circ}{n}\right) - \left(90^\circ - \frac{180^\circ}{n}\right) = \frac{90^\circ}{n}.$$

$$\frac{\frac{1}{2}c}{b} = \cos\frac{90^\circ}{n}.$$

$$\therefore \tfrac{1}{2}c = b\cos\frac{90^\circ}{n}.$$

$$\therefore b = \frac{\frac{1}{2}c}{\cos\frac{90^\circ}{n}} = \frac{c}{2\cos\frac{90^\circ}{n}}.$$

11. Compute the difference between the areas of a regular octagon and a regular nonagon if the perimeter of each is 16.

$$\tfrac{1}{2}c = \frac{p}{2n} = \frac{16}{16} = 1.$$

$$A = \frac{180^\circ}{n} = 22^\circ\ 30'.$$

$$\log h = \log \tfrac{1}{2}c + \log\cot A.$$

$$\begin{aligned} \log \tfrac{1}{2}c &= 0.00000 \\ \log\cot A &= \underline{10.38278 - 10} \\ \log h &= 0.38278 \end{aligned}$$

$$\log F = \log h + \log \tfrac{1}{2}p.$$

$$\begin{aligned} \log h &= 0.38278 \\ \log \tfrac{1}{2}p &= \underline{0.90309} \\ \log F &= 1.28587 \end{aligned}$$

$$F = 19.3139.$$

$$\tfrac{1}{2}c' = \frac{p}{2n'} = \frac{16}{18} = 0.8889.$$

$$A' = \frac{180^\circ}{n'} = 20^\circ.$$

$$\log h' = \log \tfrac{1}{2}c' + \log\cot A'.$$

$$\begin{aligned} \log \tfrac{1}{2}c' &= 9.94885 - 10 \\ \log\cot A' &= \underline{10.43893 - 10} \\ \log h' &= 0.38778 \end{aligned}$$

$$\log F' = \log h' + \log \tfrac{1}{2}p.$$

$$\begin{aligned} \log h' &= 0.38778 \\ \log \tfrac{1}{2}p &= \underline{0.90309} \\ \log F' &= 1.29087 \end{aligned}$$

$$F' = 19.5377.$$

$$\begin{aligned} F' - F &= 19.5377 - 19.3139 \\ &= 0.2238. \end{aligned}$$

12. Compute the difference between the perimeters of a regular pentagon and a regular hexagon if the area of each is 12.

$$F = 12, \quad n = 5.$$

$$\tfrac{1}{2}C = \frac{180^\circ}{5} = 36^\circ.$$

$$F = \tfrac{1}{2}hp.$$

$$h = \frac{24}{p}.$$

$$\tfrac{1}{2}c = \frac{p}{2n} = \frac{p}{10}.$$

$$\tan\tfrac{1}{2}C = \frac{\frac{1}{2}c}{h} = \frac{\frac{p}{10}}{\frac{24}{p}} = \frac{p^2}{240}.$$

$$p^2 = 240\tan\tfrac{1}{2}C.$$

$$\begin{aligned} \log 240 &= 2.38021 \\ \log\tan\tfrac{1}{2}C &= \underline{9.86126 - 10} \\ & 2\overline{)2.24147} \\ \log p &= 1.12074 \end{aligned}$$

$$p = 13.205.$$

$$n' = 6,\ \tfrac{1}{2}C' = 30^\circ.$$

$$\tan\tfrac{1}{2}C' = \frac{\frac{p'}{12}}{\frac{24}{p'}} = \frac{p'^2}{288}.$$

$$p'^2 = 288\tan\tfrac{1}{2}C'.$$

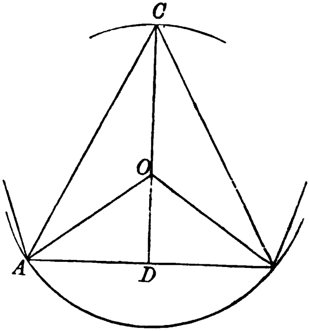
$$\log 288 = 2.45939$$
$$\log \tan \tfrac{1}{2} C' = \underline{9.76144 - 10}$$
$$2\,\overline{)\,2.22083}$$
$$\log p' = 1.11042$$
$$p' = 12.895.$$
$$p - p' = 0.310.$$

13. Find the area of the regular octagon formed by cutting away the corners of a square whose side is 1.

$$h = \tfrac{1}{2}.$$
$$\tfrac{1}{2} C = \frac{1}{2}\left(\frac{360^\circ}{8}\right) = 22^\circ\ 30'.$$
$$A = 90^\circ - 22^\circ\ 30'$$
$$= 67^\circ\ 30'.$$
$$\tan A = \frac{h}{\frac{1}{2}c}.$$
$$\therefore \tfrac{1}{2}c = \frac{h}{\tan A}.$$
$$\log \tfrac{1}{2}c = \log h + \operatorname{colog} \tan A.$$
$$\log h = 9.69897 - 10$$
$$\operatorname{colog} \tan A = \underline{9.61722 - 10}$$
$$\log \tfrac{1}{2}c = 9.31619 - 10$$
$$p = \tfrac{1}{2}c \times 2n = nc.$$
$$F = \tfrac{1}{2}ph = \tfrac{1}{2}c \times \tfrac{1}{2}n.$$
$$\log F = \log \tfrac{1}{2}c + \log \tfrac{1}{2}n.$$
$$\log \tfrac{1}{2}c = 9.31619 - 10$$
$$\log \tfrac{1}{2}n = \underline{0.60206}$$
$$\log F = 9.91825 - 10$$
$$F = 0.82842.$$

14. Find the area of a regular pentagon if its diagonals are each equal to 12.

$$\angle AOD = \frac{180^\circ}{n} = 36^\circ.$$
$$\angle AOC = 180^\circ - 36^\circ = 144^\circ.$$
$$\angle ACD = \tfrac{1}{2}(180^\circ - 144^\circ)$$
$$= 18^\circ = \angle CAO.$$
$$\angle OAD = 90^\circ - \angle AOD = 54^\circ.$$
$$\angle DAC = 54^\circ + 18^\circ = 72^\circ.$$
$$\cos DAC = \frac{AD}{AC} = \frac{\frac{1}{2}c}{12}.$$
$$\log \tfrac{1}{2}c = \log 12 + \log \cos 72^\circ.$$
$$\log \cos 72^\circ = 9.48998$$
$$\log 12 = \underline{1.07918}$$
$$\log \tfrac{1}{2}c = 0.56916$$
$$\tan DAO = \frac{h}{\frac{1}{2}c}.$$
$$\log h = \log \tfrac{1}{2}c + \log \tan 54^\circ.$$



$$\log \tfrac{1}{2}c = 0.56916$$
$$\log \tan 54^\circ = \underline{10.13874 - 10}$$
$$\log h = 0.70790$$
$$p = \tfrac{1}{2}c \times 2n.$$
$$F = \tfrac{1}{2}ph = \tfrac{1}{2}c \times nh.$$
$$\log F = \log \tfrac{1}{2}c + \log n + \log h.$$
$$\log \tfrac{1}{2}c = 0.56916$$
$$\log n = 0.69897$$
$$\log h = \underline{0.70790}$$
$$\log F = 1.97603$$
$$F = 94.63.$$

15. Find the area of a regular polygon of 11 sides inscribed in a circle, if the area of an inscribed regular pentagon is 331.8.

Let AB be a side of a regular inscribed pentagon, and AD the side of a regular inscribed polygon of 11 sides.

Let r be the radius of the circle whose centre is O, and h and h' the apothems of the two polygons respectively.

Given F the area of pentagon $= 331.8$; find F', the area of the 11-sided polygon.

Let p and p' and c and c' represent the perimeters and sides of the pentagon and the 11-sided polygon respectively.

$$F = \tfrac{1}{2}ph.$$

$$331.8 = \tfrac{1}{2}ph.$$

$$ph = 663.6.$$

$$h = \frac{663.6}{p}.$$

$$c = \frac{p}{5}.$$

$$\tfrac{1}{2}c = \frac{p}{10}.$$

$$\angle AOE = 36°.$$

$$\tan 36° = \frac{\frac{1}{2}c}{h} = \frac{p}{10} \times \frac{p}{663.6}$$

$$= \frac{p^2}{6636}.$$

$$\log p^2 = \log \tan 36° + \log 6636.$$

$$\begin{aligned} \log 6636 &= 3.82191 \\ \log \tan 36° &= \underline{9.86126 - 10} \\ \log p^2 &= 3.68317 \\ \log p &= 1.84159. \end{aligned}$$

Since $\tfrac{1}{2}c = \tfrac{1}{10}$ of p,

$$\log \tfrac{1}{2}c = 0.84159.$$

$$\sin \angle AOE = \frac{\frac{1}{2}c}{r}.$$

$$\log r = \log \tfrac{1}{2}c + \operatorname{colog} \sin 36°.$$

$$\begin{aligned} \log \tfrac{1}{2}c &= 0.84159 \\ \operatorname{colog} \sin 36° &= \underline{0.23078} \\ \log r &= 1.07237 \end{aligned}$$

$$\angle AOC = \frac{360°}{22} = 16°\,21'\,49''.$$

$$\sin \angle AOC = \tfrac{1}{2}c' \div r.$$

$$\begin{aligned} \log r &= 1.07237 \\ \log \sin AOC &= \underline{9.44984 - 10} \\ \log \tfrac{1}{2}c' &= 0.52221 \end{aligned}$$

$$\tan AOC = \frac{\frac{1}{2}c'}{h'}.$$

$$\log h' = \log \tfrac{1}{2}c' + \operatorname{colog} \tan AOC.$$

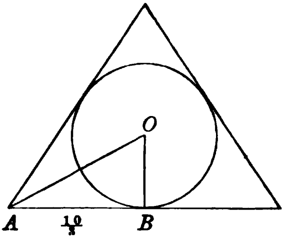
$$\begin{aligned} \log \tfrac{1}{2}c' &= 0.52221 \\ \operatorname{colog} \tan AOC &= \underline{0.53221} \\ \log h' &= 1.05442 \end{aligned}$$

$$\begin{aligned} F &= \tfrac{1}{2}p'h' \\ &= \tfrac{1}{2}c' \times 11 \times h'. \end{aligned}$$

$$\begin{aligned} \log \tfrac{1}{2}c' &= 0.52221 \\ \log 11 &= 1.04139 \\ \log h' &= \underline{1.05442} \\ \log F &= 2.61802 \end{aligned}$$

$$F = 414.97.$$

16. Find the area of a circle inscribed in an equilateral triangle whose perimeter is 20.



Perimeter $= 20$.

$$AB = \tfrac{1}{6} \times 20 = \tfrac{10}{3}.$$

$$\angle OAB = \tfrac{1}{6} \text{ of } 180° = 30°.$$

$$\tan 30° = \frac{r}{AB}.$$

$$\begin{aligned} \log AB &= 0.52288 \\ \log \tan 30° &= \underline{9.76144 - 10} \\ \log r &= 0.28432 \end{aligned}$$

$$\text{Area} = \pi r^2.$$

$$\begin{aligned}\log \pi &= 0.49715.\\ \log r^2 &= \underline{0.56864}\\ \log \text{area} &= 1.06579\end{aligned}$$

$$\text{Area} = 11.636.$$

17. Find the area of a regular polygon of 15 sides inscribed in a circle, if the area of a regular inscribed polygon of 16 sides is 100.

$$\tfrac{1}{2}C = \frac{360^\circ}{32} = 11^\circ\ 15'.$$

$$\tfrac{1}{2}C' = \frac{360^\circ}{30} = 12^\circ.$$

Let $AC = h, AB = r, BC = \tfrac{1}{2}c.$

$$\begin{aligned}F &= \tfrac{1}{2}hp.\\ 100 &= \tfrac{1}{2}hp.\\ h &= \frac{200}{p}.\end{aligned}$$

$$\tan\tfrac{1}{2}C = \frac{\dfrac{p}{32}}{\dfrac{200}{p}}.$$

$$p^2 = 6400 \tan\tfrac{1}{2}C.$$

$$\begin{aligned}\log 6400 &= 3.80618\\ \log\tan\tfrac{1}{2}C &= \underline{9.29866 - 10}\\ 2\,&\overline{)3.10484}\\ \log p &= 1.55242\end{aligned}$$

$$\begin{aligned}p &= 35.68.\\ \tfrac{1}{2}c &= 35.68 \div 32\\ &= 1.115.\end{aligned}$$

$$\sin\tfrac{1}{2}C = \frac{\tfrac{1}{2}c}{r} = \frac{1.115}{r}.$$

$$\begin{aligned}\log 1.115 &= 0.04727\\ \text{colog}\sin\tfrac{1}{2}C &= \underline{0.70976}\\ \log r &= 0.75703\end{aligned}$$

$$\frac{h'}{r} = \cos\tfrac{1}{2}C'\ (12^\circ).$$

$$\therefore h' = r \times \cos\tfrac{1}{2}C'.$$

$$\begin{aligned}\log r &= 0.75703\\ \log\cos\tfrac{1}{2}C' &= \underline{9.99040 - 10}\\ \log h' &= 0.74743\end{aligned}$$

$$\frac{\tfrac{1}{2}c'}{r} = \sin\tfrac{1}{2}C'.$$

$$\tfrac{1}{2}c' = r \times \sin\tfrac{1}{2}C'.$$

$$\begin{aligned}\log r &= 0.75703\\ \log\sin\tfrac{1}{2}C' &= \underline{9.31788}\\ \log\tfrac{1}{2}c' &= 0.07491\end{aligned}$$

$$F = \frac{1}{2}\left(\frac{c'}{2} \times 2nh'\right).$$

$$\log F = \log\tfrac{1}{2}c' + \log n + \log h'.$$

$$\begin{aligned}\log\tfrac{1}{2}c' &= 0.07491\\ \log 15 &= 1.17609\\ \log h' &= \underline{0.74743}\\ &\ 1.99843\end{aligned}$$

$$F = 99.640.$$

18. Find the perimeter of a regular dodecagon circumscribed about a circle the circumference of which is 1.

Given circumference of inscribed circle $= 1$, $n = 12$; find p.

$$2\pi r = \text{circumference}.$$

$$r = \frac{\text{circ.}}{2\pi}.$$

$$\tfrac{1}{2}C = \frac{360^\circ}{24} = 15^\circ.$$

$$\tan 15^\circ = \frac{\tfrac{1}{2}c}{r} = \pi c.$$

$$c = \frac{\tan 15^\circ}{3.1416}.$$

$$\begin{aligned}\log\tan 15^\circ &= 9.42805\\ \text{colog}\,3.1416 &= \underline{9.50285 - 10}\\ \log c &= 8.93090 - 10\\ \log 12 &= \underline{1.07918}\\ \log p &= 0.01008\end{aligned}$$

$$p = 1.0235.$$

19. The area of a regular polygon of 25 sides is 40; find the area of the ring comprised between the circumferences of the inscribed and circumscribed circles.

$$\tfrac{1}{2}ch = \frac{40}{25} = 1.6.$$

$$\tfrac{1}{2}C = 7^\circ\ 12'.$$

$$\frac{\tfrac{1}{2}c}{h} = \tan\tfrac{1}{2}C,$$

or

$$\frac{\tfrac{1}{2}ch}{h^2} = \tan\tfrac{1}{2}C.$$

$$\therefore h^2 = \frac{1.6}{\tan\tfrac{1}{2}C}.$$

$$\begin{aligned} \log 1.6 &= 0.20412 \\ \text{colog}\tan\tfrac{1}{2}C &= \underline{0.89850} \\ \log h^2 &= 1.10262 \\ \log h &= 0.55131. \end{aligned}$$

$$\frac{h}{r} = \cos\tfrac{1}{2}C.$$

$$\therefore r = \frac{h}{\cos\tfrac{1}{2}C}.$$

$$\begin{aligned} \log h &= 0.55131 \\ \text{colog}\cos\tfrac{1}{2}C &= \underline{0.00344} \\ \log r &= 0.55475 \\ \log r^2 &= 1.10950. \end{aligned}$$

πr^2 = area of circumscribed ⊙.

$$\begin{aligned} \log \pi &= 0.49715 \\ \log r^2 &= \underline{1.10950} \\ \log F &= 1.60665 \\ F &= 40.425. \\ \log \pi &= 0.49715 \\ \log h^2 &= \underline{1.10262} \\ \log \pi h^2 &= 1.59977 \end{aligned}$$

Area of inscribed ⊙ = 39.790.

$40.425 - 39.790 = 0.635.$

EXERCISE XII. PAGE 45.

1. Construct the functions of an angle in Quadrant II. What are their signs?

Sines and tangents extending upwards from the horizontal diameter are positive; downwards, negative. Cosines and cotangents extending from the vertical diameter towards the right are positive; towards the left, negative. The sign of the secant is the same as the sign of the cosine; and the sign of the cosecant is the same as the sign of the sine. Hence,

sin and csc are +.
cos and sec are −.
tan and cot are −.

2. Construct the functions of an angle in Quadrant III. What are their signs?

sin and csc are −.
cos and sec are −.
tan and cot are +.

3. Construct the functions of an angle in Quadrant IV. What are their signs?

sin and csc are −.
cos and sec are +.
tan and cot are −.

4. What are the signs of the functions of the following angles: 340°, 239°, 145°, 400°, 700°, 1200°, 3800°?

240° is in Quadrant IV.

sin = −. tan = −. sec = +.
cos = +. cot = −. csc = −.

239° is in Quadrant III.

sin $= -$. tan $= +$. sec $= -$.
cos $= -$. cot $= +$. csc $= -$.

145° is in Quadrant II.

sin $= +$. tan $= -$. sec $= -$.
cos $= -$. cot $= -$. csc $= +$.

$$400° = 360° + 40°.$$

Therefore, 400° is in Quadrant I.

sin $= +$. tan $= +$. sec $= +$.
cos $= +$. cot $= +$. csc $= +$.

$$700° = 360° + 340°.$$

Therefore, 700° is in Quadrant IV.

sin $= -$. tan $= -$. sec $= +$.
cos $= +$. cot $= -$. csc $= -$.

$$1200° = 3 \times 360° + 120°.$$

Therefore, 1200° is in Quadrant II.

sin $= +$. tan $= -$. sec $= -$.
cos $= -$. cot $= -$. csc $= +$.

$$3800° = 10 \times 360° + 200°.$$

Therefore, 3800° is in Quadrant III.

sin $= -$. tan $= +$. sec $= -$.
cos $= -$. cot $= +$. csc $= -$.

5. How many angles less than 360° have the value of the sine equal to $+\frac{5}{7}$, and in what quadrants do they lie?

Since the sine is +, by Sect. XXII, the angles can lie in but two quadrants, the first and second.

In the first quadrant, by Sect. XXII, the sine increases from 0 to 1, and in the second, decreases from 1 to 0.

Therefore, of angles less than 360° there may be two with sine equal to $+\frac{5}{7}$; one in the first quadrant, and one in the second quadrant.

6. How many values less than 720° can the angle x have if $\cos x = +\frac{2}{3}$, and in what quadrants do they lie?

720° is twice 360°; hence, the moving radius will make exactly 2 complete revolutions.

The cosine has the + sign in the first and fourth quadrants.

Hence, x can have four values: two in Quadrant I and two in Quadrant IV.

7. If we take into account only angles less than 180°, how many values can x have if $\sin x = \frac{5}{7}$? if $\cos x = \frac{1}{5}$? if $\cos x = -\frac{4}{5}$? if $\tan x = \frac{2}{3}$? if $\cot x = -7$?

(i) Sign being +, the angle can be in Quadrant I or II.

Therefore, there can be two values of x less than 180°.

(ii) Sign being +, the angle can be in Quadrant I or IV.

Therefore, there can be one value of x less than 180°.

(iii) Sign being −, the angle can be in Quadrant II or III.

Therefore, there can be one value of x less than 180°.

(iv) Sign being +, the angle can be in Quadrant I or III.

Therefore, there can be one value of x less than 180°.

(v) Sign being −, the angle can be in Quadrant II or IV.

Therefore, there can be one value or x less than 180°.

8. Within what limits must the angle x lie if $\cos x = -\frac{2}{9}$? if $\cot x = 4$? if $\sec x = 80$? if $\csc x = -3$? (If $x < 360$.)

If $\cos x = -\frac{2}{3}$, x must lie in the second or third quadrant, or between 90° and 270°.

If $\cot x = 4$, x is between 0° and 90° or 180° and 270°.

If $\sec x = 80$, x is between 0° and 90° or 270° and 360°.

If $\csc x = -3$, x is between 180° and 360°.

9. In what quadrant does an angle lie if sine and cosine are both negative? if cosine and tangent are both negative? if cotangent is positive and sine negative?

(i) Sine is negative in Quadrants II and III; cosine is negative in Quadrants III and IV.

Therefore, angles having both sine and cosine negative are in Quadrant III.

(ii) Cosine is negative in Quadrants II and III; tangent is negative in Quadrants II and IV.

Therefore, angles having both cosine and tangent negative are in Quadrant II.

(iii) Cotangent is positive in Quadrants I and III; sine is negative in Quadrants III and IV.

Therefore, angles having cotangent positive and sine negative are in Quadrant III.

10. Between 0° and 3600° how many angles are there whose sines have the absolute value $\frac{3}{5}$? Of these sines how many are positive and how many negative?

Between 0° and 3600° there are 10 revolutions, and in each there are 4 angles whose sines have the absolute value $\frac{3}{5}$. Therefore, there are 40 angles. The sine is positive in Quadrants I and II, and negative in Quadrants III and IV. Hence, there are 20 angles with the sine positive, and 20 with the sine negative.

11. In finding $\cos x$ by means of the equation $\cos x = \pm\sqrt{1 - \sin^2 x}$, when must we choose the positive sign and when the negative sign?

Since cosines of angles in Quadrants I or IV are positive, we use the sign + when angle x lies within these limits.

Also, since cosines of angles in Quadrants II and III are negative, we use the sign −, when x lies in either of these.

12. Given $\cos x = -\sqrt{\frac{1}{2}}$; find the other functions when x is an angle in Quadrant II.

$$\sin^2 x + \cos^2 x = 1.$$

$$\begin{aligned}\sin x &= \sqrt{1 - \cos^2 x}\\ &= \sqrt{1 - (-\sqrt{\tfrac{1}{2}})^2} = \sqrt{\tfrac{1}{2}}\\ &= \tfrac{1}{2}\sqrt{2}.\end{aligned}$$

$$\csc x = \frac{1}{\sin x} = \frac{1}{\frac{1}{2}\sqrt{2}} = \sqrt{2}.$$

$$\sec x = \frac{1}{\cos x} = \frac{1}{-\frac{1}{2}\sqrt{2}} = -\sqrt{2}.$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\sqrt{\frac{1}{2}}}{-\frac{1}{2}\sqrt{2}} = -1.$$

$$\cot x = \frac{1}{\tan x} = \frac{1}{-1} = -1.$$

13. Given $\tan x = \sqrt{3}$; find the other functions when x is an angle in Quadrant III.

$$\tan x = \sqrt{3}.$$

$$\cot x = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3}.$$

$$\tan x = \frac{\sin x}{\cos x}.$$

$$\tan x \times \cos x = \sin x.$$

$$\sqrt{3} \cos x = \sin x.$$

$$\begin{array}{l} 3 \cos^2 x - \sin^2 x = 0 \\ \underline{\cos^2 x + \sin^2 x = 1} \\ 4 \cos^2 x \qquad\quad = 1 \end{array}$$

$$\cos^2 x = \tfrac{1}{4}.$$

$$\cos\ x = \pm \tfrac{1}{2}.$$

The angle being in Quadrant III, the cosine is negative.

$$\therefore \cos x = -\tfrac{1}{2}.$$

$$\sin x = \sqrt{1 - (-\tfrac{1}{2})^2}$$

$$= \sqrt{\tfrac{3}{4}} = \pm \tfrac{1}{2} \sqrt{3}.$$

Sine is negative.

$$\therefore \sin x = -\tfrac{1}{2} \sqrt{3}.$$

$$\sec x = \frac{1}{-\frac{1}{2}} = -2.$$

$$\csc x = \frac{1}{-\frac{1}{2}\sqrt{3}} = -\tfrac{2}{3} \sqrt{3}.$$

14. Given $\sec x = +7$, and $\tan x$ negative; find the other functions of x.

x must be in Quadrant IV.

Hence, sine, cosecant, tangent, and cotangent are negative, and cosine positive.

$$\cos x = \frac{1}{\sec x} = \frac{1}{7}.$$

$$\sin x = \pm \sqrt{1 - \frac{1}{49}} = \pm \sqrt{\frac{48}{49}}$$

$$= -\tfrac{4}{7} \sqrt{3}.$$

$$\csc x = \frac{1}{\sin x} = \frac{1}{-\frac{4}{7}\sqrt{3}}$$

$$= -\tfrac{7}{12} \sqrt{3}.$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{-\frac{4}{7}\sqrt{3}}{\frac{1}{7}}$$

$$= -4\sqrt{3}.$$

$$\cot x = \frac{1}{\tan x} = -\frac{1}{4\sqrt{3}}$$

$$= -\tfrac{1}{12} \sqrt{3}.$$

15. Given $\cot x = -3$; find all the possible values of the other functions.

By [3] $\tan x = -\tfrac{1}{3}$, and may be in Quadrant II or IV.

By [1],

$$\sin^2 x = 1 - \cos^2 x.$$

$$\sin x = \sqrt{1 - \cos^2 x}.$$

By [2],

$$-\frac{1}{3} = \frac{\sqrt{1 - \cos^2 x}}{\cos x}.$$

$$\frac{1}{9} = \frac{1 - \cos^2 x}{\cos^2 x}.$$

$$\cos^2 x = 9 - 9 \cos^2 x.$$

$$\cos^2 x = \frac{9}{10}.$$

$$\cos x = \frac{3}{\sqrt{10}} = \tfrac{3}{10} \sqrt{10},$$

and is − in Quadrant II, + in IV.

By [1],

$$\sin x = \sqrt{1 - \frac{9}{10}} = \sqrt{\frac{1}{10}}$$

$$= \tfrac{1}{10} \sqrt{10},$$

and is + in Quadrant II, − in IV.

$$\sec x = \frac{\sqrt{10}}{3} = \tfrac{1}{3} \sqrt{10}.$$

$$\csc x = \sqrt{10},$$

and is + in Quadrant II, − in IV.

16. What functions of an angle of a triangle may be negative? In what case are they negative?

When an angle of a triangle is acute, its functions are all positive. When an angle is obtuse, its functions are those of an angle in Quadrant II.

Hence, sine and cosecant are always positive, and cosine, tangent, cotangent, and secant may be negative.

17. Why may cot 380° be considered equal either to $+\infty$ or to $-\infty$?

The nearer an acute angle is to 0°, the greater the positive value of its cotangent; and the nearer an angle is to 360°, the greater the negative value of its cotangent. When the angle is 0° or 360°, the cotangent is parallel to the horizontal diameter and cannot meet it. But the cotangent of 360° may be regarded as extending either in the positive or in the negative direction, and hence either $+\infty$ or $-\infty$.

18. Obtain by means of Formulas [1]-[3] the other functions of the angles, given:

(i) $\tan 90^\circ = \infty$.
(ii) $\cos 180^\circ = -1$.
(iii) $\cot 270^\circ = 0$.
(iv) $\csc 360^\circ = -\infty$.

(i)

$$\tan 90^\circ = \infty = \frac{1}{0}.$$

$$\cot 90^\circ = \frac{1}{\infty} = 0.$$

$$\frac{\sin 90^\circ}{\cos 90^\circ} = \frac{1}{0}.$$

$$\cos 90^\circ = 0 \sin 90^\circ = 0.$$

$$\cos^2 90^\circ + \sin^2 90^\circ = 1.$$

$$\sin^2 90^\circ = 1.$$

$$\sin 90^\circ = 1.$$

$$\sec 90^\circ = \frac{1}{\cos 90^\circ} = \frac{1}{0} = \infty.$$

$$\csc 90^\circ = \frac{1}{\sin 90^\circ} = \frac{1}{1} = 1.$$

(ii)

$$\cos 180^\circ = -1.$$

$$\sin^2 180^\circ + \cos^2 180^\circ = 1.$$

$$\sin^2 180^\circ + 1 = 1.$$

$$\sin 180^\circ = 0.$$

$$\tan 180^\circ = \frac{\sin 180^\circ}{\cos 180^\circ} = \frac{0}{-1} = -0.$$

$$\cot 180^\circ = \frac{\cos 180^\circ}{\sin 180^\circ} = \frac{-1}{0} = -\infty.$$

$$\sec 180^\circ = \frac{1}{\cos 180^\circ} = \frac{1}{-1} = -1.$$

$$\csc 180^\circ = \frac{1}{\sin 180^\circ} = \frac{1}{0} = \infty.$$

(iii)

$$\cot 270^\circ = 0.$$

$$\tan 270^\circ = \frac{1}{0} = \infty.$$

$$\frac{\cos 270^\circ}{\sin 270^\circ} = 0.$$

$$\cos 270^\circ = 0 \sin 270^\circ = 0.$$

$$\sin^2 270^\circ + \cos^2 270^\circ = 1.$$

$$\sin^2 270^\circ + 0 = 1.$$

$$\sin^2 270^\circ = 1.$$

$$\sin 270^\circ = -1.$$

$$\sec 270^\circ = \frac{1}{\cos 270^\circ} = \frac{1}{0} = \infty.$$

$$\csc 270^\circ = \frac{1}{\sin 270^\circ} = \frac{1}{-1} = -1.$$

(iv)

$$\csc 360^\circ = -\infty.$$

$$\sin 360^\circ = \frac{1}{-\infty} = -0.$$

$$\sin^2 360^\circ + \cos^2 360^\circ = 1.$$

$$\cos^2 360^\circ = 1.$$

$\cos 360° = 1.$

$\tan 360° = \frac{-0}{1} = -0.$

$\cot 360° = \frac{1}{-0} = -\infty.$

$\sec 360° = \frac{1}{\cos 360°} = \frac{1}{1} = 1.$

19. Find the values of sin 450°, tan 540°, cos 630°, cot 720°, sin 810°, csc 900°.

$$\begin{aligned}\sin 450° &= \sin(360° + 90°)\\ &= \sin 90°\\ &= 1.\end{aligned}$$

$$\begin{aligned}\tan 540° &= \tan(360° + 180°)\\ &= \tan 180°\\ &= 0.\end{aligned}$$

$$\begin{aligned}\cos 630° &= \cos(360° + 270°)\\ &= \cos 270°\\ &= 0.\end{aligned}$$

$$\begin{aligned}\cot 720° &= \cot(360° + 360°)\\ &= \cot 360°\\ &= \infty.\end{aligned}$$

$$\begin{aligned}\sin 810° &= \sin(2 \times 360° + 90°)\\ &= \sin 90°\\ &= 1.\end{aligned}$$

$$\begin{aligned}\csc 900° &= \csc(2 \times 360° + 180°)\\ &= \csc 180°\\ &= \infty.\end{aligned}$$

20. Compute the value of $a \sin 0° + b \cos 90° - c \tan 180°$.

$\sin 0° = 0.$
$\cos 90° = 0.$
$\tan 180° = 0.$

Substituting,
$a \times 0 + b \times 0 - c \times 0 = 0.$

21. Compute the value of $a \cos 90° - b \tan 180° + c \cot 90°$.

$\cos 90° = 0.$
$\tan 180° = 0.$
$\cot 90° = 0.$

Substituting,
$a \times 0 - b \times 0 + c \times 0 = 0.$

22. Compute the value of $a \sin 90° - b \cos 360° + (a - b) \cos 180°$.

$\sin 90° = 1.$
$\cos 360° = 1.$
$\cos 180° = -1.$

Substituting,
$a \times 1 - b \times 1 + (a - b) \times (-1) = 0$

23. Compute the value of $(a^2 - b^2) \cos 360° - 4ab \sin 270°$.

$\cos 360° = 1.$
$\sin 270° = -1.$

Substituting,
$$\begin{aligned}(a^2 - b^2) \times 1 - 4ab \times (-1)\\ = a^2 - b^2 + 4ab.\end{aligned}$$

EXERCISE XIII. PAGE 51.

1. Express sin 250° in terms of the functions of an acute angle less than 45°.

$$\begin{aligned}\sin 250° &= \sin(270° - 20°)\\ &= -\cos 20°.\end{aligned}$$

2. Express sin 172° in terms of the functions of an angle less than 45°.

$$\begin{aligned}\sin 172° &= \sin(180° - 8°)\\ &= \sin 8°.\end{aligned}$$

3. Express cos 100° in terms of the functions of an angle less than 45°.

$$\cos 100° = \cos(90° + 10°)$$
$$= -\sin 10°.$$

4. Express tan 125° in terms of the functions of an angle less than 45°.

$$\tan 125° = \tan(90° + 35°)$$
$$= -\cot 35°.$$

5. Express cot 91° in terms of the functions of an angle less than 45°.

$$\cot 91° = \cot(90° + 1°)$$
$$= -\tan 1°.$$

6. Express sec 110° in terms of the functions of an angle less than 45°.

$$\sec 110° = \sec(90° + 20°)$$
$$= -\csc 20°.$$

7. Express csc 157° in terms of the functions of an angle less than 45°.

$$\csc 157° = \csc(180° - 23°)$$
$$= \csc 23°.$$

8. Express sin 204° in terms of the functions of an angle less than 45°.

$$\sin 204° = \sin(180° + 24°)$$
$$= -\sin 24°.$$

9. Express cos 359° in terms of the functions of an angle less than 45°.

$$\cos 359° = \cos(360° - 1°)$$
$$= \cos 1°.$$

10. Express tan 300° in terms of the functions of an angle less than 45°.

$$\tan 300° = \tan(270° + 30°)$$
$$= -\cot 30°.$$

11. Express cot 264° in terms of the functions of an angle less than 45°.

$$\cot 264° = \cot(270° - 6°)$$
$$= \tan 6°.$$

12. Express sec 244° in terms of the functions of an angle less than 45°.

$$\sec 244° = \sec(270° - 26°)$$
$$= -\csc 26°.$$

13. Express csc 271° in terms of the functions of an angle less than 45°.

$$\csc 271° = \csc(270° + 1°)$$
$$= -\sec 1°.$$

14. Express sin 163° 49′ in terms of the functions of an angle less than 45°.

$$\sin 163° 49' = \sin(180° - 16° 11')$$
$$= \sin 16° 11'.$$

15. Express cos 195° 33′ in terms of the functions of an angle less than 45°.

$$\cos 195° 33' = \cos(180° + 15° 33')$$
$$= -\cos 15° 33'.$$

16. Express tan 269° 15′ in terms of the functions of an angle less than 45°.

$$\tan 269° 15' = \tan(270° - 45')$$
$$= \cot 45'.$$

17. Express cot 139° 17′ in terms of the functions of an angle less than 45°.

$$\cot 139° 17' = \cot(180° - 40° 43')$$
$$= -\cot 40° 43'.$$

18. Express sec 299° 45′ in terms of the functions of an angle less than 45°.

$$\sec 299° 45' = \sec(270° + 29° 45')$$
$$= \csc 29° 45'.$$

19. Express csc 92° 25′ in terms of the functions of an angle less than 45°.

$$\csc 92^\circ 25' = \csc (90^\circ + 2^\circ 25') = \sec 2^\circ 25'.$$

20. Express all the functions of − 75° in terms of the functions of a positive angle less than 45°.

$$\sin (-75^\circ) = \sin (270^\circ + 15^\circ) = -\cos 15^\circ.$$
$$\cos (-75^\circ) = \cos (270^\circ + 15^\circ) = \sin 15^\circ.$$
$$\tan (-75^\circ) = \tan (270^\circ + 15^\circ) = -\cot 15^\circ.$$
$$\cot (-75^\circ) = \cot (270^\circ + 15^\circ) = -\tan 15^\circ.$$

21. Express all the functions of − 127° in terms of the functions of a positive angle less than 45°.

$$\sin (-127^\circ) = \sin (270^\circ - 37^\circ) = -\cos 37^\circ.$$
$$\cos (-127^\circ) = \cos (270^\circ - 37^\circ) = -\sin 37^\circ.$$
$$\tan (-127^\circ) = \tan (270^\circ - 37^\circ) = \cot 37^\circ.$$
$$\cot (-127^\circ) = \cot (270^\circ - 37^\circ) = \tan 37^\circ.$$

22. Express all the functions of − 200° in terms of the functions of a positive angle less than 45°.

$$\sin (-200^\circ) = \sin (180^\circ - 20^\circ) = \sin 20^\circ.$$
$$\cos (-200^\circ) = \cos (180^\circ - 20^\circ) = -\cos 20^\circ.$$
$$\tan (-200^\circ) = \tan (180^\circ - 20^\circ) = -\tan 20^\circ.$$
$$\cot (-200^\circ) = \cot (180^\circ - 20^\circ) = -\cot 20^\circ.$$

23. Express all the functions of − 345° in terms of the functions of a positive angle less than 45°.

$$\sin (-345^\circ) = \sin 15^\circ.$$
$$\cos (-345^\circ) = \cos 15^\circ.$$
$$\tan (-345^\circ) = \tan 15^\circ.$$
$$\cot (-345^\circ) = \cot 15^\circ.$$

24. Express all the functions of − 52° 37′ in terms of the functions of a positive angle less than 45°.

$$\sin (-52^\circ 37') = \sin (270^\circ + 37^\circ 23') = -\cos 37^\circ 23'.$$
$$\cos (-52^\circ 37') = \cos (270^\circ + 37^\circ 23') = \sin 37^\circ 23'.$$
$$\tan (-52^\circ 37') = \tan (270^\circ + 37^\circ 23') = -\cot 37^\circ 23'.$$
$$\cot (-52^\circ 37') = \cot (270^\circ + 37^\circ 23') = -\tan 37^\circ 23'.$$

25. Express all the functions of − 196° 54′ in terms of the functions of a positive angle less than 45°.

$\sin(-196^\circ 54') = \sin(180^\circ - 16^\circ 54') = \sin 16^\circ 54'.$
$\cos(-196^\circ 54') = \cos(180^\circ - 16^\circ 54') = -\cos 16^\circ 54'.$
$\tan(-196^\circ 54') = \tan(180^\circ - 16^\circ 54') = -\tan 16^\circ 54'.$
$\cot(-196^\circ 54') = \cot(180^\circ - 16^\circ 54') = -\cot 16^\circ 54'.$

26. Find the functions of 120°.

$\sin 120^\circ = \sin(90^\circ + 30^\circ) = \cos 30^\circ = \frac{1}{2}\sqrt{3}.$
$\cos 120^\circ = \cos(90^\circ + 30^\circ) = -\sin 30^\circ = -\frac{1}{2}.$
$\tan 120^\circ = \tan(90^\circ + 30^\circ) = -\cot 30^\circ = -\sqrt{3}.$
$\cot 120^\circ = \cot(90^\circ + 30^\circ) = -\tan 30^\circ = -\frac{1}{3}\sqrt{3}.$

27. Find the functions of 135°.

$\sin 135^\circ = \sin(90^\circ + 45^\circ) = \cos 45^\circ = \frac{1}{2}\sqrt{2}.$
$\cos 135^\circ = \cos(90^\circ + 45^\circ) = -\sin 45^\circ = -\frac{1}{2}\sqrt{2}.$
$\tan 135^\circ = \tan(90^\circ + 45^\circ) = -\cot 45^\circ = -1.$
$\cot 135^\circ = \cot(90^\circ + 45^\circ) = -\tan 45^\circ = -1.$

28. Find the functions of 150°.

$\sin 150^\circ = \sin(180^\circ - 30^\circ) = \sin 30^\circ = \frac{1}{2}.$
$\cos 150^\circ = \cos(180^\circ - 30^\circ) = -\cos 30^\circ = -\frac{1}{2}\sqrt{3}.$
$\tan 150^\circ = \tan(180^\circ - 30^\circ) = -\tan 30^\circ = -\frac{1}{3}\sqrt{3}.$
$\cot 150^\circ = \cot(180^\circ - 30^\circ) = -\cot 30^\circ = -\sqrt{3}.$

29. Find the functions of 210°.

$\sin 210^\circ = \sin(180^\circ + 30^\circ) = -\sin 30^\circ = -\frac{1}{2}.$
$\cos 210^\circ = \cos(180^\circ + 30^\circ) = -\cos 30^\circ = -\frac{1}{2}\sqrt{3}.$
$\tan 210^\circ = \tan(180^\circ + 30^\circ) = \tan 30^\circ = \frac{1}{3}\sqrt{3}.$
$\cot 210^\circ = \cot(180^\circ + 30^\circ) = \cot 30^\circ = \sqrt{3}.$

30. Find the functions of 225°.

$\sin 225^\circ = \sin(180^\circ + 45^\circ) = -\sin 45^\circ = -\frac{1}{2}\sqrt{2}.$
$\cos 225^\circ = \cos(180^\circ + 45^\circ) = -\cos 45^\circ = -\frac{1}{2}\sqrt{2}.$
$\tan 225^\circ = \tan(180^\circ + 45^\circ) = \tan 45^\circ = 1.$
$\cot 225^\circ = \cot(180^\circ + 45^\circ) = \cot 45^\circ = 1.$

31. Find the functions of 240°.

$\sin 240^\circ = \sin(270^\circ - 30^\circ) = -\cos 30^\circ = -\frac{1}{2}\sqrt{3}.$
$\cos 240^\circ = \cos(270^\circ - 30^\circ) = -\sin 30^\circ = -\frac{1}{2}.$
$\tan 240^\circ = \tan(270^\circ - 30^\circ) = \cot 30^\circ = \sqrt{3}.$
$\cot 240^\circ = \cot(270^\circ - 30^\circ) = \tan 30^\circ = \frac{1}{3}\sqrt{3}.$

32. Find the functions of 300°.

$$\begin{aligned}
\sin 300^\circ &= \sin\ (270^\circ + 30^\circ) = -\cos 30^\circ = -\tfrac{1}{2}\sqrt{3}.\\
\cos 300^\circ &= \cos\ (270^\circ + 30^\circ) = \sin 30^\circ = \tfrac{1}{2}.\\
\tan 300^\circ &= \tan\ (270^\circ + 30^\circ) = -\cot 30^\circ = -\sqrt{3}.\\
\cot 300^\circ &= \cot\ (270^\circ + 30^\circ) = -\tan 30^\circ = -\tfrac{1}{3}\sqrt{3}.
\end{aligned}$$

33. Find the functions of -30°.

$$\begin{aligned}
\sin(-30^\circ) &= -\sin 30^\circ = -\tfrac{1}{2}.\\
\cos(-30^\circ) &= \cos 30^\circ = \tfrac{1}{2}\sqrt{3}.\\
\tan(-30^\circ) &= -\tan 30^\circ = -\tfrac{1}{3}\sqrt{3}.\\
\cot(-30^\circ) &= -\cot 30^\circ = -\sqrt{3}.
\end{aligned}$$

34. Find the functions of -225°.

$$\begin{aligned}
-225^\circ &= 90^\circ + 45^\circ.\\
\sin(-225^\circ) &= \sin\ (90^\circ + 45^\circ) = \cos 45^\circ = \tfrac{1}{2}\sqrt{2}.\\
\cos(-225^\circ) &= \cos\ (90^\circ + 45^\circ) = -\sin\ 45^\circ = -\tfrac{1}{2}\sqrt{2}.\\
\tan(-225^\circ) &= \tan\ (90^\circ + 45^\circ) = -\cot 45^\circ = -1.\\
\cot(-225^\circ) &= \cot\ (90^\circ + 45^\circ) = -\tan 45^\circ = -1.
\end{aligned}$$

35. Given $\sin x = -\sqrt{\tfrac{1}{2}}$, and $\cos x$ negative; find the other functions of x, and the value of x.

Since $\sin 45^\circ = -\sqrt{\tfrac{1}{2}}$ and the sign of the cosine is negative, the angle must be in Quadrant III, and must be, therefore,

$$\begin{aligned}
180^\circ + 45^\circ &= 225^\circ.\\
\cos\ (180^\circ + 45^\circ) &= -\cos 45^\circ = -\tfrac{1}{2}\sqrt{2}.\\
\tan\ (180^\circ + 45^\circ) &= \tan 45^\circ = 1.\\
\cot\ (180^\circ + 45^\circ) &= \cot 45^\circ = 1.
\end{aligned}$$

36. Given $\cot x = -\sqrt{3}$, and x in Quadrant II; find the other functions of x, and the value of x.

Since $\cot 30^\circ = \sqrt{3}$ and the sign is negative, the angle is $180^\circ - 30^\circ = 150^\circ$.

$$\begin{aligned}
\sin 150^\circ &= \sin\ (180^\circ - 30^\circ) = \sin 30^\circ = \tfrac{1}{2}.\\
\cos 150^\circ &= \cos(180^\circ - 30^\circ) = -\cos 30^\circ = -\tfrac{1}{2}\sqrt{3}.\\
\tan 150^\circ &= \tan(180^\circ - 30^\circ) = -\tan 30^\circ = -\tfrac{1}{3}\sqrt{3}.
\end{aligned}$$

37. Find the functions of 3540°.

$$\begin{aligned}
3540^\circ &= 9 \times 360^\circ + 300^\circ.\\
\sin 300^\circ &= \sin\ (270^\circ + 30^\circ) = -\cos 30^\circ = -\tfrac{1}{2}\sqrt{3}.\\
\cos 300^\circ &= \cos\ (270^\circ + 30^\circ) = \sin 30^\circ = \tfrac{1}{2}.\\
\tan 300^\circ &= \tan(270^\circ + 30^\circ) = -\cot 30^\circ = -\sqrt{3}.\\
\cot 300^\circ &= \cot\ (270^\circ + 30^\circ) = -\tan 30^\circ = -\tfrac{1}{3}\sqrt{3}.
\end{aligned}$$

38. What angles less than 360° have a sine equal to $-\frac{1}{2}$? a tangent equal to $-\sqrt{3}$?

(i) Since $\sin 30° = \frac{1}{2}$ and the sign is negative, the angle must be in Quadrant III or IV, and must be, therefore, $180° + 30° = 210°$, or $360° - 30° = 330°$.

(ii) Since $\tan 60° = \sqrt{3}$ and the sign is negative, the angle must be in Quadrant II or IV, and must be, therefore, $180° - 60° = 120°$, or $360° - 60° = 300°$.

39. Which of the angles mentioned in Examples 27–34 have a cosine equal to $-\frac{1}{2}\sqrt{2}$? a cotangent equal to $-\sqrt{3}$?

(i) Since $\cos 45° = \frac{1}{2}\sqrt{2}$ and the sign is negative, the angle must be in Quadrant II or III, and must be, therefore, $180° - 45° = 135°$, or $180° + 45° = 225°$. Also, the functions of $-225°$ are the same as the functions of $360° - 225° = 135°$. Hence, the angles are 135°, 225°, or $-225°$.

(ii) Since $\cot 30° = \sqrt{3}$ and the sign is negative, the angle must be in Quadrant II or IV, and must be, therefore, $180° - 30° = 150°$, or $360° - 30° = 330°$, or $-30°$. Hence, the angles are 150° or $-30°$.

40. What values of x between 0° and 720° will satisfy the equation $\sin x = +\frac{1}{2}$?

Since $\sin 30° = \frac{1}{2}$ and the sign is positive, the angle must be in Quadrant I or II, and must be, therefore, 30° or $180° - 30° = 150°$, the first revolution. In the second revolution these angles must be increased by 360°. Hence, the angles are 30°, 150°, 390°, and 510°.

41. Find the other angle between 0° and 360° for which the corresponding function (sign included) has the same value as sin 12°, cos 26°, tan 45°, cot 72°; sin 191°, cos 120°, tan 244°, cot 357°.

In order that the sign shall be the same,

sin 12° must be in Quadrant II $= \sin(180° - 12°) = \sin 168°$.
cos 26° must be in Quadrant IV $= \cos(360° - 26°) = \cos 334°$.
tan 45° must be in Quadrant III $= \tan(180° + 45°) = \tan 225°$.
cot 72° must be in Quadrant III $= \cot(180° + 72°) = \cot 252°$.
sin 191° must be in Quadrant IV $= \sin(360° - 11°) = \sin 349°$.
cos 120° must be in Quadrant III $= \cos(180° + 60°) = \cos 240°$.
tan 244° must be in Quadrant I $= \tan(244° - 180°) = \tan 64°$.
cot 357° must be in Quadrant II $= \cot(357° - 180°) = \cot 177°$.

42. Given $\tan 238° = 1.6$; find $\sin 122°$.

$$\tan 238° = \tan(180° + 58°) = \tan 58°.$$
$$\sin 122° = \sin(180° - 58°) = \sin 58°.$$

But
$$\tan 238° = 1.6.$$
$$\therefore \tan 58° = 1.6.$$
$$\tan 58° = \frac{\sin 58°}{\cos 58°}.$$
$$1.6 = \frac{\sin 58°}{\sqrt{1 - \sin^2 58°}}.$$
$$2.56 - 2.56 \sin^2 58° = \sin^2 58°.$$
$$3.56 \sin^2 58° = 2.56.$$
$$\sin 58° = \sqrt{\frac{2.56}{3.56}} = 0.8480.$$

43. Given $\cos 333° = 0.89$; find $\tan 117°$.

$$\cos 333° = 0.89 = \cos(270° + 63°) = \sin 63°.$$
$$\tan 117° = \tan(180° - 63°) = -\tan 63°.$$
$$\sin^2 63° + \cos^2 63° = 1.$$
$$(0.89)^2 + \cos^2 63° = 1.$$
$$\cos^2 63° = 0.2079.$$
$$\cos 63° = 0.4559.$$
$$-\tan 63° = -\frac{\sin 63°}{\cos 63°} = -\frac{0.89}{0.4559} = -1.9522.$$

44. Simplify the expression

$$a \cos(90° - x) + b \cos(90° + x).$$
$$a \cos(90° - x) + b \cos(90° + x)$$
$$= a \sin x + b(-\sin x)$$
$$= (a - b) \sin x.$$

45. Simplify the expression

$$m \cos(90° - x) \sin(90° - x).$$
$$m \cos(90° - x) \sin(90° - x)$$
$$= m \sin x \cos x.$$

46. Simplify the expression

$$(a - b) \tan(90° - x) + (a + b) \cot(90° + x).$$
$$\tan(90° - x) = \cot x.$$
$$\cot(90° + x) = -\tan x.$$

Hence, the expression $= (a - b) \cot x - (a + b) \tan x$.

47. Simplify the expression

$$a^2 + b^2 - 2ab \cos(180° - x).$$
$$a^2 + b^2 - 2ab \cos(180° - x) = a^2 + b^2 - 2ab(-\cos x)$$
$$= a^2 + b^2 + 2ab \cos x.$$

48. Simplify the expression

$$\sin(90° + x) \sin(180° + x) + \cos(90° + x) \cos(180° - x).$$
$$\sin(90° + x) \sin(180° + x) + \cos(90° + x) \cos(180° - x)$$
$$= (\cos x)(-\sin x) + (-\sin x)(-\cos x)$$
$$= -\sin x \cos x + \sin x \cos x = 0.$$

49. Simplify the expression

$$\cos(180^\circ + x)\cos(270^\circ - y) - \sin(180^\circ + x)\sin(270^\circ - y).$$

$$\begin{aligned}\cos(180^\circ + x) &= -\cos x.\\ \cos(270^\circ - y) &= -\sin y.\\ \sin(180^\circ + x) &= -\sin x.\\ \sin(270^\circ - y) &= -\cos y.\end{aligned}$$

Hence, the expression $= \cos x \sin y - \sin x \cos y$.

50. Simplify the expression

$$\tan x + \tan(-y) - \tan(180^\circ - y).$$

$$\begin{aligned}\tan(-y) &= -\tan y.\\ -\tan(180^\circ - y) &= \tan y.\end{aligned}$$

Hence, the expression $= \tan x$.

51. For what values of x is the expression $\sin x + \cos x$ positive, and for what values negative?

If x is in Quadrant I, $\sin x + \cos x$ must be positive, since both the sine and cosine are positive. In Quadrant II the sine is positive and cosine negative; hence, so long as the sine is greater than, or equal to, the cosine, the expression $\sin x + \cos x$ is positive; but after passing the middle of Quadrant II, viz., 135°, the cosine of x is greater than sine, and the expression is negative. In Quadrant III both sine and cosine are negative, and hence their sum must be negative. In Quadrant IV the sine is negative and cosine positive. The sine and cosine are equal at 315°, after which the cosine is greater than sine. Hence, the expression $\sin x + \cos x$ is negative from 135° to 315°, and positive between 0° and 135°, and 315° and 360°.

52. Answer the questions of Example 51 for $\sin x - \cos x$.

As x increases from 0° to 45°, the sine increases in value, and cosine decreases, until at 45° sine = cosine. Hence, up to this point $\sin x - \cos x$ is negative. For the remainder of Quadrant I sine is greater than cosine, and consequently the expression $\sin x - \cos x$ is positive. In Quadrant II sine is positive and cosine negative, so the expression $\sin x - \cos x$ is uniformly positive. In Quadrant III sine is negative and cosine negative; hence, so long as sine is less than cosine, the expression is positive, viz., to 225°; after that point, sine is greater than cosine, and $\sin x - \cos x$ is negative. In Quadrant IV sine is negative and cosine positive; therefore, $\sin x - \cos x$ is uniformly negative. The expression is, then, negative between 0° and 45°, and 225° and 360°; positive between 45° and 225°.

53. Find the functions of $x - 90°$ in functions of x.

$$x - 90° = 360° - (90° - x) = 270° + x.$$
$$\sin (x - 90°) = \sin (270° + x) = - \cos x.$$
$$\cos (x - 90°) = \cos (270° + x) = \sin x.$$
$$\tan (x - 90°) = \tan (270° + x) = - \cot x.$$
$$\cot (x - 90°) = \cot (270° + x) = - \tan x.$$

54. Find the functions of $x - 180°$ in functions of x.

$$x - 180° = 360° - (180° - x) = 180° + x.$$
$$\sin (x - 180°) = \sin (180° + x) = - \sin x.$$
$$\cos (x - 180°) = \cos (180° + x) = - \cos x.$$
$$\tan (x - 180°) = \tan (180° + x) = \tan x.$$
$$\cot (x - 180°) = \cot (180° + x) = \cot x.$$

EXERCISE XIV. PAGE 60.

1. Find the value of $\sin (x + y)$ and $\cos (x + y)$ when $\sin x = \frac{3}{5}$, $\cos x = \frac{4}{5}$, $\sin y = \frac{5}{13}$, $\cos y = \frac{12}{13}$.

$$\sin (x + y) = \sin x \cos y + \cos x \sin y$$
$$= \left(\frac{3}{5} \times \frac{12}{13}\right) + \left(\frac{4}{5} \times \frac{5}{13}\right) = \frac{36}{65} + \frac{20}{65} = \frac{56}{65}.$$
$$\cos (x + y) = \cos x \cos y - \sin x \sin y$$
$$= \left(\frac{4}{5} \times \frac{12}{13}\right) - \left(\frac{3}{5} \times \frac{5}{13}\right) = \frac{48}{65} - \frac{15}{65} = \frac{33}{65}.$$

2. Find $\sin (90° - y)$ and $\cos (90° - y)$ by making $x = 90°$ in Formulas [8] and [9].

$$\sin (90° - y) = \sin 90° \cos y - \cos 90° \sin y$$
$$= (1 \times \cos y) - (0 \times \sin y) = \cos y.$$
$$\cos (90° - y) = \cos 90° \cos y + \sin 90° \sin y$$
$$= (0 \times \cos y) + (1 \times \sin y) = \sin y.$$

3. Find, by Formulas [4]–[11], the first four functions of $90° + y$.

$$\sin (90° + y) = \sin 90° \cos y + \cos 90° \sin y$$
$$= (1 \times \cos y) + (0 \times \sin y) = \cos y.$$
$$\cos (90° + y) = \cos 90° \cos y - \sin 90° \sin y$$
$$= (0 \times \cos y) - (1 \times \sin y) = - \sin y.$$
$$\tan (90° + y) = - \frac{\cos y}{\sin y} = - \cot y.$$
$$\cot (90° + y) = - \frac{\sin y}{\cos y} = - \tan y.$$

4. **Find, by Formulas [4]–[11], the first four functions of 180° – y.**

$$\begin{aligned}\sin(180^\circ - y) &= \sin 180^\circ \cos y - \cos 180^\circ \sin y\\ &= (0 \times \cos y) - (-1 \times \sin y) = \sin y.\\ \cos(180^\circ - y) &= \cos 180^\circ \cos y + \sin 180^\circ \sin y\\ &= (-1 \times \cos y) + (0 \times \sin y) = -\cos y.\\ \tan(180^\circ - y) &= -\frac{\sin y}{\cos y} = -\tan y.\\ \cot(180^\circ - y) &= -\frac{\cos y}{\sin y} = -\cot y.\end{aligned}$$

5. **Find, by Formulas [4]–[11], the first four functions of 180° + y.**

$$\begin{aligned}\sin(180^\circ + y) &= \sin 180^\circ \cos y + \cos 180^\circ \sin y\\ &= (0 \times \cos y) + (-1 \times \sin y) = -\sin y.\\ \cos(180^\circ + y) &= \cos 180^\circ \cos y - \sin 180^\circ \sin y\\ &= (-1 \times \cos y) - (0 \times \sin y) = -\cos y.\\ \tan(180^\circ + y) &= \frac{-\sin y}{-\cos y} = \tan y.\\ \cot(180^\circ + y) &= \frac{-\cos y}{-\sin y} = \cot y.\end{aligned}$$

6. **Find, by Formulas [4]–[11], the first four functions of 270° – y.**

$$\begin{aligned}\sin(270^\circ - y) &= \sin 270^\circ \cos y - \cos 270^\circ \sin y\\ &= (-1 \times \cos y) - (0 \times \sin y) = -\cos y.\\ \cos(270^\circ - y) &= \cos 270^\circ \cos y + \sin 270^\circ \sin y\\ &= (0 \times \cos y) + (-1 \times \sin y) = -\sin y.\\ \tan(270^\circ - y) &= \frac{-\cos y}{-\sin y} = \cot y.\\ \cot(270^\circ - y) &= \frac{-\sin y}{-\cos y} = \tan y.\end{aligned}$$

7. **Find, by Formulas [4]–[11], the first four functions of 270° + y.**

$$\begin{aligned}\sin(270^\circ + y) &= \sin 270^\circ \cos y + \cos 270^\circ \sin y\\ &= (-1 \times \cos y) + (0 \times \sin y) = -\cos y.\\ \cos(270^\circ + y) &= \cos 270^\circ \cos y - \sin 270^\circ \sin y\\ &= (0 \times \cos y) - (-1 \times \sin y) = \sin y.\\ \tan(270^\circ + y) &= \frac{-\cos y}{\sin y} = -\cot y.\\ \cot(270^\circ + y) &= \frac{\sin y}{-\cos y} = -\tan y.\end{aligned}$$

8. Find, by Formulas [4]–[11], the first four functions of $360^\circ - y$.

$$\begin{aligned}
\sin(360^\circ - y) &= \sin 360^\circ \cos y - \cos 360^\circ \sin y \\
&= (0 \times \cos y) - (1 \times \sin y) = -\sin y. \\
\cos(360^\circ - y) &= \cos 360^\circ \cos y + \sin 360^\circ \sin y \\
&= (1 \times \cos y) + (0 \times \sin y) = \cos y. \\
\tan(360^\circ - y) &= \frac{-\sin y}{\cos y} = -\tan y. \\
\cot(360^\circ - y) &= \frac{\cos y}{-\sin y} = -\cot y.
\end{aligned}$$

9. Find, by Formulas [4]–[11], the first four functions of $360^\circ + y$.

$$\begin{aligned}
\sin(360^\circ + y) &= \sin 360^\circ \cos y + \cos 360^\circ \sin y \\
&= (0 \times \cos y) + (1 \times \sin y) = \sin y. \\
\cos(360^\circ + y) &= \cos 360^\circ \cos y - \sin 360^\circ \sin y \\
&= (1 \times \cos y) - (0 \times \sin y) = \cos y. \\
\tan(360^\circ + y) &= \frac{\sin y}{\cos y} = \tan y. \\
\cot(360^\circ + y) &= \frac{\cos y}{\sin y} = \cot y.
\end{aligned}$$

10. Find, by Formulas [4]–[11], the first four functions of $x - 90^\circ$.

$$\begin{aligned}
\sin(x - 90^\circ) &= \sin x \cos 90^\circ - \cos x \sin 90^\circ \\
&= (0 \times \sin x) - (1 \times \cos x) = -\cos x. \\
\cos(x - 90^\circ) &= \cos x \cos 90^\circ + \sin x \sin 90^\circ \\
&= (0 \times \cos x) + (1 \times \sin x) = \sin x. \\
\tan(x - 90^\circ) &= \frac{-\cos x}{\sin x} = -\cot x. \\
\cot(x - 90^\circ) &= \frac{\sin x}{-\cos x} = -\tan x.
\end{aligned}$$

11. Find, by Formulas [4]–[11], the first four functions of $x - 180^\circ$.

$$\begin{aligned}
\sin(x - 180^\circ) &= \sin x \cos 180^\circ - \cos x \sin 180^\circ \\
&= \sin x(-1) - \cos x \times 0 = -\sin x. \\
\cos(x - 180^\circ) &= \cos x \cos 180^\circ + \sin x \sin 180^\circ \\
&= \cos x(-1) + \sin x \times 0 = -\cos x. \\
\tan(x - 180^\circ) &= \frac{-\sin x}{-\cos x} = \tan x. \\
\cot(x - 180^\circ) &= \frac{-\cos x}{-\sin x} = \cot x.
\end{aligned}$$

12. Find, by Formulas [4]–[11], the first four functions of $x - 270°$.

$$\begin{aligned}\sin(x - 270°) &= \sin x \cos 270° - \cos x \sin 270° \\ &= \sin x \times 0 - \cos x \times (-1) = \cos x. \\ \cos(x - 270°) &= \cos x \cos 270° + \sin x \sin 270° \\ &= \cos x \times 0 + \sin x(-1) = -\sin x. \\ \tan(x - 270°) &= \frac{\cos x}{-\sin x} = -\cot x. \\ \cot(x - 270°) &= \frac{-\sin x}{\cos x} = -\tan x.\end{aligned}$$

13. Find, by Formulas [4]–[11], the first four functions of $-y$.

$$\begin{aligned}\sin(0° - y) &= \sin 0° \cos y - \cos 0° \sin y \\ &= (0 \times \cos y) - (1 \times \sin y) = -\sin y. \\ \cos(0° - y) &= \cos 0° \cos y + \sin 0° \sin y \\ &= (1 \times \cos y) + (0 \times \sin y) = \cos y. \\ \tan(0° - y) &= \frac{-\sin y}{\cos y} = -\tan y. \\ \cot(0° - y) &= \frac{\cos y}{-\sin y} = -\cot y.\end{aligned}$$

14. Find, by Formulas [4]–[11], the first four functions of $45° - y$.

$$\begin{aligned}\sin(45° - y) &= \sin 45° \cos y - \cos 45° \sin y \\ &= \tfrac{1}{2}\sqrt{2}\cos y - \tfrac{1}{2}\sqrt{2}\sin y = \tfrac{1}{2}\sqrt{2}(\cos y - \sin y). \\ \cos(45° - y) &= \cos 45° \cos y + \sin 45° \sin y \\ &= \tfrac{1}{2}\sqrt{2}\cos y + \tfrac{1}{2}\sqrt{2}\sin y = \tfrac{1}{2}\sqrt{2}(\cos y + \sin y). \\ \tan(45° - y) &= \frac{\cos y - \sin y}{\cos y + \sin y} = \frac{1 - \tan y}{1 + \tan y}. \\ \cot(45° - y) &= \frac{\cos y + \sin y}{\cos y - \sin y} = \frac{\cot y + 1}{\cot y - 1}.\end{aligned}$$

15. Find, by Formulas [4]–[11], the first four functions of $45° + y$.

$$\begin{aligned}\sin(45° + y) &= \sin 45° \cos y + \cos 45° \sin y \\ &= \tfrac{1}{2}\sqrt{2}\cos y + \tfrac{1}{2}\sqrt{2}\sin y = \tfrac{1}{2}\sqrt{2}(\cos y + \sin y). \\ \cos(45° + y) &= \cos 45° \cos y - \sin 45° \sin y \\ &= \tfrac{1}{2}\sqrt{2}\cos y - \tfrac{1}{2}\sqrt{2}\sin y = \tfrac{1}{2}\sqrt{2}(\cos y - \sin y). \\ \tan(45° + y) &= \frac{\cos y + \sin y}{\cos y - \sin y} = \frac{1 + \tan y}{1 - \tan y}. \\ \cot(45° + y) &= \frac{\cos y - \sin y}{\cos y + \sin y} = \frac{\cot y - 1}{\cot y + 1}.\end{aligned}$$

16. Find, by Formulas [4]–[11], the first four functions of $30^\circ + y$.

$$\sin(30^\circ + y) = \sin 30^\circ \cos y + \cos 30^\circ \sin y = \tfrac{1}{2}(\cos y + \sqrt{3}\sin y).$$
$$\cos(30^\circ + y) = \cos 30^\circ \cos y - \sin 30^\circ \sin y = \tfrac{1}{2}(\sqrt{3}\cos y - \sin y).$$
$$\tan(30^\circ + y) = \frac{\cos y + \sqrt{3}\sin y}{\sqrt{3}\cos y - \sin y};\text{ divide each term by } \sqrt{3}\cos y,$$
$$= \frac{\tfrac{1}{3}\sqrt{3} + \tan y}{1 - \tfrac{1}{3}\sqrt{3}\tan y}.$$
$$\cot(30^\circ + y) = \frac{\sqrt{3}\cos y - \sin y}{\cos y + \sqrt{3}\sin y};\text{ divide each term by } \sin y,$$
$$= \frac{\sqrt{3}\cot y - 1}{\cot y + \sqrt{3}}.$$

17. Find, by Formulas [4]–[11], the first four functions of $60^\circ - y$.

$$\sin(60^\circ - y) = \sin 60^\circ \cos y - \cos 60^\circ \sin y = \tfrac{1}{2}(\sqrt{3}\cos y - \sin y).$$
$$\cos(60^\circ - y) = \cos 60^\circ \cos y + \sin 60^\circ \sin y = \tfrac{1}{2}(\cos y + \sqrt{3}\sin y).$$
$$\tan(60^\circ - y) = \frac{\sqrt{3}\cos y - \sin y}{\cos y + \sqrt{3}\sin y} = \frac{\sqrt{3} - \tan y}{1 + \sqrt{3}\tan y}.$$
$$\cot(60^\circ - y) = \frac{\cos y + \sqrt{3}\sin y}{\sqrt{3}\cos y - \sin y} = \frac{\tfrac{1}{3}\sqrt{3}\cot y + 1}{\cot y - \tfrac{1}{3}\sqrt{3}}.$$

18. Find $\sin 3x$ in terms of $\sin x$.

$$\sin 3x = \sin(2x + x)$$
$$= \sin 2x \cos x + \cos 2x \sin x.$$
$$\sin 2x = 2\sin x \cos x.$$
$$\cos 2x = \cos^2 x - \sin^2 x.$$
$$\therefore \sin 3x$$
$$= 2\sin x \cos^2 x + \sin x \cos^2 x - \sin^3 x$$
$$= 3\sin x \cos^2 x - \sin^3 x.$$

But $\cos^2 x = 1 - \sin^2 x$.

$$\therefore \sin 3x$$
$$= 3\sin x - 3\sin^3 x - \sin^3 x$$
$$= 3\sin x - 4\sin^3 x.$$

19. Find $\cos 3x$ in terms of $\cos x$.

$$\cos 3x = \cos(2x + x)$$
$$= \cos 2x \cos x - \sin 2x \sin x.$$
$$\sin 2x = 2\sin x \cos x.$$
$$\cos 2x = \cos^2 x - \sin^2 x.$$
$$\therefore \cos 3x$$
$$= \cos^3 x - \sin^2 x \cos x - 2\sin^2 x \cos x$$
$$= \cos^3 x - 3\sin^2 x \cos x.$$

But $\sin^2 x = 1 - \cos^2 x$.

$$\therefore \cos 3x$$
$$= \cos^3 x - 3\cos x + 3\cos^3 x$$
$$= 4\cos^3 x - 3\cos x.$$

20. Given $\tan \tfrac{1}{2}x = 1$; find $\cos x$.

$$\tan \tfrac{1}{2}x = \sqrt{\frac{1 - \cos x}{1 + \cos x}}.$$
$$1 = \sqrt{\frac{1 - \cos x}{1 + \cos x}}.$$
$$1 = \frac{1 - \cos x}{1 + \cos x}.$$

$$1+\cos x = 1-\cos x.$$
$$2\cos x = 0.$$
$$\cos x = 0.$$

21. Given $\cot \frac{1}{2}x = \sqrt{3}$; find $\sin x$.

$$\cot \tfrac{1}{2}x = \sqrt{\frac{1+\cos x}{1-\cos x}}.$$
$$\sqrt{3} = \sqrt{\frac{1+\cos x}{1-\cos x}}.$$
$$3 = \frac{1+\cos x}{1-\cos x}.$$
$$3 - 3\cos x = 1 + \cos x.$$
$$-4\cos x = -2.$$
$$\cos x = \frac{1}{2}.$$
$$\sin^2 x = 1 - \cos^2 x$$
$$= 1 - \frac{1}{4} = \frac{3}{4}.$$
$$\sin x = \sqrt{\frac{3}{4}} = \tfrac{1}{2}\sqrt{3}.$$

22. Given $\sin x = 0.2$; find $\sin \frac{1}{2}x$ and $\cos \frac{1}{2}x$.

$$\sin x = 0.2.$$
$$\cos^2 x = 1 - \sin^2 x$$
$$= 1 - 0.04.$$
$$\cos x = \sqrt{0.96}.$$
$$\sin \tfrac{1}{2}x = \sqrt{\frac{1-\cos x}{2}}$$
$$= \sqrt{\frac{1-\sqrt{0.96}}{2}}$$
$$= \sqrt{\frac{1-0.4\sqrt{6}}{2}}$$
$$= 0.10051.$$
$$\cos \tfrac{1}{2}x = \sqrt{\frac{1+\cos x}{2}}$$
$$= \sqrt{\frac{1+0.4\sqrt{6}}{2}}$$
$$= 0.99493.$$

23. Given $\cos x = 0.5$; find $\cos 2x$ and $\tan 2x$.

$$\cos 2x = \cos^2 x - \sin^2 x.$$
$$\sin x = \sqrt{1-0.5^2} = \tfrac{1}{2}\sqrt{3}.$$
$$\therefore \cos 2x = 0.25 - 0.75$$
$$= -0.50 = -\frac{1}{2}.$$
$$\tan x = \frac{\sin x}{\cos x} = \frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}} = \sqrt{3}.$$
$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} = \frac{2\sqrt{3}}{1-3} = -\sqrt{3}.$$

24. Given $\tan 45° = 1$; find the functions of $22° 30'$.

Let $$x = 45°.$$
$$\tan x = \frac{\sin x}{\cos x} = 1.$$
$$\therefore \sin x = \cos x.$$
$$\sin^2 x + \cos^2 x = 1.$$
$$2\sin^2 x = 1.$$
$$\sin^2 x = \tfrac{1}{2}.$$
$$\sin x = \tfrac{1}{2}\sqrt{2} = \cos x.$$

$$\sin 22^\circ\,30' = \sin\tfrac{1}{2}x = \sqrt{\frac{1-\frac{1}{2}\sqrt{2}}{2}} = \tfrac{1}{2}\sqrt{2-\sqrt{2}} = 0.3827.$$

$$\cos 22^\circ\,30' = \cos\tfrac{1}{2}x = \sqrt{\frac{1+\frac{1}{2}\sqrt{2}}{2}} = \tfrac{1}{2}\sqrt{2+\sqrt{2}} = 0.9239.$$

$$\tan 22^\circ\,30' = \tan\tfrac{1}{2}x = \sqrt{\frac{1-\frac{1}{2}\sqrt{2}}{1+\frac{1}{2}\sqrt{2}}} = \sqrt{\frac{(1-\frac{1}{2}\sqrt{2})^2}{1-\frac{1}{2}}} = \sqrt{2\,(1-\tfrac{1}{2}\sqrt{2})^2}$$
$$= (1-\tfrac{1}{2}\sqrt{2})\sqrt{2} = \sqrt{2}-1 = 0.4142.$$

$$\cot 22^\circ\,30' = \cot\tfrac{1}{2}x = \sqrt{\frac{1+\frac{1}{2}\sqrt{2}}{1-\frac{1}{2}\sqrt{2}}} = \sqrt{\frac{(1+\frac{1}{2}\sqrt{2})^2}{1-\frac{1}{2}}} = \sqrt{2\,(1+\tfrac{1}{2}\sqrt{2})^2}$$
$$= (1+\tfrac{1}{2}\sqrt{2})\sqrt{2} = \sqrt{2}+1 = 2.4142.$$

25. Given $\sin 30^\circ = 0.5$; find the functions of 15°.

$$\sin 30^\circ = 0.5 = \frac{1}{2}.$$

$$\therefore \cos 30^\circ = \sqrt{1-\frac{1}{4}} = \sqrt{\frac{3}{4}} = \tfrac{1}{2}\sqrt{3}.$$

$$\sin\tfrac{1}{2}x = \sqrt{\frac{1-\cos x}{2}}.$$

$$\therefore \sin 15^\circ = \sqrt{\frac{1-\frac{1}{2}\sqrt{3}}{2}} = \tfrac{1}{2}\sqrt{2-\sqrt{3}} = 0.2588.$$

$$\cos 15^\circ = \sqrt{\frac{1+\frac{1}{2}\sqrt{3}}{2}} = \tfrac{1}{2}\sqrt{2+\sqrt{3}} = 0.9659.$$

$$\tan 15^\circ = \sqrt{\frac{1-\frac{1}{2}\sqrt{3}}{1+\frac{1}{2}\sqrt{3}}} = \sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}$$
$$= \sqrt{\frac{(2-\sqrt{3})^2}{4-3}} = 2-\sqrt{3} = 0.2679.$$

$$\cot 15^\circ = \sqrt{\frac{1+\frac{1}{2}\sqrt{3}}{1-\frac{1}{2}\sqrt{3}}} = 2+\sqrt{3} = 3.7321.$$

26. Prove that

$$\tan 18^\circ = \frac{\sin 33^\circ + \sin 3^\circ}{\cos 33^\circ + \cos 3^\circ}.$$

Let $x = 18^\circ$,
and $y = 15^\circ$.

Then

(1) $2\sin x\cos y$
$= \sin(x+y) + \sin(x-y)$.

(2) $2\cos x\cos y$
$= \cos(x+y) + \cos(x-y)$.

Divide (1) by (2),

$$\tan x = \frac{\sin(x+y)+\sin(x-y)}{\cos(x+y)+\cos(x-y)}.$$

Substitute values of x and y,

$$\tan 18^\circ = \frac{\sin 33^\circ + \sin 3^\circ}{\cos 33^\circ + \cos 3^\circ}.$$

27. Prove the formula

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}.$$

$$\begin{aligned}\sin 2x &= 2\sin x \cos x\\ &= \frac{2\sin x}{\cos x}\cos^2 x\\ &= \frac{2\sin x}{\cos x}\times\frac{1}{\sec^2 x}.\end{aligned}$$

But by Prob. 2, Ex. V,

$$\sec^2 x = 1+\tan^2 x.$$

$$\therefore \sin 2x = \frac{2\tan x}{1+\tan^2 x}.$$

28. Prove the formula

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}.$$

$$\begin{aligned}\cos 2x &= \frac{\sin 2x}{\tan 2x}\\ &= \frac{2\sin x\cos x}{\dfrac{2\tan x}{1-\tan^2 x}}\\ &= \frac{(1-\tan^2 x)(2\sin x\cos x)}{2\tan x}\\ &= (1-\tan^2 x)\cos^2 x.\end{aligned}$$

But by Prob. 2, Ex. V,

$$\sec^2 x = 1+\tan^2 x = \frac{1}{\cos^2 x}.$$

$$\therefore \cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}.$$

29. Prove the formula

$$\tan\tfrac{1}{2}x = \frac{\sin x}{1+\cos x}.$$

$$\begin{aligned}\tan\tfrac{1}{2}x &= \sqrt{\frac{1-\cos x}{1+\cos x}}.\\ &= \sqrt{\frac{(1-\cos x)(1+\cos x)}{(1+\cos x)^2}}\\ &= \frac{\sqrt{1-\cos^2 x}}{1+\cos x}\\ &= \frac{\sin x}{1+\cos x}.\end{aligned}$$

30. Prove the formula

$$\cot\tfrac{1}{2}x = \frac{\sin x}{1-\cos x}.$$

$$\begin{aligned}\cot\tfrac{1}{2}x &= \sqrt{\frac{1+\cos x}{1-\cos x}}\\ &= \sqrt{\frac{(1+\cos x)(1-\cos x)}{(1-\cos x)^2}}\\ &= \frac{\sqrt{1-\cos^2 x}}{1-\cos x}\\ &= \frac{\sin x}{1-\cos x}.\end{aligned}$$

31. Prove the formula

$$\sin\tfrac{1}{2}x \pm \cos\tfrac{1}{2}x = \sqrt{1\pm\sin x}.$$

$$\begin{aligned}\sin\tfrac{1}{2}x \pm \cos\tfrac{1}{2}x &= \sqrt{(\sin\tfrac{1}{2}x \pm \cos\tfrac{1}{2}x)^2}\\ &= \sqrt{\sin^2\tfrac{1}{2}x + \cos^2\tfrac{1}{2}x \pm 2\sin\tfrac{1}{2}x\cos\tfrac{1}{2}x}\\ &= \sqrt{1\pm 2\sin\tfrac{1}{2}x\cos\tfrac{1}{2}x}\\ &= \sqrt{1\pm\sin x}.\end{aligned}$$

32. Prove the formula

$$\frac{\tan x \pm \tan y}{\cot x \pm \cot y} = \pm\tan x\tan y.$$

$$\frac{\tan x \pm \tan y}{\cot x \pm \cot y} = \frac{\dfrac{\sin x}{\cos x} \pm \dfrac{\sin y}{\cos y}}{\dfrac{\cos x}{\sin x} \pm \dfrac{\cos y}{\sin y}}$$

$$= \frac{\sin x \cos y \pm \cos x \sin y}{\cos x \cos y} \times \frac{\sin x \sin y}{\cos x \sin y \pm \sin x \cos y}$$

$$= \pm \frac{\sin x \sin y}{\cos x \cos y} = \pm \tan x \tan y.$$

33. Prove the formula

$$\tan(45^\circ - x) = \frac{1 - \tan x}{1 + \tan x}.$$

$$\tan(45^\circ - x) = \frac{\tan 45^\circ - \tan x}{1 + \tan 45^\circ \tan x} = \frac{1 - \tan x}{1 + \tan x}.$$

34. If A, B, C are the angles of a triangle, prove that

$$\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C.$$

$\sin A + \sin B + \sin C$

$= \sin A + \sin B + \sin[180^\circ - (A + B)]$

$= \sin A + \sin B + \sin(A + B)$

By [20] and [12],

$= 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B) + 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A + B)$

$= 2 \sin \tfrac{1}{2}(A + B)[\cos \tfrac{1}{2}(A - B) + \cos \tfrac{1}{2}(A + B)]$

By [22], $= 2 \sin \tfrac{1}{2}(A + B)\, 2 \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B$

$= 4 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B.$

But $\cos \tfrac{1}{2} C = \cos[90^\circ - \tfrac{1}{2}(A + B)] = \sin \tfrac{1}{2}(A + B).$

$\therefore \sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C.$

35. If A, B, C are the angles of a triangle, prove that

$$\cos A + \cos B + \cos C = 1 + 4 \sin \tfrac{1}{2} A \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C.$$

$\cos C = \cos[180^\circ - (A + B)] = -\cos(A + B).$

$\therefore \cos A + \cos B + \cos C$

$= \cos A + \cos B - \cos(A + B)$

By [22], $= 2 \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B) - \cos(A + B)$

By [13], $= 2 \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B) - 2 \cos^2 \tfrac{1}{2}(A + B) + 1$

$= [2 \cos \tfrac{1}{2}(A + B)][\cos \tfrac{1}{2}(A - B) - \cos \tfrac{1}{2}(A + B)] + 1$

By [23], $= [2 \cos \tfrac{1}{2}(A + B)] \times [2 \sin \tfrac{1}{2} A \sin \tfrac{1}{2} B] + 1$

$= (2 \sin \tfrac{1}{2} C)(2 \sin \tfrac{1}{2} A \sin \tfrac{1}{2} B) + 1$

$= 1 + 4 \sin \tfrac{1}{2} A \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C.$

36. If A, B, C are the angles of a triangle, prove that

$$\tan A + \tan B + \tan C = \tan A \times \tan B \times \tan C.$$

Since $A + B + C = 180°$, $C = 180° - (A + B)$.

$$\therefore \tan C = \tan[180° - (A + B)] = -\tan(A + B).$$

By [6], $\tan A + \tan B = \tan(A + B)(1 - \tan A \tan B)$

$$= \tan(A + B) - \tan(A + B)\tan A \tan B.$$

$$\begin{aligned}\therefore \tan A + \tan B + \tan C &= \tan(A + B) - \tan(A + B) \\ &\quad - \tan(A + B)\tan A \tan B \\ &= -\tan(A + B)\tan A \tan B \\ &= \tan A \tan B \tan C.\end{aligned}$$

37. If A, B, C are the angles of a triangle, prove that

$$\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C = \cot\tfrac{1}{2}A \times \cot\tfrac{1}{2}B \times \cot\tfrac{1}{2}C.$$

Since $\tfrac{1}{2}A + \tfrac{1}{2}B + \tfrac{1}{2}C = 90°$, $\tfrac{1}{2}C = 90° - \tfrac{1}{2}(A + B)$.

$$\therefore \cot\tfrac{1}{2}C = \tan\tfrac{1}{2}(A + B),$$

$$\cot\tfrac{1}{2}B = \tan\tfrac{1}{2}(A + C),$$

and $\cot\tfrac{1}{2}A = \tan\tfrac{1}{2}(B + C)$.

$$\therefore \cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C$$

$$= \tan\tfrac{1}{2}(A + B) + \tan\tfrac{1}{2}(A + C) + \tan\tfrac{1}{2}(B + C)$$

By Prob. 36, $= \tan\tfrac{1}{2}(A + B) \times \tan\tfrac{1}{2}(A + C) \times \tan\tfrac{1}{2}(B + C)$

$$= \cot\tfrac{1}{2}A \times \cot\tfrac{1}{2}B \times \cot\tfrac{1}{2}C.$$

38. Change $\cot x + \tan x$ to a form more convenient for logarithmic computation.

$$\begin{aligned}&\cot x + \tan x \\ &= \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} \\ &= \frac{\cos^2 x + \sin^2 x}{\sin x \cos x} \\ &= \frac{2(\cos^2 x + \sin^2 x)}{2 \sin x \cos x}\end{aligned}$$

By [12], $= \dfrac{2}{\sin 2x}$.

39. Change $\cot x - \tan x$ to a form more convenient for logarithmic computation.

$$\begin{aligned}&\cot x - \tan x \\ &= \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} \\ &= \frac{\cos^2 x - \sin^2 x}{\sin x \cos x}\end{aligned}$$

By [13], $= \dfrac{\cos 2x}{\sin x \cos x}$

$$= \frac{2 \cos 2x}{2 \sin x \cos x}$$

By [12], $= \dfrac{2 \cos 2x}{\sin 2x}$

$$= 2 \cot 2x.$$

40. Change $\cot x + \tan y$ to a form more convenient for logarithmic computation.

$$\begin{aligned}&\cot x + \tan y \\ &= \frac{\cos x}{\sin x} + \frac{\sin y}{\cos y} \\ &= \frac{\cos x \cos y + \sin x \sin y}{\sin x \cos y}\end{aligned}$$

By [9], $= \dfrac{\cos(x - y)}{\sin x \cos y}$.

41. Change $\cot x - \tan y$ to a form more convenient for logarithmic computation.

$$\cot x - \tan y$$
$$= \frac{\cos x}{\sin x} - \frac{\sin y}{\cos y}$$
$$= \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y}$$

By [5], $= \dfrac{\cos (x + y)}{\sin x \cos y}.$

42. Change $\dfrac{1 - \cos 2x}{1 + \cos 2x}$ to a form more convenient for logarithmic computation.

$$\frac{1 - \cos 2x}{1 + \cos 2x} = \frac{\dfrac{1 - \cos 2x}{2}}{\dfrac{1 + \cos 2x}{2}}$$

By [16], [17], $= \dfrac{\sin^2 x}{\cos^2 x}$

$$= \tan^2 x.$$

43. Change $1 + \tan x \tan y$ to a form more convenient for logarithmic computation.

$$1 + \tan x \tan y$$
$$= 1 + \frac{\sin x}{\cos x} \times \frac{\sin y}{\cos y}$$
$$= \frac{\cos x \cos y + \sin x \sin y}{\cos x \cos y}$$

By [9], $= \dfrac{\cos (x - y)}{\cos x \cos y}.$

44. Change $1 - \tan x \tan y$ to a form more convenient for logarithmic computation.

$$1 - \tan x \tan y$$
$$= 1 - \frac{\sin x \sin y}{\cos x \cos y}$$
$$= \frac{\cos x \cos y - \sin x \sin y}{\cos x \cos y}$$

By [5], $= \dfrac{\cos (x + y)}{\cos x \cos y}.$

45. Change $\cot x \cot y + 1$ to a form more convenient for logarithmic computation.

$$\cot x \cot y + 1$$
$$= \frac{\cos x}{\sin x} \times \frac{\cos y}{\sin y} + 1$$
$$= \frac{\cos x \cos y + \sin x \sin y}{\sin x \sin y}$$

By [9], $= \dfrac{\cos (x - y)}{\sin x \sin y}.$

46. Change $\cot x \cot y - 1$ to a form more convenient for logarithmic computation.

$$\cot x \cot y - 1$$
$$= \frac{\cos x \cos y}{\sin x \sin y} - 1$$
$$= \frac{\cos x \cos y - \sin x \sin y}{\sin x \sin y}$$

By [5], $= \dfrac{\cos (x + y)}{\sin x \sin y}.$

47. Change $\dfrac{\tan x + \tan y}{\cot x + \cot y}$ to a form more convenient for logarithmic computation.

$$\frac{\tan x + \tan y}{\cot x + \cot y}$$
$$= \frac{\dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y}}{\dfrac{\cos x}{\sin x} + \dfrac{\cos y}{\sin y}}$$
$$= \frac{\dfrac{\sin x \cos y + \cos x \sin y}{\cos x \cos y}}{\dfrac{\sin x \cos y + \cos x \sin y}{\sin x \sin y}}$$
$$= \frac{\sin x \sin y}{\cos x \cos y}$$
$$= \tan x \tan y.$$

EXERCISE XV. PAGE 63.

1. Find all the values of the following functions: $\sin^{-1}\frac{1}{2}\sqrt{3}$, $\text{an}^{-1}\frac{1}{3}\sqrt{3}$, $\text{vers}^{-1}\frac{1}{2}$, $\cos^{-1}(-\frac{1}{2}\sqrt{2})$, $\csc^{-1}\sqrt{2}$, $\tan^{-1}\infty$, $\sec^{-1}2$, $\text{os}^{-1}(-\frac{1}{2}\sqrt{3})$.

$$\begin{aligned}
\sin^{-1}\tfrac{1}{2}\sqrt{3} &= 60^\circ + 2n\pi \quad \text{or} \quad 120^\circ + 2n\pi.\\
\tan^{-1}\tfrac{1}{3}\sqrt{3} &= 30^\circ + 2n\pi \quad \text{or} \quad 210^\circ + 2n\pi.\\
\text{vers}^{-1}\tfrac{1}{2} &= 60^\circ + 2n\pi \quad \text{or} \quad 300^\circ + 2n\pi.\\
\cos^{-1}(-\tfrac{1}{2}\sqrt{2}) &= 135^\circ + 2n\pi \quad \text{or} \quad 225^\circ + 2n\pi.\\
\csc^{-1}\sqrt{2} &= 45^\circ + 2n\pi \quad \text{or} \quad 135^\circ + 2n\pi.\\
\tan^{-1}\infty &= 90^\circ + 2n\pi \quad \text{or} \quad 270^\circ + 2n\pi.\\
\sec^{-1}2 &= 60^\circ + 2n\pi \quad \text{or} \quad 300^\circ + 2n\pi.\\
\cos^{-1}(-\tfrac{1}{2}\sqrt{3}) &= 150^\circ + 2n\pi \quad \text{or} \quad 210^\circ + 2n\pi.
\end{aligned}$$

2. Prove that $\sin^{-1}(-x) = -\sin^{-1}x$; $\cos^{-1}(-x) = \pi - \cos^{-1}x$.

$$\begin{aligned}
\sin^{-1}(-x) &= \text{the angle whose sine is } -x\\
&= -\text{ the angle whose sine is } x\\
&= -\sin^{-1}x.\\
\cos^{-1}(-x) &= \text{the angle whose cosine is } -x\\
&= \pi - \text{the angle whose cosine is } x\\
&= \pi - \cos^{-1}x.
\end{aligned}$$

3. If $\sin^{-1}x + \sin^{-1}y = \pi$, prove that $x = y$.

If the sum of two angles is 180°, the sine of the one is equal to that of the other.

Hence, $\sin(\sin^{-1}x) = \sin(\sin^{-1}y)$.

$$\therefore x = y.$$

4. If $y = \sin^{-1}\frac{1}{3}$, find $\tan y$.

$$y = \sin^{-1}\tfrac{1}{3}.$$

$$\therefore \sin y = \tfrac{1}{3}.$$

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{\tfrac{8}{9}} = \tfrac{2}{3}\sqrt{2}.$$

$$\tan y = \frac{\frac{1}{3}}{\frac{2}{3}\sqrt{2}} = \frac{1}{2\sqrt{2}} = \tfrac{1}{4}\sqrt{2}.$$

5. Prove that

$$\cos(\sin^{-1}x) = \sqrt{1 - x^2}.$$

Let $\sin^{-1}x = y$;

then

$$\begin{aligned}
x &= \sin y,\\
\sqrt{1 - x^2} &= \cos y\\
&= \cos(\sin^{-1}x).
\end{aligned}$$

6. Prove that

$$\cos(2\sin^{-1}x) = 1 - 2x^2.$$

Let $\sin^{-1}x = y$;

then

$$\begin{aligned}
x &= \sin y,\\
\cos(2\sin^{-1}x) &= \cos 2y\\
&= \cos^2 y - \sin^2 y\\
&= 1 - \sin^2 y - \sin^2 y\\
&= 1 - 2\sin^2 y\\
&= 1 - 2x^2.
\end{aligned}$$

7. Prove that

$$\tan(\tan^{-1}x + \tan^{-1}y) = \frac{x + y}{1 - xy}.$$

Let $\tan^{-1}x = u$,

and $\tan^{-1}y = v$;

then
$$x = \tan u,$$
$$y = \tan v,$$
$$\tan(\tan^{-1}x + \tan^{-1}y)$$
$$= \tan(u + v)$$
$$= \frac{\tan u + \tan v}{1 - \tan u \tan v}$$
$$= \frac{x + y}{1 - xy}.$$

8. If $x = \sqrt{\frac{1}{2}}$, find all the values of $\sin^{-1}x + \cos^{-1}x$.

$$\sin^{-1}\sqrt{\tfrac{1}{2}} = 45^\circ + 2n\pi \text{ or } 135^\circ + 2n\pi.$$
$$\cos^{-1}\sqrt{\tfrac{1}{2}} = 45^\circ + 2n\pi \text{ or } -45^\circ + 2n\pi.$$
$$\therefore \sin^{-1}\sqrt{\tfrac{1}{2}} + \cos^{-1}\sqrt{\tfrac{1}{2}}$$
$$= 90^\circ + 2n\pi,\ 2n\pi.$$
$$180^\circ + 2n\pi, \text{ or } 90^\circ + 2n\pi$$
$$= 0^\circ,\ 90^\circ, \text{ or } 180^\circ.$$

9. Prove that
$$\tan^{-1}\left(\frac{x}{\sqrt{1 - x^2}}\right) = \sin^{-1}x.$$

Let
$$\tan^{-1}\left(\frac{x}{\sqrt{1 - x^2}}\right) = y;$$

then
$$\frac{x}{\sqrt{1 - x^2}} = \tan y.$$

By Prob. 2, Ex. V,
$$\sec^2 y = 1 + \tan^2 y$$
$$= 1 + \left(\frac{x}{\sqrt{1 - x^2}}\right)^2$$
$$= \frac{1}{1 - x^2}.$$
$$\therefore \cos^2 y = \frac{1}{\sec^2 y} = 1 - x^2.$$
$$\therefore \sin^2 y = 1 - \cos^2 y = x^2.$$
$$\therefore \sin y = x.$$
$$\therefore y = \sin^{-1}x.$$
$$\therefore \tan^{-1}\left(\frac{x}{\sqrt{1 - x^2}}\right) = \sin^{-1}x.$$

10. Find the value of
$$\sin(\tan^{-1}\tfrac{5}{12}).$$

Let $\tan^{-1}\frac{5}{12} = x$;

then $\frac{5}{12} = \tan x$.

By Prob. 2, Ex. V,
$$\sec^2 x = 1 + \tan^2 x$$
$$= 1 + (\tfrac{5}{12})^2$$
$$= \tfrac{169}{144}.$$
$$\therefore \cos^2 x = \tfrac{144}{169}.$$
$$\therefore \sin^2 x = \tfrac{25}{169}.$$
$$\therefore \sin x = \pm\tfrac{5}{13}.$$

11. Find the value of
$$\cot(2\sin^{-1}\tfrac{3}{5}).$$

Let $\sin^{-1}\frac{3}{5} = x$;

then
$$\sin x = \tfrac{3}{5},$$
$$\cos x = \pm\tfrac{4}{5},$$
$$\cot x = \pm\tfrac{4}{3}.$$
$$\cot(2\sin^{-1}\tfrac{3}{5}) = \cot 2x$$

By [5],
$$= \frac{\cot^2 x - 1}{2\cot x}$$
$$= \frac{\frac{16}{9} - 1}{\pm\frac{8}{3}}$$
$$= \pm\frac{7}{24}.$$

12. Find the value of
$$\sin(\tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{3}).$$

Let
$$\tan^{-1}\tfrac{1}{2} = x,$$
$$\tan^{-1}\tfrac{1}{3} = y;$$

then $\tan x = \frac{1}{2}$,

and $\tan y = \frac{1}{3}$.

Now
$$\sec^2 x = 1 + \tan^2 x$$
$$= \tfrac{5}{4},$$
$$\cos^2 x = \tfrac{4}{5},$$
$$\cos x = \frac{\pm 2}{\sqrt{5}},$$
$$\sin x = \frac{\pm 1}{\sqrt{5}};$$

and
$$\cos y = \frac{\pm 3}{\sqrt{10}},$$

$$\sin\ y = \frac{\pm 1}{\sqrt{10}}.$$

$$\therefore \sin(\tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{3})$$
$$= \sin(x+y)$$
$$= \sin x \cos y + \cos x \sin y$$
$$= \frac{1}{\sqrt{5}} \times \frac{3}{\sqrt{10}} + \frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{10}}$$
$$= \frac{\pm 5}{\sqrt{50}}$$
$$= \pm \tfrac{1}{2}\sqrt{2}.$$

13. If $\sin^{-1}x = 2\cos^{-1}x$, find x.

$$\sin^{-1}x = 90° - \cos^{-1}x.$$
$$\therefore 90° - \cos^{-1}x = 2\cos^{-1}x,$$
$$3\cos^{-1}x = 90°,$$
$$\cos^{-1}x = 30°, 150°, \text{ or } 270°,$$
$$x = \pm\tfrac{1}{2}\sqrt{3}, \text{ or } 0.$$

14. Prove that

$$\tan(2\tan^{-1}x) = \frac{2x}{1-x^2}.$$

Let $\tan^{-1}x = y$;

then $x = \tan y$,

$$\tan(2\tan^{-1}x) = \tan 2y$$
$$= \frac{2\tan y}{1-\tan^2 y}$$
$$= \frac{2x}{1-x^2}.$$

15. Prove that

$$\sin(2\tan^{-1}x) = \frac{2x}{1+x^2}.$$

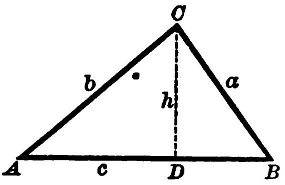
Let $\tan^{-1}x = y$;

then $x = \tan y$,

$$\sin(2\tan^{-1}x) = \sin 2y$$
$$= 2\sin y \cos y$$
$$= 2\frac{\sin y}{\cos y}\cos^2 y$$
$$= \frac{2\tan y}{\sec^2 y}$$
$$= \frac{2\tan y}{1+\tan^2 y}.$$
$$= \frac{2x}{1+x^2}.$$

EXERCISE XVI. PAGE 67.

1. What do the formulas of Sect. XXXIV, p. 64, become when one of the angles is a right angle?



If angle C is a right angle,

$$\frac{a}{c} = \frac{\sin A}{\sin C} = \sin A;$$
$$\frac{b}{c} = \frac{\sin B}{\sin C} = \sin B;$$
$$\frac{a}{b} = \frac{\sin A}{\sin B} = \tan A;$$
$$\frac{a}{\sin A} = \frac{c}{\sin C} = c;$$
$$\frac{b}{\sin B} = \frac{c}{\sin C} = c.$$

2. Prove by means of the Law of Sines that the bisector of an angle of a triangle divides the opposite side into parts proportional to the adjacent sides.

Let CD bisect angle C;

then $$\frac{AD}{CD}=\frac{\sin\frac{1}{2}C}{\sin A},$$

and $$\frac{DB}{CD}=\frac{\sin\frac{1}{2}C}{\sin B}.$$

By division,

$$\frac{AD}{DB}=\frac{\sin B}{\sin A}.$$

But $$\frac{\sin B}{\sin A}=\frac{b}{a}.$$

$$\therefore \frac{AD}{DB}=\frac{b}{a}.$$

3. What does Formula [26] become when $A=90°$? when $A=0°$? when $A=180°$? What does the triangle become in each of these cases?

Formula [26] is

$$a^2=b^2+c^2-2bc\cos A.$$

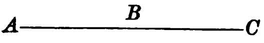
1. When $A=90°$, $\cos A=0$.
$\therefore a^2=b^2+c^2.$

2. When $A=0°$, $\cos A=1$.
$\therefore a^2=b^2+c^2-2bc.$

3. When $A=180°$, $\cos A=-1$.
$\therefore a^2=b^2+c^2+2bc.$

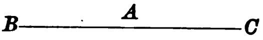
4. A right triangle.

5. A straight line.



$a=BC.$ $c=AB.$
$b=AC.$ $a=b-c.$

6. A straight line.



$a=BC.$ $c=BA.$
$b=AC.$ $a=b+c.$

4. Prove (Figs. 56 and 57) that whether the angle B is acute or obtuse $c=a\cos B+b\cos A$. What are the two symmetrical formulas obtained by changing the letters? What does the formula become when $B=90°$?

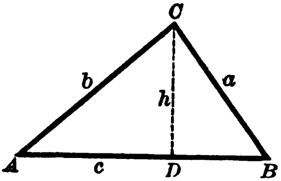


FIG. 1.

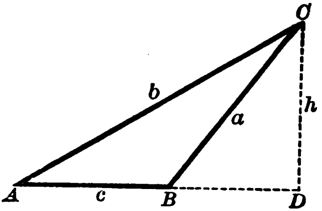


FIG. 2.

CASE I. When angle B is acute (Fig. 1).

$$\cos B=\frac{DB}{a}.$$

$$\cos A=\frac{AD}{b}.$$

$$\therefore DB=a\cos B,$$

and $$AD=b\cos A.$$

$$\therefore DB+AD=a\cos B+b\cos A.$$

But $$DB+AD=c.$$

$$\therefore c=a\cos B+b\cos A.$$

CASE II. When angle B is obtuse (Fig. 2).

$$\frac{AD}{b}=\cos A.$$

$$\frac{BD}{a}=\cos(180°-B)$$

$$=-\cos B.$$

$$\therefore AD=b\cos A,$$

and $$BD=-a\cos B.$$

$$\therefore AD - BD = b\cos A + a\cos B.$$

But $AD - BD = c.$

$$\therefore c = a\cos B + b\cos A.$$

The symmetrical formulas are

$$b = a\cos C + c\cos A,$$

and $a = b\cos C + c\cos B.$

When $B = 90°$,

$$\cos A = \frac{c}{b}.$$

$$\therefore c = b\cos A.$$

5. From the three following equations (found in the last example) prove the theorem of Sect. XXXV, p. 66:

$$c = a\cos B + b\cos A,$$
$$b = a\cos C + c\cos A,$$
$$a = b\cos C + c\cos B.$$
$$c^2 = ac\cos B + bc\cos A. \quad (1)$$
$$b^2 = ab\cos C + bc\cos A. \quad (2)$$
$$a^2 = ab\cos C + ac\cos B. \quad (3)$$

Add (2) and (3),

$$a^2 + b^2 = 2\,ab\cos C + bc\cos A + ac\cos B. \quad (4)$$

Subtract (4) from (1),

$$c^2 - a^2 - b^2 = -2\,ab\cos C.$$
$$\therefore c^2 = a^2 + b^2 - 2\,ab\cos C.$$

In a similar manner it may be proved that

$$a^2 = b^2 + c^2 - 2\,bc\cos A,$$

and $b^2 = a^2 + c^2 - 2\,ac\cos B.$

6. In Formula [27] what is the maximum value of $\frac{1}{2}(A - B)$?

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}.$$

The limit of $A - B$ is 180°.

Therefore, the limit of the maximum value of $\frac{1}{2}(A - B)$

$$= \frac{180°}{2} = 90°.$$

7. Find the form to which Formula [27] reduces, and describe the nature of the triangle, when

(i) $C = 90$;

(ii) $A - B = 90°$, and $B = C$.

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}.$$

(i) When $C = 90°$.

$$A + B = 90°.$$
$$B = 90° - A.$$
$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}[A - (90° - A)]}{\tan 45°}$$
$$= \frac{\tan(A - 45°)}{1}$$
$$= \tan(A - 45°).$$

Since C is a right angle, the triangle is a right triangle.

(ii) When $A - B = 90°$, and $B = C$.

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}.$$

$$A + B + C = 180°,$$

or

$$A + 2B = 180°$$
$$\underline{A - B = 90°}$$
$$\therefore 3B = 90°,$$
$$B = 30°,$$
$$C = 30°,$$

and $A = 120°.$

$$\frac{a-b}{a+b} = \frac{\tan 45°}{\tan 75°}$$
$$= \frac{\tan 45°}{\cot 15°}$$
$$= \frac{1}{2 + \sqrt{3}}.$$
$$\therefore a + b = (a - b)(2 + \sqrt{3}).$$

Since $B = C$, the triangle is isosceles.

EXERCISE XVII. PAGE 69.

1.

Given	find
$a = 500$,	$C = 123° 12'$,
$A = 10° 12'$,	$b = 2051.5$,
$B = 46° 36'$;	$c = 2362.6$.

$$\begin{aligned} a &= 500. \\ A &= 10° 12' \\ B &= \underline{46° 36'} \\ A + B &= 56° 48' \\ \therefore C &= 123° 12'. \end{aligned}$$

$$\begin{aligned} \log a &= 2.69897 \\ \text{colog} \sin A &= 0.75182 \\ \log \sin B &= \underline{9.86128} \\ \log b &= 3.31207 \\ b &= 2051.5. \end{aligned}$$

$$\begin{aligned} \log a &= 2.69897 \\ \text{colog} \sin A &= 0.75182 \\ \log \sin C &= \underline{9.92260} \\ \log c &= 3.37339 \\ c &= 2362.6. \end{aligned}$$

2.

Given	find
$a = 795$,	$C = 55° 20'$,
$A = 79° 59'$,	$b = 567.69$,
$B = 44° 41'$;	$c = 663.99$.

$$\begin{aligned} a &= 795. \\ A &= 79° 59' \\ B &= \underline{44° 41'} \\ A + B &= 124° 40' \\ \therefore C &= 55° 20'. \end{aligned}$$

$$\begin{aligned} \log a &= 2.90037 \\ \text{colog} \sin A &= 0.00667 \\ \log \sin B &= \underline{9.84707} \\ \log b &= 2.75411 \\ b &= 567.69. \end{aligned}$$

$$\begin{aligned} \log a &= 2.90037 \\ \text{colog} \sin A &= 0.00667 \\ \log \sin C &= \underline{9.91512} \\ \log c &= 2.82216 \\ c &= 663.99. \end{aligned}$$

3.

Given	find
$a = 804$,	$C = 35° 4'$,
$A = 99° 55'$,	$b = 577.31$,
$B = 45° 1'$;	$c = 468.93$.

$$\begin{aligned} a &= 804. \\ A &= 99° 55' \\ B &= \underline{45° \ 1'} \\ A + B &= 144° 56' \\ \therefore C &= 35° \ 4'. \end{aligned}$$

$$\begin{aligned} \log a &= 2.90526 \\ \text{colog} \sin A &= 0.00654 \\ \log \sin B &= \underline{9.84961} \\ \log b &= 2.76141 \\ b &= 577.31. \end{aligned}$$

$$\begin{aligned} \log a &= 2.90526 \\ \text{colog} \sin A &= 0.00654 \\ \log \sin C &= \underline{9.75931} \\ \log c &= 2.67111 \\ c &= 468.93. \end{aligned}$$

4.

Given	find
$a = 820$,	$C = 25° 12'$,
$A = 12° 49'$,	$b = 2276.6$,
$B = 141° 59'$;	$c = 1573.9$.

$$\begin{aligned} a &= 820. \\ A &= 12° 49' \\ B &= \underline{141° 59'} \\ A + B &= 154° 48' \\ \therefore C &= 25° 12'. \end{aligned}$$

$$\begin{aligned} \log a &= 2.91381 \\ \text{colog} \sin A &= 0.65398 \\ \log \sin B &= \underline{9.78950} \\ \log b &= 3.35729 \\ b &= 2276.6. \end{aligned}$$

$$\begin{aligned} \log a &= 2.91381 \\ \text{colog} \sin A &= 0.65398 \\ \log \sin C &= \underline{9.62918} \\ \log c &= 3.19697 \\ c &= 1573.9. \end{aligned}$$

5. Given	find
$c = 1005$,	$C = 47° 14'$,
$A = 78° 19'$,	$a = 1340.6$,
$B = 54° 27'$;	$b = 1113.8$.

$$c = 1005.$$

$$\begin{aligned} A &= 78° 19' \\ B &= \underline{54° 27'} \\ A + B &= 132° 46' \\ \therefore C &= 47° 14'. \end{aligned}$$

$$\begin{aligned} \log c &= 3.00217 \\ \text{colog} \sin C &= 0.13423 \\ \log \sin A &= \underline{9.99091} \\ \log a &= 3.12731 \end{aligned}$$

$$a = 1340.6.$$

$$\begin{aligned} \log c &= 3.00217 \\ \text{colog} \sin C &= 0.13423 \\ \log \sin B &= \underline{9.91042} \\ \log b &= 3.04682 \end{aligned}$$

$$b = 1113.8.$$

6. Given	find
$b = 13.57$,	$A = 108° 50'$,
$B = 13° 57'$,	$a = 53.276$,
$C = 57° 13'$;	$c = 47.324$.

$$b = 13.57.$$

$$\begin{aligned} B &= 13° 57' \\ C &= \underline{57° 13'} \\ B + C &= 71° 10' \\ \therefore A &= 108° 50'. \end{aligned}$$

$$\begin{aligned} \log b &= 1.13258 \\ \text{colog} \sin B &= 0.61785 \\ \log \sin A &= \underline{9.97610} \\ \log a &= 1.72653 \end{aligned}$$

$$a = 53.276.$$

$$\begin{aligned} \log b &= 1.13258 \\ \text{colog} \sin B &= 0.61785 \\ \log \sin C &= \underline{9.92465} \\ \log c &= 1.67508 \end{aligned}$$

$$c = 47.324.$$

7. Given	find
$a = 6412$,	$B = 56° 56'$,
$A = 70° 55'$,	$b = 5685.9$,
$C = 52° 9'$;	$c = 5357.5$.

$$a = 6412.$$

$$\begin{aligned} A &= 70° 55' \\ C &= \underline{52° \ \ 9'} \\ A + C &= 123° \ \ 4' \\ \therefore B &= 56° 56'. \end{aligned}$$

$$\begin{aligned} \log a &= 3.80699 \\ \text{colog} \sin A &= 0.02455 \\ \log \sin B &= \underline{9.92326} \\ \log b &= 3.75480 \end{aligned}$$

$$b = 5685.9.$$

$$\begin{aligned} \log a &= 3.80699 \\ \text{colog} \sin A &= 0.02455 \\ \log \sin C &= \underline{9.89742} \\ \log c &= 3.72896 \end{aligned}$$

$$c = 5357.5.$$

8. Given	find
$b = 999$,	$B = 77°$,
$A = 37° 58'$,	$a = 630.77$,
$C = 65° 2'$;	$c = 929.48$.

$$b = 999.$$

$$\begin{aligned} A &= 37° 58' \\ C &= \underline{65° \ \ 2'} \\ A + C &= 103° \\ \therefore B &= 77°. \end{aligned}$$

$$\begin{aligned} \log b &= 2.99957 \\ \text{colog} \sin B &= 0.01128 \\ \log \sin A &= \underline{9.78902} \\ \log a &= 2.79987 \end{aligned}$$

$$a = 630.77.$$

$$\begin{aligned} \log b &= 2.99957 \\ \text{colog} \sin B &= 0.01128 \\ \log \sin C &= \underline{9.95739} \\ \log c &= 2.96824 \end{aligned}$$

$$c = 929.48.$$

9. In order to determine the distance of a hostile fort A from a place B, a line BC and the angles ABC and BCA were measured, and found to be 322.55 yards, 60° 34′, and 56° 10′, respectively. Find the distance AB.

$$\begin{aligned} a &= 322.55. \\ B &= 60^\circ\ 34' \\ C &= \underline{56^\circ\ 10'} \\ B + C &= 116^\circ\ 44' \\ \therefore A &= 63^\circ\ 16' \end{aligned}$$

$$\begin{aligned} \log a &= 2.50860 \\ \text{colog} \sin A &= 0.04910 \\ \log \sin C &= \underline{9.91942} \\ \log c &= 2.47712 \\ c &= 300. \\ \therefore AB &= 300 \text{ yards.} \end{aligned}$$

10. The angles B and C of a triangle ABC are 50° 30′ and 122° 9′, respectively, and BC is 9 miles. Find AB and AC.

$$\begin{aligned} C &= 122^\circ\ 9' \\ B &= \underline{50^\circ\ 30'} \\ B + C &= 172^\circ\ 39' \\ \therefore A &= 7^\circ\ 21'. \end{aligned}$$

$$\begin{aligned} \log BC &= 0.95424 \\ \text{colog} \sin A &= 0.89303 \\ \log \sin B &= \underline{9.88741} \\ \log b &= 1.73468 \\ b &= 54.285. \\ \therefore AC &= 54.285 \text{ miles.} \end{aligned}$$

$$\begin{aligned} \log BC &= 0.95424 \\ \text{colog} \sin A &= 0.89303 \\ \log \sin C &= \underline{9.92771} \\ \log c &= 1.77498 \\ c &= 59.564. \\ \therefore AB &= 59.564 \text{ miles.} \end{aligned}$$

11. Two observers 5 miles apart on a plain, and facing each other, find that the angles of elevation of a balloon in the same vertical plane with themselves are 55° and 58°, respectively. Find the distance from the balloon to each observer, and also the height of the balloon above the plain.

$$\begin{aligned} B &= 58^\circ \\ A &= \underline{55^\circ} \\ A + B &= 113^\circ \\ \therefore C &= 67^\circ. \end{aligned}$$

$$\begin{aligned} \log c &= 0.69897 \\ \text{colog} \sin C &= 0.03597 \\ \log \sin A &= \underline{9.91336} \\ \log a &= 0.64830 \\ a &= 4.4494. \\ \therefore BC &= 4.4494 \text{ miles.} \end{aligned}$$

$$\begin{aligned} \log c &= 0.69897 \\ \text{colog} \sin C &= 0.03597 \\ \log \sin B &= \underline{9.92842} \\ \log b &= 0.66336 \\ b &= 4.6064. \\ \therefore AC &= 4.6064 \text{ miles.} \end{aligned}$$

To find h.

$$\frac{h}{a} = \sin B.$$

$$\therefore h = a \sin B.$$

$$\begin{aligned} \log a &= 0.64830 \\ \log \sin B &= \underline{9.92842} \\ \log h &= 0.57672 \\ h &= 3.7733. \\ \therefore \text{height} &= 3.7733 \text{ miles.} \end{aligned}$$

12. In a parallelogram, given a diagonal d and the angles x and y which this diagonal makes with the sides. Find the sides. Compute the results if $d = 11.237$, $x = 19^\circ\ 1'$, and $y = 42^\circ\ 54'$.

$$\frac{a}{d}=\frac{\sin x}{\sin z}.$$

$$\therefore a=\frac{d\sin x}{\sin z}.$$

$$\frac{c}{d}=\frac{\sin y}{\sin z}.$$

$$\therefore c=\frac{d\sin y}{\sin z}.$$

$$\begin{aligned}
d &= 11.237.\\
x &= 19^\circ\ 1'\\
y &= \underline{42^\circ\ 54'}\\
x+y &= 61^\circ\ 55'\\
\therefore z &= 118^\circ\ 5'
\end{aligned}$$

$$\begin{aligned}
\log d &= 1.05065\\
\operatorname{colog}\sin z &= 0.05440\\
\log\sin x &= \underline{9.51301}\\
\log a &= 0.61806\\
a &= 4.1501.
\end{aligned}$$

$$\begin{aligned}
\log d &= 1.05065\\
\operatorname{colog}\sin z &= 0.05440\\
\log\sin y &= \underline{9.83297}\\
\log c &= 0.93802\\
c &= 8.67.
\end{aligned}$$

13. A lighthouse was observed from a ship to bear N. 34° E.; after the ship sailed due south 3 miles, it bore N. 23° E. Find the distance from the lighthouse to the ship in each position.

$$\begin{aligned}
c &= 3.\\
A &= 23^\circ\\
B=(180^\circ-34^\circ) &= \underline{146^\circ}\\
A+B &= 169^\circ\\
\therefore C &= 11^\circ.
\end{aligned}$$

$$\begin{aligned}
\log c &= 0.47712\\
\operatorname{colog}\sin C &= 0.71940\\
\log\sin A &= \underline{9.59188}\\
\log a &= 0.78840\\
a &= 6.1433.
\end{aligned}$$

$$\begin{aligned}
\log c &= 0.47712\\
\operatorname{colog}\sin C &= 0.71940\\
\log\sin B &= \underline{9.74756}\\
\log b &= 0.94408\\
b &= 8.7918.
\end{aligned}$$

Therefore, the required distances are 6.1433 miles and 8.7918 miles.

14. In a trapezoid, given the parallel sides a and b, and the angles x and y at the ends of one of the parallel sides. Find the non-parallel sides. Compute the results when $a=15$, $b=7$, $x=70^\circ$, $y=40^\circ$.

Given parallel sides,

$$AB=7 \quad\text{and}\quad DC=15;$$

also, $ADC=40^\circ$ and $BCD=70^\circ$; required AD and BC.

Draw $AE \parallel BC$;

then $AB=EC$ ($\parallel_s$ comp. bet. $\parallel_s$),

and $DE=DC-AB$

$$=15-7=8.$$

Also $AED=BCD=70^\circ$ (ext. int. $\angle$s).

Now

$$DAE=180^\circ-(40^\circ+70^\circ)=70^\circ.$$

But since

$$AED=DAE=70^\circ,$$

the $\triangle AED$ is isosceles, and

$$DA=DE=8.$$

Now $AE=BC$, and we are to find BC.

$$\frac{AE}{DE}=\frac{\sin ADE}{\sin DAE}.$$

$$\begin{aligned}
\log DE &= 0.90309\\
\log\sin ADE &= 9.80807\\
\operatorname{colog}\sin DAE &= \underline{0.02701}\\
\log AE &= 0.73817\\
AE=BC &= 5.4723.
\end{aligned}$$

15. Given $b=7.07107$, $A=30^\circ$, $C=105^\circ$; find a and c without using logarithms.

Let p and q denote the segments of c made by the $\perp$ dropped from C.

$$A = 30°,$$
$$C = 105°.$$
$$\therefore B = 45°.$$
$$\therefore \frac{a}{b} = \frac{\sin A}{\sin B} = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{2}}.$$
$$a = \frac{b}{\sqrt{2}} = \frac{7.07107}{1,41421} = 5.$$
$$\frac{p}{b} = \cos A = \tfrac{1}{2}\sqrt{3} = 0.86603.$$
$$p = b \times 0.86603$$
$$= 7.07107 \times 0.86603$$
$$= 6.12376.$$
$$\frac{q}{a} = \cos B = \tfrac{1}{2}\sqrt{2} = 0.70711.$$
$$q = a \times 0.70711$$
$$= 5 \times 0.70711 = 3.53555.$$
$$c = p + q$$
$$= 6.12376 + 3.53555$$
$$= 9.6593.$$

16. Given $c = 9.562$, $A = 45°$, $B = 60°$; find a and b without using logarithms.

$$C = 75°.$$
$$a = \frac{c \sin A}{\sin C}.$$
$$\sin C = \sin(45° + 30°)$$
$$= \sin 45° \cos 30° + \cos 45° \sin 30°$$
$$= \tfrac{1}{2}\sqrt{2} \times \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{2} \times \tfrac{1}{2}$$
$$= \tfrac{1}{4}(\sqrt{6} + \sqrt{2}).$$
$$\therefore a = \frac{9.562 \times \frac{1}{2}\sqrt{2}}{\frac{1}{4}(\sqrt{6} + \sqrt{2})}$$
$$= \frac{19.124 \times \sqrt{2}}{\sqrt{6} + \sqrt{2}}$$
$$= \frac{(19.124 \times \sqrt{2})(\sqrt{6} - \sqrt{2})}{6 - 2}$$
$$= 9.562(\sqrt{3} - 1)$$
$$= 6.99986 = 7.$$
$$b = \frac{a \sin B}{\sin A} = \frac{7 \times \frac{1}{2}\sqrt{3}}{\frac{1}{2}\sqrt{2}}$$
$$= \frac{7\sqrt{3}}{\sqrt{2}} = \frac{7\sqrt{6}}{2}$$
$$= 3.5\sqrt{6} = 8.573.$$

17. The base of a triangle is 600 feet, and the angles at the base are 30° and 120°. Find the other sides and the altitude without using logarithms.

$$AB = 600.$$
$$A = 30°.$$
$$B = 120°.$$
$$\therefore C = 30°.$$
$$\therefore a = c = 600 \text{ feet.}$$
$$b = \frac{a \sin B}{\sin A}$$
$$= \frac{600 \times \sin(180° - 60°)}{\sin 30°}$$
$$= \frac{600 \times \frac{1}{2}\sqrt{3}}{\frac{1}{2}}$$
$$= 600 \times 1.732051$$
$$= 1039.2.$$
$$h = a \sin B = 600 \times \tfrac{1}{2}\sqrt{3}$$
$$= 519.6 \text{ feet.}$$

18. Two angles of a triangle are, the one 20°, the other 40°. Find the ratio of the opposite sides without using logarithms.

Let $x = 20°$,
$y = 40°$,

and a and b be opposite sides.

Then $$\frac{\sin x}{\sin y} = \frac{a}{b}.$$

$$\text{nat} \sin x = 0.3420.$$
$$\text{nat} \sin y = 0.6428.$$
$$\therefore a : b = 3420 : 6428$$
$$= 855 : 1607.$$

19. The angles of a triangle are as 5 : 10 : 21, and the side opposite the smallest angle is 3. Find the other sides without using logarithms.

Since the angles A, B, C are as 5 : 10 : 21,

$$A = \tfrac{5}{36} \text{ of } 180^\circ = 25^\circ.$$
$$B = \tfrac{10}{36} \text{ of } 180^\circ = 50^\circ.$$
$$C = \tfrac{21}{36} \text{ of } 180^\circ = 105^\circ.$$
$$b = \frac{a \sin B}{\sin A} = \frac{3 \times 0.7660}{0.4226}$$
$$= 5.438.$$
$$c = \frac{a \sin C}{\sin A} = \frac{3 \times 0.9659}{0.4226}$$
$$= 6.857.$$

20. Given one side of a triangle equal to 27, the adjacent angles equal each to 30°; find the radius of the circumscribed circle without using logarithms.

$$2R = \frac{a}{\sin A}.$$
$$\sin A = \sin 120^\circ$$
$$= \sin(180^\circ - 120^\circ) = \sin 60^\circ.$$
$$\sin 60^\circ = \tfrac{1}{2}\sqrt{3}.$$
$$\therefore 2R = \frac{27}{\frac{1}{2}\sqrt{3}} = \frac{54}{\sqrt{3}} = 18\sqrt{3}.$$
$$\therefore R = 9\sqrt{3} = 15.588.$$

Exercise XVIII. Page 74.

1. Find the number of solutions of the following:

(i) $a = 80,\ b = 100,\ A = 30^\circ.$

$\therefore a < b,$

but $a > b \sin A = 100 \times \frac{1}{2},$

and $A < 90^\circ.$

$\therefore$ two solutions.

(ii) $a = 50,\ b = 100,\ A = 30^\circ.$

$\therefore a = b \sin A = 100 \times \frac{1}{2}.$

$\therefore$ one solution.

(iii) $a = 40,\ b = 100,\ A = 30^\circ.$

$\therefore a < b \sin A = 100 \times \frac{1}{2},$

and $A < 90^\circ.$

$\therefore$ no solution.

(iv) $a = 13.4,\ b = 11.46,\ A = 77^\circ\ 20'.$

$\therefore a > b.$

$\therefore$ one solution.

(v) $a = 70,\ b = 75,\ A = 60^\circ.$

$\therefore a < b,$

but $a > b \sin A = 75 \times \frac{1}{2}\sqrt{3},$

and $A < 90^\circ.$

$\therefore$ two solutions.

(vi) $a = 134.16,\ b = 84.54,$
$B = 52^\circ\ 9'\ 11''.$

$b < a,$

$B < 90^\circ,$

$\text{nat} \sin B = 0.7896.$

$84.54 < 134.16 \times 0.7896.$

$\therefore b < a \sin B.$

$\therefore$ no solution.

(vii) $a = 200,\ b = 100,\ A = 30^\circ.$

$a > b.$

$\therefore$ one solution.

2. Given	find
$a = 840,$	$B = 12^\circ\ 13'\ 34'',$
$b = 485,$	$C = 146^\circ\ 15'\ 26'',$
$A = 21^\circ\ 31';$	$c = 1272.1.$

$$\begin{aligned}
\operatorname{colog} a &= 7.07572 - 10\\
\log b &= 2.68574\\
\log \sin A &= \underline{9.56440}\\
\log \sin B &= 9.32586\\
B &= 12^\circ\, 13'\, 34''.\\
\therefore\ C &= 146^\circ\, 15'\, 26''.
\end{aligned}$$

$$\begin{aligned}
\log a &= 2.92428\\
\log \sin C &= 9.74466\\
\operatorname{colog} \sin A &= \underline{0.43560}\\
\log c &= 3.10454\\
c &= 1272.1.
\end{aligned}$$

3.

Given	find
$a = 9.399,$	$B = 57^\circ\, 23'\, 40'',$
$b = 9.197,$	$C = 2^\circ\ \ 1'\, 20'',$
$A = 120^\circ\, 35'\,;$	$c = 0.38525.$

$$\begin{aligned}
\operatorname{colog} a &= 9.02692 - 10\\
\log b &= 0.96365\\
\log \sin A &= \underline{9.93495}\\
\log \sin B &= 9.92552\\
B &= 57^\circ\, 23'\, 40''.\\
\therefore\ C &= 2^\circ\ \ 1'\, 20''.
\end{aligned}$$

$$\begin{aligned}
\log a &= 0.97308\\
\log \sin C &= 8.54761\\
\operatorname{colog} \sin A &= \underline{0.06505}\\
\log c &= 9.58574 - 10\\
c &= 0.38525.
\end{aligned}$$

4.

Given	find
$a = 91.06,$	$B = 41^\circ\, 13',$
$b = 77.04,$	$C = 87^\circ\, 37'\, 54'',$
$A = 51^\circ\, 9'\, 6''\,;$	$c = 116.82.$

$$\begin{aligned}
\operatorname{colog} a &= 8.04067 - 10\\
\log b &= 1.88672\\
\log \sin A &= \underline{9.89143}\\
\log \sin B &= 9.81882\\
B &= 41^\circ\, 13'.\\
\therefore\ C &= 87^\circ\, 37'\, 54''.
\end{aligned}$$

$$\begin{aligned}
\log a &= 1.95933\\
\log \sin C &= 9.99963\\
\operatorname{colog} \sin A &= \underline{0.10857}\\
\log c &= 2.06753\\
c &= 116.82.
\end{aligned}$$

5.

Given	find
$a = 55.55,$	$A = 54^\circ\, 31'\, 13'',$
$b = 66.66,$	$C = 47^\circ\, 44'\ \ 7'',$
$B = 77^\circ\, 44'\, 40''\,;$	$c = 50.481.$

$$\begin{aligned}
\log a &= 1.74468\\
\log \sin B &= 9.98999\\
\operatorname{colog} b &= \underline{8.17613 - 10}\\
\log \sin A &= 9.91080\\
A &= 54^\circ\, 31'\, 13''.\\
\therefore\ C &= 47^\circ\, 44'\ \ 7''.
\end{aligned}$$

$$\begin{aligned}
\log a &= 1.74468\\
\log \sin C &= 9.86925\\
\operatorname{colog} \sin A &= \underline{0.08920}\\
\log c &= 1.70313\\
c &= 50.481.
\end{aligned}$$

6. Given

$a = 309,\ b = 360,\ A = 21^\circ\, 14'\, 25''\,;$

find

$$\begin{aligned}
B &= 24^\circ\, 57'\, 54'',\\
C &= 133^\circ\, 47'\, 41'',\\
c &= 615.67,\\
B' &= 155^\circ\ \ 2'\ \ 6'',\\
C' &= 3^\circ\, 43'\, 29'',\\
c' &= 55.41.
\end{aligned}$$

There are two solutions,

for $a < b$,
but $a > b \sin A$,
and $A < 90^\circ$.

$$\begin{aligned}
\log b &= 2.55630\\
\log \sin A &= 9.55904\\
\operatorname{colog} a &= \underline{7.51004 - 10}\\
\log \sin B &= 9.62538\\
B &= 24^\circ\, 57'\, 54''.\\
\therefore\ C &= 133^\circ\, 47'\, 41''.
\end{aligned}$$

$$\begin{aligned} \log a &= 2.48996 \\ \log \sin C &= 9.85843 \\ \text{colog} \sin A &= \underline{0.44096} \\ \log c &= 2.78935 \end{aligned}$$

$$c = 615.67.$$

Second Solution.

$B' = 180° - B = 155° \; 2' \; 6''.$
$C' = B - A \quad = \quad 8° 43' 29''.$

$$\begin{aligned} \log a &= 2.48996 \\ \log \sin C' &= 8.81267 \\ \text{colog} \sin A &= \underline{0.44096} \\ \log c' &= 1.74359 \end{aligned}$$

$$c' = 55.41.$$

7. Given $a = 8.716, b = 9.787,$
$A = 38° 14' 12'';$
find $B = 44° \; 1' 28'',$
$C = 97° 44' 20'',$
$c = 13.954,$
$B' = 135° 58' 32'',$
$C' = 5° 47' 16'',$
$c' = 1.4202.$

There are two solutions, for
$A < 90°$, $a < b$, and $> b \sin A$.

$$\begin{aligned} \text{colog}\, a &= 9.05968 - 10 \\ \log b &= 0.99065 \\ \log \sin A &= \underline{9.79163 \qquad\;} \\ \log \sin B &= 9.84196 \end{aligned}$$

$B = 44° \; 1' 28''.$
$\therefore B' = 135° 58' 32''.$
$\therefore C = 97° 44' 20''.$
$\therefore C' = 5° 47' 16''.$

$$\begin{aligned} \log a &= 0.94032 \\ \log \sin C &= 9.99602 \\ \text{colog} \sin A &= \underline{0.20837} \\ \log c &= 1.14471 \end{aligned}$$

$$c = 13.954.$$

$$\begin{aligned} \log a &= 0.94032 \\ \log \sin C' &= 9.00365 \\ \text{colog} \sin A &= \underline{0.20837} \\ \log c' &= 0.15234 \end{aligned}$$

$$c' = 1.4202.$$

8. Given find
$a = 4.4,$ $B = 90°,$
$b = 5.21,$ $C = 32° 22' 43'',$
$A = 57° 37' 17'';$ $c = 2.7901.$

$$\begin{aligned} \log \sin A &= 9.92661 \\ \log b &= 0.71684 \\ \text{colog}\, a &= \underline{9.35655 - 10} \\ \log \sin B &= 10.00000 \end{aligned}$$

$B = 90°.$
$\therefore C = 32° 22' 43''.$

$$\begin{aligned} \log b &= 0.71684 \\ \log \cos A &= \underline{9.72877} \\ \log c &= 0.44561 \end{aligned}$$

$$c = 2.7901.$$

9. Given $a = 34,$
$b = 22,$
$B = 30° 20';$
find $A = 51° 18' 27'',$
$C = 98° 21' 33'',$
$c = 43.098,$
$A' = 128° 41' 33'',$
$C' = 20° 58' 27'',$
$c' = 15.593.$

Here $b < a$, but $> a \sin B$, and $B < 90°$.

$\therefore$ two solutions.

$$\begin{aligned} \log a &= 1.53148 \\ \log \sin B &= 9.70332 \\ \text{colog}\, b &= \underline{8.65758 - 10} \\ \log \sin A &= 9.89238 \end{aligned}$$

$A = 51° 18' 27''.$
$\therefore A' = 128° 41' 33''.$
$\therefore C = 98° 21' 33''.$
$\therefore C' = 20° 58' 27''.$

$$\begin{aligned}
\log a &= 1.53148\\
\log\sin C &= 9.99536\\
\operatorname{colog}\sin A &= \underline{0.10762}\\
\log c &= 1.63446\\
c &= 43.098.
\end{aligned}$$

$$\begin{aligned}
\log a &= 1.53148\\
\log\sin C' &= 9.55382\\
\operatorname{colog}\sin A &= \underline{0.10762}\\
\log c' &= 1.19292\\
c' &= 15.593.
\end{aligned}$$

10. Given

$b = 19$, $c = 18$, $C = 15^\circ 49'$;

find

$$\begin{aligned}
B &= 16^\circ 43' 13'',\\
A &= 147^\circ 27' 47'',\\
a &= 35.519,\\
B' &= 163^\circ 16' 47'',\\
A' &= 0^\circ 54' 13'',\\
a' &= 1.0415.
\end{aligned}$$

There are two solutions,

for $c < b$,

but $c > b \sin C$,

and $C < 90^\circ$.

$$\begin{aligned}
\log b &= 1.27875\\
\log\sin C &= 9.43546\\
\operatorname{colog} c &= \underline{8.74473 - 10}\\
\log\sin B &= 9.45894\\
B &= 16^\circ 43' 13''.\\
\therefore B' &= 163^\circ 16' 47''.\\
\therefore A &= 147^\circ 27' 47''.\\
\therefore A' &= 0^\circ 54' 13''.
\end{aligned}$$

$$\begin{aligned}
\log b &= 1.27875\\
\operatorname{colog}\sin B &= 0.54106\\
\log\sin A &= \underline{9.73065}\\
\log a &= 1.55046\\
a &= 35.519.
\end{aligned}$$

$$\begin{aligned}
\log b &= 1.27875\\
\operatorname{colog}\sin B' &= 0.54106\\
\log\sin A' &= \underline{8.19784}\\
\log a' &= 0.01765\\
a' &= 1.0415.
\end{aligned}$$

11. Given $a = 75$, $b = 29$, $B = 16^\circ 15' 36''$; find the difference between the areas of the two corresponding triangles.

The triangle which is the difference of the two triangles has for its altitude $a \sin B$, and two of its sides are of length 29.

$$\begin{aligned}
\log a &= 1.87506\\
\log\sin B &= \underline{9.44715}\\
\log(a\sin B) &= 1.32221\\
a\sin B &= 21.\\
29^2 - 21^2 &= (29-21)(29+21)\\
&= 8 \times 50 = 400.\\
\therefore \sqrt{29^2 - 21^2} &= 20.
\end{aligned}$$

Hence, the base of the triangle is $2 \times 20 = 40$, and its altitude 21. Its area is therefore $\frac{1}{2} \times 40 \times 21 = 420$.

12. Given in a parallelogram the side a, a diagonal d, and the angle A made by the two diagonals; find the other diagonal.

Special case: $a = 35$, $d = 63$, $A = 21^\circ 36' 30''$.

$$\begin{aligned}
a &= 35.\\
\tfrac{1}{2} d &= 31.5.\\
A &= 21^\circ 36' 30''.
\end{aligned}$$

$$\begin{aligned}
\operatorname{colog} a &= 8.45593 - 10\\
\log \tfrac{1}{2} d &= 1.49831\\
\log\sin A &= \underline{9.56615}\\
\log\sin B &= 9.52039\\
B &= 19^\circ 21' 20''.\\
C &= 139^\circ\ 2' 10''.
\end{aligned}$$

$$\begin{aligned}
\log a &= 1.54407\\
\log\sin C &= 9.81663\\
\operatorname{colog}\sin A &= \underline{0.43385}\\
\log \tfrac{1}{2} d' &= 1.79455\\
\tfrac{1}{2} d' &= 62.3086.\\
d' &= 124.617.
\end{aligned}$$

EXERCISE XIX. PAGE 78.

1. Given	find
$a = 77.99$,	$A = 51° 15'$,
$b = 83.39$,	$B = 56° 30'$.
$C = 72° 15'$;	$c = 95.24$.

$$\begin{aligned} b + a &= 161.38. \\ b - a &= 5.4. \\ B + A &= 107° 45'. \\ \tfrac{1}{2}(B + A) &= 53° 52' 30''. \end{aligned}$$

$$\begin{aligned} \log(b - a) &= 0.73239 \\ \text{colog}(b + a) &= 7.79215 - 10 \\ \log\tan\tfrac{1}{2}(B + A) &= \underline{0.13675} \\ \log\tan\tfrac{1}{2}(B - A) &= 8.66129 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(B - A) &= 2° 37' 30''. \\ A &= 51° 15'. \\ B &= 56° 30'. \end{aligned}$$

$$\begin{aligned} \log b &= 1.92111 \\ \log\sin C &= 9.97882 \\ \text{colog}\sin B &= \underline{0.07889} \\ \log c &= 1.97882 \end{aligned}$$

$$c = 95.24.$$

2. Given	find
$b = 872.5$,	$B = 60° 45' \ 2''$,
$c = 632.7$,	$C = 39° 14' 58''$,
$A = 80°$;	$a = 984.83$.

$$\begin{aligned} b - c &= 239.8. \\ b + c &= 1505.2. \\ B + C &= 100°. \\ \tfrac{1}{2}(B + C) &= 50°. \end{aligned}$$

$$\begin{aligned} \log(b - c) &= 2.37985 \\ \text{colog}(b + c) &= 6.82240 - 10 \\ \log\tan\tfrac{1}{2}(B + C) &= \underline{0.07619} \\ \log\tan\tfrac{1}{2}(B - C) &= 9.27844 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(B - C) &= 10° 45' \ 2''. \\ B &= 60° 45' \ 2''. \\ C &= 39° 14' 58''. \end{aligned}$$

$$\begin{aligned} \log b &= 2.94077 \\ \log\sin A &= 9.99335 \\ \text{colog}\sin B &= \underline{0.05924} \\ \log a &= 2.99336 \end{aligned}$$

$$a = 984.83.$$

3. Given	find
$a = 17$,	$A = 77° 12' 53''$,
$b = 12$,	$B = 43° 30' \ 7''$,
$C = 59° 17'$;	$c = 14.987$.

$$\begin{aligned} a + b &= 29. \\ a - b &= 5. \\ A + B &= 120° 43'. \\ \tfrac{1}{2}(A + B) &= 60° 21' 30''. \end{aligned}$$

$$\begin{aligned} \log(a - b) &= 0.69897 \\ \text{colog}(a + b) &= 8.53760 - 10 \\ \log\tan\tfrac{1}{2}(A + B) &= \underline{10.24486} \\ \log\tan\tfrac{1}{2}(A - B) &= 9.48143 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(A - B) &= 16° 51' 23''. \\ A &= 77° 12' 53''. \\ B &= 43° 30' \ 7''. \end{aligned}$$

$$\begin{aligned} \log b &= 1.07918 \\ \log\sin C &= 9.93435 \\ \text{colog}\sin B &= \underline{0.16217} \\ \log c &= 1.17570 \end{aligned}$$

$$c = 14.987.$$

4. Given	find
$b = \sqrt{5}$,	$B = 93° 28' 36''$,
$c = \sqrt{3}$,	$C = 50° 38' 24''$,
$A = 35° 53'$;	$a = 1.3131$.

$$\begin{aligned} \sqrt{5} &= 2.2361. \\ \sqrt{3} &= 1.7321. \\ b + c &= 3.9681. \\ b - c &= 0.5040. \\ B + C &= 144° 7'. \\ \tfrac{1}{2}(B + C) &= 72° 3' 30''. \end{aligned}$$

$$\begin{aligned}
\log(b-c) &= 9.70243-10\\
\operatorname{colog}(b+c) &= 9.40142-10\\
\log\tan\tfrac{1}{2}(B+C) &= \underline{10.48973}\\
\log\tan\tfrac{1}{2}(B-C) &= 9.59358
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}(B-C) &= 21^\circ\,25'\,6''.\\
B &= 93^\circ\,28'\,36''.\\
C &= 50^\circ\,38'\,24''.
\end{aligned}$$

$$\begin{aligned}
\log c &= 0.23856\\
\log\sin A &= 9.76800\\
\operatorname{colog}\sin C &= \underline{0.11172}\\
\log a &= 0.11828\\
a &= 1.3131.
\end{aligned}$$

5.

Given	find
$a = 0.917$,	$A = 132^\circ\,18'\,27''$,
$b = 0.312$,	$B = 14^\circ\,34'\,24''$,
$C = 33^\circ\,7'\,9''$;	$c = 0.6775$.

$$\begin{aligned}
a+b &= 1.229.\\
a-b &= 0.605.\\
A+B &= 146^\circ\,52'\,51''.\\
\tfrac{1}{2}(A+B) &= 73^\circ\,26'\,25''.
\end{aligned}$$

$$\begin{aligned}
\log(a-b) &= 9.78176-10\\
\operatorname{colog}(a+b) &= 9.91045-10\\
\log\tan\tfrac{1}{2}(A+B) &= \underline{10.52674}\\
\log\tan\tfrac{1}{2}(A-B) &= 10.21895
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}(A-B) &= 58^\circ\,52'\,2''.\\
A &= 132^\circ\,18'\,27''.\\
B &= 14^\circ\,34'\,24''.
\end{aligned}$$

$$\begin{aligned}
\log b &= 9.49415-10\\
\log\sin C &= 9.73750\\
\operatorname{colog}\sin B &= \underline{0.59926}\\
\log c &= 9.83091-10\\
c &= 0.6775.
\end{aligned}$$

6.

Given	find
$a = 13.715$,	$A = 118^\circ\,55'\,49''$,
$c = 11.214$,	$C = 45^\circ\,41'\,35''$,
$B = 15^\circ\,22'\,36''$;	$b = 4.1554$.

$$\begin{aligned}
a-c &= 2.501.\\
a+c &= 24.929.\\
A+C &= 164^\circ\,37'\,24''.\\
\tfrac{1}{2}(A+C) &= 82^\circ\,18'\,42''.
\end{aligned}$$

$$\begin{aligned}
\log(a-c) &= 0.39811\\
\operatorname{colog}(a+c) &= 8.60330-10\\
\log\tan\tfrac{1}{2}(A+C) &= \underline{10.86968}\\
\log\tan\tfrac{1}{2}(A-C) &= 9.87109
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}(A-C) &= 36^\circ\,37'\,7''.\\
A &= 118^\circ\,55'\,49''.\\
C &= 45^\circ\,41'\,35''.
\end{aligned}$$

$$\begin{aligned}
\log a &= 1.13720\\
\log\sin B &= 9.42352\\
\operatorname{colog}\sin A &= \underline{0.05789}\\
\log b &= 0.61861\\
b &= 4.1554.
\end{aligned}$$

7.

Given	find
$b = 3000.9$,	$B = 65^\circ\,13'\,51''$,
$c = 1587.2$,	$C = 28^\circ\,42'\,5''$,
$A = 86^\circ\,4'\,4''$;	$a = 3297.2$.

$$\begin{aligned}
b+c &= 4588.1.\\
b-c &= 1413.7.\\
B+C &= 93^\circ\,55'\,56''.\\
\tfrac{1}{2}(B+C) &= 46^\circ\,57'\,58''.
\end{aligned}$$

$$\begin{aligned}
\log(b-c) &= 3.15036\\
\operatorname{colog}(b+c) &= 6.33837-10\\
\log\tan\tfrac{1}{2}(B+C) &= \underline{10.02983}\\
\log\tan\tfrac{1}{2}(B-C) &= 9.51856
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}(B-C) &= 18^\circ\,15'\,53''.\\
C &= 28^\circ\,42'\,5''.\\
B &= 65^\circ\,13'\,51''.
\end{aligned}$$

$$\begin{aligned}
\log b &= 3.47726\\
\log\sin A &= 9.99898\\
\operatorname{colog}\sin B &= \underline{0.04191}\\
\log a &= 3.51815\\
a &= 3297.2.
\end{aligned}$$

8.

Given	find
$a = 4527$,	$A = 68^\circ\,29'\,15''$,
$b = 3465$,	$B = 45^\circ\,24'\,18''$,
$C = 66^\circ\,6'\,27''$;	$c = 4449$.

$$\begin{aligned} a+b &= 7992. \\ a-b &= 1062. \\ A+B &= 113^\circ\ 53'\ 33''. \\ \tfrac{1}{2}(A+B) &= 56^\circ\ 56'\ 47''. \end{aligned}$$

$$\begin{aligned} \log(a-b) &= 3.02612 \\ \text{colog}(a+b) &= 6.09734-10 \\ \log\tan\tfrac{1}{2}(A+B) &= \underline{10.18659} \\ \log\tan\tfrac{1}{2}(A-B) &= 9.31005 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(A-B) &= 11^\circ\ 32'\ 28''. \\ A &= 68^\circ\ 29'\ 15''. \\ B &= 45^\circ\ 24'\ 18''. \end{aligned}$$

$$\begin{aligned} \log a &= 3.65581 \\ \log\sin C &= 9.96109 \\ \text{colog}\sin A &= \underline{0.03136} \\ \log c &= 3.64826 \end{aligned}$$

$$c = 4449.$$

9.

Given	find
$a = 55.14,$	$A = 117^\circ\ 24'\ 32'',$
$b = 33.09,$	$B = 32^\circ\ 11'\ 28'',$
$C = 30^\circ\ 24';$	$c = 31.431.$

$$\begin{aligned} a+b &= 88.23. \\ a-b &= 22.05. \\ A+B &= 149^\circ\ 36'. \\ \tfrac{1}{2}(A+B) &= 74^\circ\ 48'. \end{aligned}$$

$$\begin{aligned} \log(a-b) &= 1.34341 \\ \text{colog}(a+b) &= 8.05438-10 \\ \log\tan\tfrac{1}{2}(A+B) &= \underline{10.56592} \\ \log\tan\tfrac{1}{2}(A-B) &= 9.96371 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(A-B) &= 42^\circ\ 36'\ 32''. \\ A &= 117^\circ\ 24'\ 32''. \\ B &= 32^\circ\ 11'\ 28''. \end{aligned}$$

$$\begin{aligned} \log b &= 1.51970 \\ \log\sin C &= 9.70418 \\ \text{colog}\sin B &= \underline{0.27348} \\ \log c &= 1.49736 \end{aligned}$$

$$c = 31.431.$$

10.

Given	find
$a = 47.99,$	$A = 2^\circ\ 46'\ 8'',$
$b = 33.14,$	$B = 1^\circ\ 54'\ 42'',$
$C = 175^\circ\ 19'\ 10'';$	$c = 81.066.$

$$\begin{aligned} a+b &= 81.13. \\ a-b &= 14.85. \\ A+B &= 4^\circ\ 40'\ 50''. \\ \tfrac{1}{2}(A+B) &= 2^\circ\ 20'\ 25''. \end{aligned}$$

$$\begin{aligned} \log(a-b) &= 1.17173 \\ \text{colog}(a+b) &= 8.09082-10 \\ \log\tan\tfrac{1}{2}(A+B) &= \underline{8.61138} \\ \log\tan\tfrac{1}{2}(A-B) &= 7.87393 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(A-B) &= 0^\circ\ 25'\ 43''. \\ A &= 2^\circ\ 46'\ 8''. \\ B &= 1^\circ\ 54'\ 42''. \end{aligned}$$

$$\begin{aligned} \log b &= 1.52035 \\ \log\sin C &= 8.91169 \\ \text{colog}\sin B &= \underline{1.47680} \\ \log c &= 1.90884 \end{aligned}$$

$$c = 81.066.$$

11. If two sides of a triangle are each equal to 6, and the included angle is 60°, find the third side.

Since $a = b,$

$$\begin{aligned} A &= B, \\ A+B &= 120^\circ. \\ \therefore A &= B = C = 60^\circ. \\ \therefore a &= b = c = 6. \end{aligned}$$

12. If two sides of a triangle are each equal to 6, and the included angle is 120°, find the third side.

$$\begin{aligned} A+B &= 60^\circ. \\ \therefore A &= B = 30^\circ, \\ a &= 6 = b. \end{aligned}$$

$$\begin{aligned} \log a &= 0.77815 \\ \log\sin C &= 9.93753 \\ \text{colog}\sin A &= \underline{0.30103} \\ \log c &= 1.01671 \end{aligned}$$

$$c = 10.392.$$

13. Apply Solution I to the case in which a is equal to b; that is, the case in which the triangle is isosceles.

If $a = b$, the formula

$$\tan \tfrac{1}{2}(A-B) = \frac{a-b}{a+b} \times \tan \tfrac{1}{2}(A+B)$$

becomes

$$\begin{aligned} \tan \tfrac{1}{2}(A - B) &= 0. \\ \therefore A - B &= 0. \\ A &= B \\ &= \tfrac{1}{2}(180^\circ - C) \\ &= 90^\circ - \tfrac{1}{2}C. \\ c &= \frac{a \sin C}{\sin A}. \end{aligned}$$

14. If two sides of a triangle are 10 and 11, and the included angle is 50°, find the third side.

$$\begin{aligned} a + b &= 21. \\ a - b &= 1. \\ A + B &= 130^\circ. \\ \tfrac{1}{2}(A + B) &= 65^\circ. \\ \log(a - b) &= 0.00000 \\ \operatorname{colog}(a + b) &= 8.67778 - 10 \\ \log \tan \tfrac{1}{2}(A + B) &= \underline{10.33133} \\ \log \tan \tfrac{1}{2}(A - B) &= 9.00911 \\ \tfrac{1}{2}(A - B) &= 5^\circ\ 49'\ 51''. \\ A &= 70^\circ\ 49'\ 51''. \\ B &= 59^\circ\ 10'\ 9''. \\ \log b &= 1.00000 \\ \log \sin C &= 9.88425 \\ \operatorname{colog} \sin B &= \underline{0.06617} \\ \log c &= 0.95042 \\ c &= 8.9212. \end{aligned}$$

15. If two sides of a triangle are 43.301 and 25, and the included angle is 30°, find the third side.

$$\begin{aligned} a + b &= 68.301. \\ a - b &= 18.301. \\ A + B &= 150^\circ. \\ \tfrac{1}{2}(A + B) &= 75^\circ. \\ \log(a - b) &= 1.26247 \\ \operatorname{colog}(a + b) &= 8.16557 - 10 \\ \log \tan \tfrac{1}{2}(A + B) &= \underline{10.57195} \\ \log \tan \tfrac{1}{2}(A - B) &= 9.99999 \\ \tfrac{1}{2}(A - B) &= 45^\circ. \\ A &= 120^\circ. \\ B &= 30^\circ. \end{aligned}$$

$\therefore$ in isosceles triangle ABC

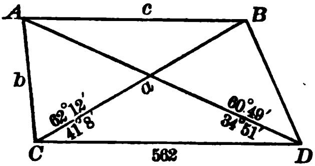
$$c = b = 25.$$

16. In order to find the distance between two objects A and B separated by a swamp, a station C was chosen, and the distances $CA = 3825$ yards, $CB = 3475.6$ yards, together with the angle $ACB = 62^\circ\ 31'$, were measured. Find the distance from A to B.

$$\begin{aligned} b + a &= 7300.6. \\ b - a &= 349.4. \\ B + A &= 117^\circ\ 29'. \\ \tfrac{1}{2}(B + A) &= 58^\circ\ 44'\ 30''. \\ \log(b - a) &= 2.54332 \\ \operatorname{colog}(b + a) &= 6.13664 - 10 \\ \log \tan \tfrac{1}{2}(B + A) &= \underline{10.21680} \\ \log \tan \tfrac{1}{2}(B - A) &= 8.89676 \\ \tfrac{1}{2}(B - A) &= 4^\circ\ 30'\ 29''. \\ B &= 63^\circ\ 14'\ 59''. \\ A &= 54^\circ\ 14'\ 1''. \\ \log b &= 3.58263 \\ \log \sin C &= 9.94799 \\ \operatorname{colog} \sin B &= \underline{0.04916} \\ \log c &= 3.57978 \\ c &= 3800. \\ \therefore AB &= 3800 \text{ yards.} \end{aligned}$$

17. Two inaccessible objects A and B are each viewed from two

stations, C and D, on the same side of AB and 562 yards apart. The angle ACB is 62° 12′, BCD 41° 8′, ADB 60° 49′, and ADC 34° 51′; required the distance AB.



In triangle ACD

$$A = 180° - (C + D) = 41° 49'.$$

$$\frac{b}{562} = \frac{\sin 34° 51'}{\sin 41° 49'}.$$

$$\therefore b = \frac{562 \sin 34° 51'}{\sin 41° 49'}.$$

$$\begin{aligned} \log 562 &= 2.74974 \\ \log \sin 34° 51' &= 9.75696 \\ \text{colog} \sin 41° 49' &= \underline{0.17604} \\ \log b &= 2.68274 \\ b &= 481.66. \end{aligned}$$

In triangle CBD

$$B = 180° - (C + D) = 43° 12'.$$

$$\frac{a}{562} = \frac{\sin 95° 40'}{\sin 43° 12'}.$$

$$\therefore a = \frac{562 \cos 5° 40'}{\sin 43° 12'}.$$

$$\begin{aligned} \log 562 &= 2.74974 \\ \log \cos 5° 40' &= 9.99787 \\ \text{colog} \sin 43° 12' &= \underline{0.16460} \\ \log a &= 2.91221 \\ a &= 816.98. \end{aligned}$$

In triangle ACB

$$\tan \tfrac{1}{2}(A - B) = \frac{a-b}{a+b} \times \tan \tfrac{1}{2}(A+B)$$

$$\begin{aligned} \tfrac{1}{2}(A + B) &= \tfrac{1}{2}(180° - C) \\ &= 58° 54'. \end{aligned}$$

$$\begin{aligned} a - b &= 816.98 - 481.66 \\ &= 335.32. \end{aligned}$$

$$\begin{aligned} a + b &= 816.98 + 481.66 \\ &= 1298.64. \end{aligned}$$

$$\begin{aligned} \log (a - b) &= 2.52548 \\ \text{colog} (a + b) &= 6.88651 - 10 \\ \log \tan \tfrac{1}{2}(A + B) &= \underline{10.21951} \\ \log \tan \tfrac{1}{2}(A - B) &= 9.63150 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(A - B) &= 23° 10' 26''. \\ A &= 82° \ 4' 26''. \end{aligned}$$

$$\begin{aligned} \log a &= 2.91221 \\ \log \sin C &= 9.94674 \\ \text{colog} \sin A &= \underline{0.00417} \\ \log c &= 2.86312 \\ c &= 729.67. \end{aligned}$$

$$\therefore AB = 729.67 \text{ yards}.$$

18. Two trains start at the same time from the same station, and move along straight tracks that form an angle of 30°, one train at the rate of 30 miles an hour, the other at the rate of 40 miles an hour. How far apart are the trains at the end of half an hour?

$$\begin{aligned} a + b &= 35. \\ a - b &= 5. \\ A + B &= 150°. \\ \tfrac{1}{2}(A + B) &= 75°. \end{aligned}$$

$$\begin{aligned} \log (a - b) &= 0.69897 \\ \text{colog} (a + b) &= 8.45593 - 10 \\ \log \tan \tfrac{1}{2}(A + B) &= \underline{10.57195} \\ \log \tan \tfrac{1}{2}(A - B) &= 9.72685 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(A - B) &= 28° \ 3' 52''. \\ B &= 46° 56' \ 8''. \\ A &= 103° \ 3' 52''. \end{aligned}$$

$$\begin{aligned}\log b &= 1.17609\\ \log\sin C &= 9.69897\\ \text{colog}\sin B &= \underline{0.13633}\\ \log c &= 1.01139\\ c &= 10.266.\end{aligned}$$

Therefore, the trains are 10.266 miles apart.

19. In a parallelogram given the two diagonals 5 and 6, and the angle that they form 49° 18′; find the sides.

In the parallelogram $ABDE$

let $EB = 6$, and $AD = 5$,

and $\angle BCA = 49° 18'$.

In triangle ACB

let $BC = a = 3.$

$AC = b = 2.5.$

Find $AB = c.$

$$\begin{aligned}a - b &= 0.5.\\ a + b &= 5.5.\\ A + B &= 130° 42'.\\ \tfrac{1}{2}(A + B) &= 65° 21'.\\ \log(a - b) &= 9.69897 - 10\\ \text{colog}(a + b) &= 9.25964 - 10\\ \log\tan\tfrac{1}{2}(A + B) &= \underline{10.33829}\\ \log\tan\tfrac{1}{2}(A - B) &= 9.29690\\ \tfrac{1}{2}(A - B) &= 11° 12' 20''.\\ A &= 76° 33' 20''.\\ B &= 54° \ 8' 40''.\end{aligned}$$

$$\begin{aligned}\log a &= 0.47712\\ \log\sin C &= 9.87975\\ \text{colog}\sin A &= \underline{0.01207}\\ \log c &= 0.36894\\ c = AB &= 2.3385.\end{aligned}$$

In triangle AEC

$$\begin{aligned}EC &= a = 3,\\ AC &= b = 2.5,\\ \angle ACE &= 130° 42'.\end{aligned}$$

$$\begin{aligned}A + E &= 49° 18'.\\ \tfrac{1}{2}(A + E) &= 24° 39'.\\ \log(a - b) &= 9.69897 - 10\\ \text{colog}(a + b) &= 9.25964 - 10\\ \log\tan\tfrac{1}{2}(A + E) &= \underline{9.66171}\\ \log\tan\tfrac{1}{2}(A - E) &= 8.62032\\ \tfrac{1}{2}(A - E) &= 2° 23' 20''.\\ A &= 27° \ 2' 20''.\end{aligned}$$

$$\begin{aligned}\log a &= 0.47712\\ \log\sin C &= 9.87975\\ \text{colog}\sin A &= \underline{0.34238}\\ \log c &= 0.69925\\ c = EA &= 5.0032.\end{aligned}$$

20. In a triangle one angle is 139° 54′, and the sides forming the angle have the ratio 5 : 9. Find the other two angles.

$$\begin{aligned}a &= 9.\\ b &= 5.\\ a + b &= 14.\\ a - b &= 4.\\ A + B &= 40° 6'.\\ \tfrac{1}{2}(A + B) &= 20° 3'.\\ \log(a - b) &= 0.60206\\ \text{colog}(a + b) &= 8.85387 - 10\\ \log\tan\tfrac{1}{2}(A + B) &= \underline{9.56224}\\ \log\tan\tfrac{1}{2}(A - B) &= 9.01817\\ \tfrac{1}{2}(A - B) &= 5° 57' 10''.\\ A &= 26° \ 0' 10''.\\ B &= 14° \ 5' 50''.\end{aligned}$$

21. In order to find the distance between two objects A and B separated by a pond, a station C was chosen, and the distances $CA = 426$ yards, $CB = 322.4$ yards, together with the angle $ACB = 68° 42'$, were measured. Find the distance from A to B.

$$\begin{aligned}
b + a &= 748.4.\\
b - a &= 103.6.\\
B + A &= 111^\circ\ 18'.\\
\tfrac{1}{2}(B + A) &= 55^\circ\ 39'.\\
\log(b - a) &= 2.01536\\
\operatorname{colog}(b + a) &= 7.12587 - 10\\
\log\tan\tfrac{1}{2}(B + A) &= \underline{10.16530}\\
\log\tan\tfrac{1}{2}(B - A) &= 9.30653
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}(B - A) &= 11^\circ\ 27'\ 1''.\\
\therefore B &= 67^\circ\ 6'\ 1''.\\
\log b &= 2.62941\\
\log\sin C &= 9.96927\\
\operatorname{colog}\sin B &= \underline{0.03565}\\
\log c &= 2.63433\\
c &= 430.85.\\
\therefore AB &= 430.85 \text{ yards.}
\end{aligned}$$

EXERCISE XX. PAGE 83.

1. Given $a = 51$, $b = 65$, $c = 20$; find the angles.

$$\begin{aligned}
a &= 51\\
b &= 65\\
c &= \underline{20}\\
2s &= 136\\
s &= 68.\\
s - a &= 17.\\
s - b &= 3.\\
s - c &= 48.
\end{aligned}$$

$$\begin{aligned}
\operatorname{colog} s &= 8.16749 - 10\\
\operatorname{colog}(s - a) &= 8.76955 - 10\\
\log(s - b) &= 0.47712\\
\log(s - c) &= \underline{1.68124}\\
&\ \ 2\overline{)19.09540 - 20}\\
\log\tan\tfrac{1}{2}A &= 9.54770\\
\tfrac{1}{2}A &= 19^\circ\ 26'\ 24''.\\
A &= 38^\circ\ 52'\ 48''.
\end{aligned}$$

$$\begin{aligned}
\operatorname{colog} s &= 8.16749 - 10\\
\operatorname{colog}(s - b) &= 9.52288 - 10\\
\log(s - a) &= 1.23045\\
\log(s - c) &= \underline{1.68124}\\
&\ \ 2\overline{)20.60206 - 20}\\
\log\tan\tfrac{1}{2}B &= 10.30103\\
\tfrac{1}{2}B &= 63^\circ\ 26'\ 6''.\\
B &= 126^\circ\ 52'\ 12''.\\
A + B &= 165^\circ\ 45'.\\
\therefore C &= 14^\circ\ 15'.
\end{aligned}$$

2. Given $a = 78$, $b = 101$, $c = 29$; find the angles.

$$\begin{aligned}
a &= 78\\
b &= 101\\
c &= \underline{29}\\
2s &= 208\\
s &= 104.\\
s - a &= 26.\\
s - b &= 3.\\
s - c &= 75.
\end{aligned}$$

$$\begin{aligned}
\operatorname{colog} s &= 7.98297 - 10\\
\operatorname{colog}(s - a) &= 8.58503 - 10\\
\log(s - b) &= 0.47712\\
\log(s - c) &= \underline{1.87506}\\
&\ \ 2\overline{)18.92018 - 20}\\
\log\tan\tfrac{1}{2}A &= 9.46009\\
\tfrac{1}{2}A &= 16^\circ\ 5'\ 27''.\\
A &= 32^\circ\ 10'\ 55''.
\end{aligned}$$

$$\begin{aligned}
\operatorname{colog} s &= 7.98297 - 10\\
\operatorname{colog}(s - b) &= 9.52288 - 10\\
\log(s - a) &= 1.41497\\
\log(s - c) &= \underline{1.87506}\\
&\ \ 2\overline{)20.79588 - 20}\\
\log\tan\tfrac{1}{2}B &= 10.39794\\
\tfrac{1}{2}B &= 68^\circ\ 11'\ 55''.\\
B &= 136^\circ\ 23'\ 50''.\\
A + B &= 168^\circ\ 34'\ 45''.\\
\therefore C &= 11^\circ\ 25'\ 15''.
\end{aligned}$$

3. Given $a = 111$, $b = 145$, $c = 40$; find the angles.

$$\begin{aligned} a &= 111 \\ b &= 145 \\ c &= \underline{\ 40} \\ 2s &= 296 \end{aligned}$$

$$\begin{aligned} s &= 148. \\ s - a &= 37. \\ s - b &= 3. \\ s - c &= 108. \end{aligned}$$

$$\begin{aligned} \text{colog}\, s &= 7.82974 - 10 \\ \text{colog}\,(s - a) &= 8.43180 - 10 \\ \log (s - b) &= 0.47712 \\ \log (s - c) &= \underline{2.03342} \\ & 2\,\overline{)18.77208 - 20} \\ \log \tan \tfrac{1}{2} A &= 9.38604 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2} A &= 13^\circ\, 40'\, 16''. \\ A &= 27^\circ\, 20'\, 32''. \end{aligned}$$

$$\begin{aligned} \text{colog}\, s &= 7.82974 - 10 \\ \log (s - a) &= 1.56820 \\ \text{colog}\,(s - b) &= 9.52288 - 10 \\ \log (s - c) &= \underline{2.03342} \\ & 2\,\overline{)20.95424 - 20} \\ \log \tan \tfrac{1}{2} B &= 10.47712 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2} B &= 71^\circ\, 33'\, 54''. \\ B &= 143^\circ\ \ 7'\, 48''. \\ B + A &= 170^\circ\, 28'\, 20''. \\ \therefore C &= 9^\circ\, 31'\, 40''. \end{aligned}$$

4. Given $a = 21$, $b = 26$, $c = 31$; find the angles.

$$\begin{aligned} a &= 21 \\ b &= 26 \\ c &= \underline{31} \\ 2s &= 78 \end{aligned}$$

$$\begin{aligned} s &= 39. \\ s - a &= 18. \\ s - b &= 13. \\ s - c &= 8. \end{aligned}$$

$$\begin{aligned} \text{colog}\, s &= 8.40894 - 10 \\ \text{colog}\,(s - a) &= 8.74473 - 10 \\ \log (s - b) &= 1.11394 \\ \log (s - c) &= \underline{0.90309} \\ & 2\,\overline{)19.17070 - 20} \\ \log \tan \tfrac{1}{2} A &= 9.58535 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2} A &= 21^\circ\, 3'\ \ 6.3''. \\ \therefore A &= 42^\circ\, 6'\, 13''. \end{aligned}$$

$$\begin{aligned} \text{colog}\, s &= 8.40894 - 10 \\ \log (s - a) &= 1.25527 \\ \text{colog}\,(s - b) &= 8.88606 - 10 \\ \log (s - c) &= \underline{0.90309} \\ & 2\,\overline{)19.45336 - 20} \\ \log \tan \tfrac{1}{2} B &= 9.72668 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2} B &= 28^\circ\ \ 3'\, 18''. \\ \therefore B &= 56^\circ\ \ 6'\, 36''. \\ A + B &= 98^\circ\, 12'\, 49''. \\ \therefore C &= 81^\circ\, 47'\, 11''. \end{aligned}$$

5. Given $a = 19$, $b = 34$, $c = 49$; find the angles.

$$\begin{aligned} a &= 19 \\ b &= 34 \\ c &= \underline{49} \\ 2s &= 102 \end{aligned}$$

$$\begin{aligned} s &= 51. \\ s - a &= 32. \\ s - b &= 17. \\ s - c &= 2. \end{aligned}$$

$$\begin{aligned} \text{colog}\, s &= 8.29243 - 10 \\ \text{colog}\,(s - a) &= 8.49485 - 10 \\ \log (s - b) &= 1.23045 \\ \log (s - c) &= \underline{0.30103} \\ & 2\,\overline{)18.31876 - 20} \\ \log \tan \tfrac{1}{2} A &= 9.15938 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2} A &= 8^\circ\, 12'\, 48''. \\ A &= 16^\circ\, 25'\, 36''. \end{aligned}$$

$$\begin{aligned}
\operatorname{colog} s &= 8.29243 - 10\\
\log(s-a) &= 1.50515\\
\operatorname{colog}(s-b) &= 8.76955 - 10\\
\log(s-c) &= \underline{0.30103}\\
&\; 2\,\overline{)\,18.86816 - 20}\\
\log\tan\tfrac{1}{2}B &= 9.43408
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}B &= 15^\circ\, 12'.\\
B &= 30^\circ\, 24'.\\
\therefore C &= 133^\circ\, 10'\, 24''.
\end{aligned}$$

6. Given $a = 43$, $b = 50$, $c = 57$; find the angles.

$$\begin{aligned}
a &= 43\\
b &= 50\\
c &= \underline{57}\\
2s &= 150\\
s &= 75.\\
s-a &= 32.\\
s-b &= 25.\\
s-c &= 18.
\end{aligned}$$

$$\begin{aligned}
\operatorname{colog} s &= 8.12494 - 10\\
\operatorname{colog}(s-a) &= 8.49485 - 10\\
\log(s-b) &= 1.39794\\
\log(s-c) &= \underline{1.25527}\\
&\; 2\,\overline{)\,19.27300 - 20}\\
\log\tan\tfrac{1}{2}A &= 9.63650
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}A &= 23^\circ\, 24'\, 47.6''.\\
A &= 46^\circ\, 49'\, 35''.
\end{aligned}$$

$$\begin{aligned}
\operatorname{colog} s &= 8.12494 - 10\\
\log(s-a) &= 1.50515\\
\operatorname{colog}(s-b) &= 8.60206 - 10\\
\log(s-c) &= \underline{1.25527}\\
&\; 2\,\overline{)\,19.48742 - 20}\\
\log\tan\tfrac{1}{2}B &= 9.74371
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}B &= 28^\circ\, 59'\, 52''.\\
B &= 57^\circ\, 59'\, 44''.\\
\therefore C &= 75^\circ\, 10'\, 41''.
\end{aligned}$$

7. Given $a = 37$, $b = 58$, $c = 79$; find the angles.

$$\begin{aligned}
a &= 37\\
b &= 58\\
c &= \underline{79}\\
2s &= 174\\
s &= 87.\\
s-a &= 50.\\
s-b &= 29.\\
s-c &= 8.
\end{aligned}$$

$$\begin{aligned}
\operatorname{colog} s &= 8.06048 - 10\\
\operatorname{colog}(s-a) &= 8.30103 - 10\\
\log(s-b) &= 1.46240\\
\log(s-c) &= \underline{0.90309}\\
&\; 2\,\overline{)\,18.72700 - 20}\\
\log\tan\tfrac{1}{2}A &= 9.36350
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}A &= 13^\circ\, 0'\, 14.5''.\\
A &= 26^\circ\, 0'\, 29''.
\end{aligned}$$

$$\begin{aligned}
\operatorname{colog} s &= 8.06048 - 10\\
\log(s-a) &= 1.69897\\
\operatorname{colog}(s-b) &= 8.53760 - 10\\
\log(s-c) &= \underline{0.90309}\\
&\; 2\,\overline{)\,19.20014 - 20}\\
\log\tan\tfrac{1}{2}B &= 9.60007
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}B &= 21^\circ\, 42'\, 40''.\\
B &= 43^\circ\, 25'\, 20''.\\
\therefore C &= 110^\circ\, 34'\, 11''.
\end{aligned}$$

8. Given $a = 73$, $b = 82$, $c = 91$; find the angles.

$$\begin{aligned}
a &= 73\\
b &= 82\\
c &= \underline{91}\\
2s &= 246\\
s &= 123.\\
s-a &= 50.\\
s-b &= 41.\\
s-c &= 32.
\end{aligned}$$

$$\begin{aligned}
\text{colog}\, s &= 7.91009 - 10\\
\text{colog}\,(s-a) &= 8.30103 - 10\\
\log(s-b) &= 1.61278\\
\log(s-c) &= \underline{1.50515}\\
2&\overline{)\,19.32905 - 20}\\
\log\tan\tfrac{1}{2}A &= 9.66453
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}A &= 24^\circ\, 47'\, 29''.\\
A &= 49^\circ\, 34'\, 58''.
\end{aligned}$$

$$\begin{aligned}
\text{colog}\, s &= 7.91009 - 10\\
\log(s-a) &= 1.69897\\
\text{colog}\,(s-b) &= 8.38722 - 10\\
\log(s-c) &= \underline{1.50515}\\
2&\overline{)\,19.50143 - 20}\\
\log\tan\tfrac{1}{2}B &= 9.75072
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}B &= 29^\circ\, 23'\, 29''.\\
B &= 58^\circ\, 46'\, 58''.\\
\therefore C &= 71^\circ\, 38'\, 4''.
\end{aligned}$$

9. Given $a = 14.493$, $b = 55.4363$, $c = 66.9129$; find the angles.

$$\begin{aligned}
a &= 14.493\\
b &= 55.4363\\
c &= \underline{66.9129}\\
2s &= 136.8422
\end{aligned}$$

$$\begin{aligned}
s &= 68.4211.\\
s-a &= 53.9281.\\
s-b &= 12.9848.\\
s-c &= 1.5082.
\end{aligned}$$

$$\begin{aligned}
\text{colog}\, s &= 8.16481 - 10\\
\text{colog}\,(s-a) &= 8.26819 - 10\\
\log(s-b) &= 1.11343\\
\log(s-c) &= \underline{0.17846}\\
2&\overline{)\,17.72489 - 20}\\
\log\tan\tfrac{1}{2}A &= 8.86245
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}A &= 4^\circ\, 10'\, 0.7''.\\
A &= 8^\circ\, 20'\, 1''.
\end{aligned}$$

$$\begin{aligned}
\text{colog}\, s &= 8.16481 - 10\\
\log(s-a) &= 1.73181\\
\text{colog}\,(s-b) &= 8.88657 - 10\\
\log(s-c) &= \underline{0.17846}\\
2&\overline{)\,18.96165 - 20}\\
\log\tan\tfrac{1}{2}B &= 9.48082
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}B &= 16^\circ\, 50'\, 2.6''.\\
B &= 33^\circ\, 40'\, 5''.\\
\therefore C &= 137^\circ\, 59'\, 54''.
\end{aligned}$$

10. Given $a = \sqrt{5}$, $b = \sqrt{6}$, $c = \sqrt{7}$; find the angles.

$$\begin{aligned}
a &= \sqrt{5} = 2.2361\\
b &= \sqrt{6} = 2.4495\\
c &= \sqrt{7} = \underline{2.6458}\\
2s &= 7.3314
\end{aligned}$$

$$\begin{aligned}
s &= 3.6657.\\
s-a &= 1.4296.\\
s-b &= 1.2162.\\
s-c &= 1.0199.
\end{aligned}$$

$$\begin{aligned}
\text{colog}\, s &= 9.43585 - 10\\
\text{colog}\,(s-a) &= 9.84478 - 10\\
\log(s-b) &= 0.08500\\
\log(s-c) &= \underline{0.00856}\\
2&\overline{)\,19.37419 - 20}\\
\log\tan\tfrac{1}{2}A &= 9.68709
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}A &= 25^\circ\, 56'\, 36''.\\
A &= 51^\circ\, 53'\, 12''.
\end{aligned}$$

$$\begin{aligned}
\text{colog}\,(s-b) &= 9.91500 - 10\\
\log(s-c) &= 0.00856\\
\text{colog}\, s &= 9.43585 - 10\\
\log(s-a) &= \underline{0.15522}\\
2&\overline{)\,19.51463 - 20}\\
\log\tan\tfrac{1}{2}B &= 9.75732
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}B &= 29^\circ\, 45'\, 54''.\\
B &= 59^\circ\, 31'\, 48''.\\
\therefore C &= 68^\circ\, 35'.
\end{aligned}$$

11. Given $a = 6$, $b = 8$, $c = 10$; find the angles.

$$\begin{aligned} a &= 6 \\ b &= 8 \\ c &= \underline{10} \\ 2s &= 24 \\ s &= 12. \\ s - a &= 6. \\ s - b &= 4. \\ s - c &= 2. \end{aligned}$$

$$\begin{aligned} \text{colog}\, s &= 8.92082 - 10 \\ \text{colog}\,(s - a) &= 9.22185 - 10 \\ \log\,(s - b) &= 0.60206 \\ \log\,(s - c) &= \underline{0.30103} \\ 2\,)\, &\underline{19.04576 - 20} \\ \log\tan \tfrac{1}{2}A &= 9.52288 \\ \tfrac{1}{2}A &= 18^\circ\, 26'\ \ 6''. \\ A &= 36^\circ\, 52'\, 12''. \end{aligned}$$

Since $10^2 = 6^2 + 8^2$, the triangle is a right triangle,

and $C = 90^\circ.$

$$\therefore B = 53^\circ\, 7'\, 48''.$$

12. Given $a = 6$, $b = 6$, $c = 10$; find the angles.

$$\begin{aligned} a &= 6 \\ b &= 6 \\ c &= \underline{10} \\ 2s &= 22 \\ s &= 11. \\ s - a &= 5. \\ s - b &= 5. \\ s - c &= 1. \end{aligned}$$

$$\begin{aligned} \text{colog}\, s &= 8.95861 - 10 \\ \log\,(s - a) &= 0.69897 \\ \log\,(s - b) &= 0.69897 \\ \text{colog}\,(s - c) &= \underline{0.00000} \\ 2\,)\, &\underline{20.35655 - 20} \\ \log\tan \tfrac{1}{2}C &= 10.17827 \\ \tfrac{1}{2}C &= 56^\circ\, 26'\, 33''. \\ C &= 112^\circ\, 53'\ \ 6''. \end{aligned}$$

Since this is an isosceles triangle,

$$\begin{aligned} A = B &= \tfrac{1}{2}(180^\circ - C) \\ &= 33^\circ\, 33'\, 27''. \end{aligned}$$

13. Given $a = 6$, $b = 6$, $c = 6$; find the angles.

The triangle is equilateral and is therefore also equiangular.

$$\therefore A = B = C = \tfrac{1}{3} \text{ of } 180^\circ = 60^\circ.$$

14. Given $a = 6$, $b = 9$, $c = 12$; find the angles.

$$\begin{aligned} a &= 6 \\ b &= 9 \\ c &= \underline{12} \\ 2s &= 27 \\ s &= 13.5. \\ s - a &= 7.5. \\ s - b &= 4.5. \\ s - c &= 1.5. \end{aligned}$$

$$\begin{aligned} \text{colog}\, s &= 8.86967 - 10 \\ \text{colog}\, s - a &= 9.12494 - 10 \\ \log s - b &= 0.65321 \\ \log s - c &= \underline{0.17609} \\ 2\,)\, &\underline{18.82391 - 20} \\ \log\tan \tfrac{1}{2}A &= 9.41196 - 10 \\ \tfrac{1}{2}A &= 14^\circ\, 28'\, 39''. \\ A &= 28^\circ\, 57'\, 18''. \end{aligned}$$

$$\begin{aligned} \text{colog}\, s &= 8.86967 - 10 \\ \log s - a &= 0.87506 \\ \text{colog}\, s - b &= 9.34679 - 10 \\ \log s - c &= \underline{0.17609} \\ 2\,)\, &\underline{19.26761 - 20} \\ \log\tan \tfrac{1}{2}B &= 9.63380 - 10 \\ \tfrac{1}{2}B &= 23^\circ\, 17'\ \ 3''. \\ B &= 46^\circ\, 34'\ \ 6''. \\ \therefore C &= 104^\circ\, 28'\, 36''. \end{aligned}$$

15. Given $a = 2$, $b = \sqrt{6}$, $c = \sqrt{3} - 1$; find the angles.

$$\begin{aligned}
a &= 2. \\
b = \sqrt{6} &= 2.44947 \\
c = \sqrt{3} - 1 &= \underline{0.73205} \\
2s &= 5.18152
\end{aligned}$$

$$\begin{aligned}
s &= 2.59076. \\
s - a &= 0.59076. \\
s - b &= 0.14129. \\
s - c &= 1.85871.
\end{aligned}$$

$$\begin{aligned}
\log(s - a) &= 9.77141 - 10 \\
\log(s - b) &= 9.15011 - 10 \\
\log(s - c) &= 0.26921 \\
\text{colog}\, s &= \underline{9.58657 - 10} \\
\log r^2 &= 18.77730 - 20
\end{aligned}$$

$$\log r = 9.38865 - 10.$$

$$\begin{aligned}
\log\tan\tfrac{1}{2}A &= 9.61724. \\
\log\tan\tfrac{1}{2}B &= 10.23854. \\
\log\tan\tfrac{1}{2}C &= 9.11944.
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}A &= 22^\circ\ 30'. \\
\tfrac{1}{2}B &= 60^\circ. \\
\tfrac{1}{2}C &= 7^\circ\ 30'.
\end{aligned}$$

$$\begin{aligned}
A &= 45^\circ. \\
B &= 120^\circ. \\
C &= 15^\circ.
\end{aligned}$$

16. Given $a = 2$, $b = \sqrt{6}$, $c = \sqrt{3} + 1$; find the angles.

$$\begin{aligned}
a &= 2. \\
b = \sqrt{6} &= 2.44947 \\
c = \sqrt{3} + 1 &= \underline{2.73205} \\
2s &= 7.18152
\end{aligned}$$

$$\begin{aligned}
s &= 3.59076. \\
s - a &= 1.59076. \\
s - b &= 1.14129. \\
s - c &= 0.85871.
\end{aligned}$$

$$\begin{aligned}
\log(s - a) &= 0.20160 \\
\log(s - b) &= 0.05740 \\
\log(s - c) &= 9.93385 - 10 \\
\text{colog}\, s &= \underline{9.44481 - 10} \\
\log r^2 &= 19.63766 - 20
\end{aligned}$$

$$\log r = 9.81883 - 10.$$

$$\begin{aligned}
\log\tan\tfrac{1}{2}A &= 9.61723. \\
\log\tan\tfrac{1}{2}B &= 9.76143. \\
\log\tan\tfrac{1}{2}C &= 9.88498.
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}A &= 22^\circ\ 30'. \\
\tfrac{1}{2}B &= 30^\circ. \\
\tfrac{1}{2}C &= 37^\circ\ 30'.
\end{aligned}$$

$$\begin{aligned}
A &= 45^\circ. \\
B &= 60^\circ. \\
C &= 75^\circ.
\end{aligned}$$

17. The distances between three cities A, B, and C are as follows: $AB = 165$ miles, $AC = 72$ miles, and $BC = 185$ miles. B is due east from A. In what direction is C from A? What two answers are admissible?

$$\begin{aligned}
a &= 185 \\
b &= 72 \\
c &= \underline{165} \\
2s &= 422
\end{aligned}$$

$$\begin{aligned}
s &= 211. \\
s - a &= 26. \\
s - b &= 139. \\
s - c &= 46.
\end{aligned}$$

$$\begin{aligned}
\text{colog}\, s &= 7.67572 - 10 \\
\text{colog}(s - a) &= 8.58503 - 10 \\
\log(s - b) &= 2.14301 \\
\log(s - c) &= \underline{1.66276} \\
2\,&\overline{)20.06652 - 20} \\
\log\tan\tfrac{1}{2}A &= 10.03326
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}A &= 47^\circ\ 11'\ 31''. \\
A &= 94^\circ\ 23'\ 2''.
\end{aligned}$$

Angle $BAC = 94° 23' 2''$. Subtract 90° of the quadrant E to N, and we obtain 4° 23′ 2″ W. of N.

But C may be to the southward of A. Hence two answers are admissible: W. of N. or W. of S.

18. Under what visual angle is an object 7 feet long seen by an observer whose eye is 5 feet from one end of the object and 8 feet from the other end?

$$\begin{aligned} a &= 5 \\ b &= 8 \\ c &= \underline{7} \\ 2s &= 20 \\ s &= 10. \\ s - a &= 5. \\ s - b &= 2. \\ s - c &= 3. \end{aligned}$$

$$\begin{aligned} \text{colog}\, s &= 9.00000 - 10 \\ \log (s - a) &= 0.69897 \\ \log (s - b) &= 0.30103 \\ \text{colog}\, (s - c) &= \underline{9.52288 - 10} \\ 2 &\overline{)19.52288 - 20} \\ \log \tan \tfrac{1}{2} C &= 9.76144 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2} C &= 30°. \\ C &= 60°. \end{aligned}$$

19. When Formula [28] is used for finding the value of an angle, why does the ambiguity that occurs in Case II not exist?

When Formula [28] is used for finding the value of an angle, the ambiguity that occurs in Case II does not exist because the sides are all known and the angle can have but one value; while in Case II the side opposite the angle is not known, and may have two values, and therefore the angle also may have two values.

20. If the sides of a triangle are 3, 4, and 6, find the sine of the largest angle.

$$\begin{aligned} a &= 3 \\ b &= 4 \\ c &= \underline{6} \\ 2s &= 13 \\ s &= 6.5. \\ s - a &= 3.5. \\ s - b &= 2.5. \\ s(s - c) &= 3.25. \end{aligned}$$

$$\begin{aligned} \log (s - a) &= 0.54407 \\ \log (s - b) &= 0.39794 \\ \text{colog}\, s(s - c) &= \underline{9.48812 - 10} \\ 2 &\overline{)20.43013 - 20} \\ \log \tan \tfrac{1}{2} C &= 10.21507 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2} C &= 58° 38' 25''. \\ C &= 117° 16' 50''. \\ \log \sin C &= 9.94879. \\ \sin C &= 0.88877. \end{aligned}$$

21. Of three towns A, B, and C, A is 200 miles from B and 184 miles from C, B is 150 miles due north from C. How far is A north of C?

$$\begin{aligned} a &= 150 \\ b &= 184 \\ c &= \underline{200} \\ 2s &= 534 \\ s &= 267. \\ s - a &= 117. \\ s - b &= 83. \\ s - c &= 67. \end{aligned}$$

$$\begin{aligned}
\operatorname{colog} s &= 7.57349 - 10\\
\log (s-a) &= 2.06819\\
\log (s-b) &= 1.91908\\
\operatorname{colog} (s-c) &= \underline{8.17393 - 10}\\
2\,)\!\!&\,\underline{19.73469 - 20}\\
\log \tan \tfrac{1}{2} C &= 9.86735
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2} C &= 36^\circ\ 22'\ 58''.\\
C &= 72^\circ\ 45'\ 56''.
\end{aligned}$$

Draw ⊥ from A to BC. To find a' (part cut off by ⊥ on BC from A).

$$a' = b \cos C.$$

$$\begin{aligned}
\log b &= 2.26482\\
\log \cos C &= \underline{9.47171}\\
\log a' &= 1.73653
\end{aligned}$$

$$a' = 54.516.$$

Therefore, A is 54.516 miles north of C.

22. The sides of a triangle are 78.9, 65.4, 97.3, respectively. Find the largest angle.

$$\begin{aligned}
a &= 78.9\\
b &= 65.4\\
c &= \underline{97.3}\\
2s &= 241.6
\end{aligned}$$

$$\begin{aligned}
s &= 120.8.\\
s-a &= 41.9.\\
s-b &= 55.4.\\
s-c &= 23.5.
\end{aligned}$$

$$\begin{aligned}
\operatorname{colog} s &= 7.91793 - 10\\
\log (s-a) &= 1.62221\\
\log (s-b) &= 1.74351\\
\operatorname{colog} (s-c) &= \underline{8.62893 - 10}\\
2\,)\!\!&\,\underline{19.91258 - 20}\\
\log \tan \tfrac{1}{2} C &= 9.95629 - 10
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2} C &= 42^\circ\ \ 7'\ 17''.\\
C &= 84^\circ\ 14'\ 34''.
\end{aligned}$$

23. The sides of a triangle are 487.25, 512.33, 544.37, respectively. Find the smallest angle.

$$\begin{aligned}
a &= 487.25\\
b &= 512.33\\
c &= \underline{544.37}\\
2s &= 1543.95
\end{aligned}$$

$$\begin{aligned}
s &= 771.975.\\
s-a &= 284.725.\\
s-b &= 259.645.\\
s-c &= 227.605.
\end{aligned}$$

$$\begin{aligned}
\operatorname{colog} s &= 7.11239 - 10\\
\operatorname{colog} (s-a) &= 7.54557 - 10\\
\log (s-b) &= 2.41438\\
\log (s-c) &= \underline{2.35718}\\
2\,)\!\!&\,\underline{19.42952 - 20}\\
\log \tan \tfrac{1}{2} A &= 9.71476 - 10
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2} A &= 27^\circ\ 24'\ 27''.\\
A &= 54^\circ\ 48'\ 54''.
\end{aligned}$$

24. Find the angles of a triangle whose sides are $\frac{\sqrt{3}+1}{2\sqrt{2}}$, $\frac{\sqrt{3}-1}{2\sqrt{2}}$, $\frac{\sqrt{3}}{2}$, respectively.

$$\begin{aligned}
\frac{\sqrt{3}+1}{2\sqrt{2}} &= \frac{\sqrt{6}+\sqrt{2}}{4}\\
&= \frac{2.44947 + 1.41421}{4}\\
&= \frac{3.86368}{4} = 0.96592.
\end{aligned}$$

$$\begin{aligned}
\frac{\sqrt{3}-1}{2\sqrt{2}} &= \frac{\sqrt{6}-\sqrt{2}}{4}\\
&= \frac{2.44947 - 1.41421}{4}\\
&= \frac{1.03526}{4} = 0.25882.
\end{aligned}$$

$$\frac{\sqrt{3}}{2} = \frac{1.73205}{2} = 0.86603.$$

$$\begin{aligned} a &= 0.96592 \\ b &= 0.25882 \\ c &= \underline{0.86603} \\ 2s &= 2.09077 \end{aligned}$$

$$\begin{aligned} s &= 1.04538. \\ s-a &= 0.07946. \\ s-b &= 0.78656. \\ s-c &= 0.17935. \end{aligned}$$

$$\begin{aligned} \log(s-a) &= 8.90015-10 \\ \log(s-b) &= 9.89573-10 \\ \log(s-c) &= 9.25370-10 \\ \operatorname{colog} s &= \underline{9.98072-10} \\ \log r^2 &= 18.03030-20 \end{aligned}$$

$$\log r = 9.01515-10.$$

$$\begin{aligned} \log\tan\tfrac{1}{2}A &= 10.11500-10. \\ \log\tan\tfrac{1}{2}B &= 9.11940-10. \\ \log\tan\tfrac{1}{2}C &= 9.76145-10. \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}A &= 52^\circ\,30'. \\ \tfrac{1}{2}B &= 7^\circ\,30'. \\ \tfrac{1}{2}C &= 30^\circ. \end{aligned}$$

$$\begin{aligned} A &= 105^\circ. \\ B &= 15^\circ. \\ C &= 60^\circ. \end{aligned}$$

25. The sides of a triangle are 14.6 inches, 16.7 inches, and 18.8 inches, respectively. Find the length of the perpendicular from the vertex of the largest angle upon the opposite side.

$$\begin{aligned} a &= 18.8 \\ b &= 16.7 \\ c &= \underline{14.6} \\ 2s &= 50.1 \end{aligned}$$

$$\begin{aligned} s &= 25.05. \\ s-a &= 6.25. \\ s-b &= 8.35. \\ s-c &= 10.45. \end{aligned}$$

$$\begin{aligned} \operatorname{colog} s &= 8.60119-10 \\ \log(s-a) &= 0.79588 \\ \operatorname{colog}(s-b) &= 9.07831-10 \\ \log(s-c) &= \underline{1.01912} \\ 2\,)\,&\overline{19.49450-20} \\ \log\tan\tfrac{1}{2}B &= 9.74725-10 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}B &= 29^\circ\,11'\,46''. \\ B &= 58^\circ\,23'\,32''. \end{aligned}$$

Let AD be the perpendicular from A upon BC.

$$\frac{AD}{14.6} = \sin B.$$

$$\therefore AD = 14.6 \sin B.$$

$$\begin{aligned} \log 14.6 &= 1.16435 \\ \log\sin B &= \underline{9.93026} \\ \log AD &= 1.09461 \end{aligned}$$

$$AD = 12.434.$$

Therefore, the required perpendicular is 12.434 inches.

Exercise XXI. Page 87.

1. Given $a = 4474.5$, $b = 2164.5$, $C = 116^\circ\,30'\,20''$; find the area.

$$F = \tfrac{1}{2}ab \sin C.$$

$$\begin{aligned} \log a &= 3.65075 \\ \log b &= 3.33536 \\ \operatorname{colog} 2 &= 9.69897-10 \\ \log\sin C &= \underline{9.95177} \\ \log F &= 6.63685 \\ F &= 4{,}333{,}600. \end{aligned}$$

2. Given $b = 21.66$, $c = 36.94$, $A = 66^\circ\,4'\,19''$; find the area.

$$F = \tfrac{1}{2}bc \sin A.$$

$$\begin{aligned} \log b &= 1.33566 \\ \log c &= 1.56750 \\ \operatorname{colog} 2 &= 9.69897-10 \\ \log\sin A &= \underline{9.96097} \\ \log F &= 2.56310 \\ F &= 365.68. \end{aligned}$$

3. Given $a = 510$, $c = 173$, $B = 162° 30' 28''$; find the area.

$$\begin{aligned}
\log a &= 2.70757\\
\log c &= 2.23805\\
\log\sin B &= 9.47795\\
\text{colog}\, 2 &= \underline{9.69897 - 10}\\
\log F &= 4.12254
\end{aligned}$$

$$F = 13{,}260.$$

4. Given $a = 408$, $b = 41$, $c = 401$; find the area.

$$\begin{aligned}
a &= 408\\
b &= 41\\
c &= \underline{401}\\
2s &= 850
\end{aligned}$$

$$\begin{aligned}
s &= 425.\\
s - a &= 17.\\
s - b &= 384.\\
s - c &= 24.
\end{aligned}$$

$$\begin{aligned}
\log s &= 2.62839\\
\log(s - a) &= 1.23045\\
\log(s - b) &= 2.58433\\
\log(s - c) &= \underline{1.38021}\\
&\;2\overline{)7.82338}\\
\log F &= 3.91169
\end{aligned}$$

$$F = 8160.$$

5. Given $a = 40$, $b = 13$, $c = 37$; find the area.

$$\begin{aligned}
a &= 40\\
b &= 13\\
c &= \underline{37}\\
2s &= 90
\end{aligned}$$

$$\begin{aligned}
s &= 45.\\
s - a &= 5.\\
s - b &= 32.\\
s - c &= 8.
\end{aligned}$$

$$\begin{aligned}
\log s &= 1.65321\\
\log(s - a) &= 0.69897\\
\log(s - b) &= 1.50515\\
\log(s - c) &= \underline{0.90309}\\
&\;2\overline{)4.76042}\\
\log F &= 2.38021
\end{aligned}$$

$$F = 240.$$

6. Given $a = 624$, $b = 205$, $c = 445$; find the area.

$$\begin{aligned}
a &= 624\\
b &= 205\\
c &= \underline{445}\\
2s &= 1274
\end{aligned}$$

$$\begin{aligned}
s &= 637.\\
s - a &= 13.\\
s - b &= 432.\\
s - c &= 192.
\end{aligned}$$

$$\begin{aligned}
\log s &= 2.80414\\
\log(s - a) &= 1.11394\\
\log(s - b) &= 2.63548\\
\log(s - c) &= \underline{2.28330}\\
2\log F &= 8.83686
\end{aligned}$$

$$\begin{aligned}
\log F &= 4.41843\\
F &= 26{,}208.
\end{aligned}$$

7. Given $b = 149$, $A = 70° 42' 30''$, $B = 39° 18' 28''$; find the area.

$$\begin{aligned}
A &= 70° 42' 30''.\\
B &= 39° 18' 28''.\\
\therefore C &= 69° 59' \ \ 2''.
\end{aligned}$$

$$\begin{aligned}
\log b &= 2.17319\\
\log\sin A &= 9.97490\\
\text{colog}\sin B &= \underline{0.19827}\\
\log a &= 2.34636
\end{aligned}$$

$$\begin{aligned}
\text{colog}\, 2 &= 9.69897 - 10\\
\log a &= 2.34636\\
\log b &= 2.17319\\
\log\sin C &= \underline{9.97294\qquad\;}\\
\log F &= 4.19146
\end{aligned}$$

$$F = 15{,}540.$$

8. Given $a = 215.9$, $c = 307.7$, $A = 25° 9' 31''$; find the area.

$a < c$ and $> c \sin A$.

$A < 90°$. $\therefore$ two solutions.

$$\begin{aligned}
\log c &= 2.48813\\
\log \sin A &= 9.62852\\
\text{colog}\, a &= \underline{7.66575 - 10}\\
\log \sin C &= 9.78240\\
C &= 37° 17' 38''.\\
\therefore B &= 117° 32' 51''.\\
\text{Or,}\quad C' &= 142° 42' 22''.\\
\therefore B' &= 12° \ 8' \ 7''.
\end{aligned}$$

$$\begin{aligned}
\text{colog}\, 2 &= 9.69897 - 10\\
\log a &= 2.33425\\
\log c &= 2.48813\\
\log \sin B &= \underline{9.94774}\\
\log F &= 4.46909\\
F &= 29{,}450.
\end{aligned}$$

$$\begin{aligned}
\text{colog}\, 2 &= 9.69897 - 10\\
\log a &= 2.33425\\
\log c &= 2.48813\\
\log \sin B' &= \underline{9.32268}\\
\log F' &= 3.84403\\
F' &= 6982.8.
\end{aligned}$$

9. Given $b = 8$, $c = 5$, $A = 60°$; find the area.

$$\begin{aligned}
F &= \tfrac{1}{2} bc \sin A\\
&= \tfrac{1}{2}(8 \times 5)(0.86603)\\
&= 20 \times 0.86603\\
&= 17.3206.
\end{aligned}$$

10. Given $a = 7$, $c = 3$, $A = 60°$; find the area.

$$\begin{aligned}
\log c &= 0.47712\\
\log \sin A &= 9.93753\\
\text{colog}\, a &= \underline{9.15490 - 10}\\
\log \sin C &= 9.56955\\
C &= 21° 47' 12''.\\
\therefore B &= 98° 12' 48''.
\end{aligned}$$

$$\begin{aligned}
\text{colog}\, 2 &= 9.69897 - 10\\
\log a &= 0.84510\\
\log c &= 0.47712\\
\log \sin B &= \underline{9.99552}\\
\log F &= 1.01671\\
F &= 10.392.
\end{aligned}$$

11. Given $a = 60$, $B = 40° 35' 12''$, area $= 12$; find the radius of the inscribed circle.

$$\begin{aligned}
\tfrac{1}{2} ac \sin B &= 12.\\
c &= \frac{24}{a \sin B}.
\end{aligned}$$

$$\begin{aligned}
\log 24 &= 1.38021\\
\text{colog}\, a &= 8.22185 - 10\\
\text{colog} \sin B &= \underline{0.18669}\\
\log c &= 9.78875 - 10\\
c &= 0.61483.\\
a - c &= 59.38517.\\
a + c &= 60.61483.\\
A + C &= 139° 24' 48''.\\
\tfrac{1}{2}(A + C) &= 69° 42' 24''.
\end{aligned}$$

$$\begin{aligned}
\log (a - c) &= 1.77368\\
\text{colog}\,(a + c) &= 8.21742 - 10\\
\log \tan \tfrac{1}{2}(A + C) &= \underline{10.43206}\\
\log \tan \tfrac{1}{2}(A - C) &= 10.42316\\
\tfrac{1}{2}(A - C) &= 69° 19' 19''.\\
\therefore A &= 139° \ 1' 48''.\\
b &= \frac{a \sin B}{\sin A}.
\end{aligned}$$

$$\begin{aligned}
\log a &= 1.77815\\
\log \sin B &= 9.81331\\
\text{colog} \sin A &= \underline{0.18331}\\
\log B &= 1.77477\\
b &= 59.534.
\end{aligned}$$

$$\begin{aligned}
a &= 60.\\
b &= 59.534\\
c &= \underline{\ 0.61483}\\
2s &= 120.14883
\end{aligned}$$

$$s = 60.07442.$$

$$F = rs.$$

$$\therefore r = \frac{F}{s} = \frac{12}{60.07442}.$$

$$\begin{aligned} \log F &= 1.07918 \\ \operatorname{colog} s &= \underline{8.22131 - 10} \\ \log r &= 9.30049 - 10 \end{aligned}$$

$$r = 0.19975.$$

12. Obtain a formula for the area of a parallelogram in terms of two adjacent sides and the included angle.

By Geometry, area of parallelogram = base × height.

In this case, area $= bh$.

But $h = a \sin A$.

$\therefore$ area of $\square = ab \sin A$.

13. Obtain a formula for the area of an isosceles trapezoid in terms of the two parallel sides and an acute angle.

Let a = greater base,
b = smaller base,
h = altitude,
$p = \frac{1}{2}(a - b)$,
and A = angle at lower base.

Now $F = \frac{1}{2}(a + b)h$.

But $\dfrac{h}{\frac{1}{2}(a-b)} = \tan A$.

$$\therefore h = \tfrac{1}{2}(a - b)\tan A.$$

$$\begin{aligned} \therefore F &= \tfrac{1}{2}(a + b) \times \tfrac{1}{2}(a - b)\tan A \\ &= \tfrac{1}{4}(a^2 - b^2)\tan A. \end{aligned}$$

14. Two sides and included angle of a triangle are 2416, 1712, and 30°; and two sides and included angle of another triangle are 1948, 2848, and 150°; find the sum of their areas.

Let $a = 2416$, $c = 1712$, $B = 30°$.

$$F = \tfrac{1}{2}ac \sin B.$$

$$\begin{aligned} \log a &= 3.38310 \\ \log c &= 3.23350 \\ \operatorname{colog} 2 &= 9.69897 - 10 \\ \log \sin B &= \underline{9.69897} \\ \log F &= 6.01454 \end{aligned}$$

$$F = 1{,}034{,}000.$$

Let $a' = 1948$, $c' = 2848$, $B' = 150°$.

$$F' = \tfrac{1}{2}a'c' \sin B'.$$

$$\begin{aligned} \log a' &= 3.28959 \\ \log c' &= 3.45454 \\ \operatorname{colog} 2 &= 9.69897 - 10 \\ \log \sin B' &= \underline{9.69897} \\ \log F' &= 6.14207 \end{aligned}$$

$$F' = 1{,}387{,}000.$$

$$\therefore F + F' = 2{,}421{,}000.$$

15. The base of an isosceles triangle is 20, and its area is $100 \div \sqrt{3}$; find its angles.

$$a = b.$$

$$c = 20.$$

$$F = 100 \div \sqrt{3}.$$

$$\tfrac{1}{2}ch = \frac{100}{\sqrt{3}}.$$

$$10h = \frac{100}{\sqrt{3}}.$$

$$h = \frac{10}{\sqrt{3}}.$$

$$\frac{h}{\frac{1}{2}c} = \tan A.$$

$$\begin{aligned} \log h &= 0.76144 \\ \operatorname{colog} \tfrac{1}{2}c &= \underline{9.00000 - 10} \\ \log \tan A &= 9.76144 \end{aligned}$$

$$A = 30°.$$

$$\therefore B = 30°.$$

$$\therefore C = 120°.$$

16. Show that the area of a quadrilateral is equal to one-half the product of its diagonals into the sine of their included angle.

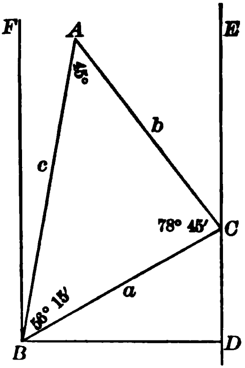
Let the lengths of the diagonals be denoted by a and b, and the included angle by C. Let the lengths of the segments made by their point of intersection at C be denoted by a_1, a_2, and b_1, b_2, respectively.

Now the area of the quadrilateral is equal to the sum of the areas of the four triangles.

$$\begin{aligned}\therefore F &= \tfrac{1}{2}a_1b_2 \sin C + \tfrac{1}{2}a_2b_2 \sin C \\ &\quad + \tfrac{1}{2}a_2b_1 \sin C + \tfrac{1}{2}a_1b_1 \sin C \\ &= \tfrac{1}{2}(a_1b_2 + a_2b_2 + a_2b_1 + a_1b_1)\sin C \\ &= \tfrac{1}{2}(a_1 + a_2)(b_1 + b_2)\sin C \\ &= \tfrac{1}{2}ab \sin C.\end{aligned}$$

EXERCISE XXII. PAGE 88.

1. From a ship sailing down the English Channel the Eddystone was observed to bear N. 33° 45′ W., and after the ship had sailed 18 miles S. 67° 30′ W. it bore N. 11° 15′ E. Find its distance from each position of the ship.



Let A represent the Eddystone, C the first position of the ship, and B the second.

$$\begin{aligned} a &= 18 \text{ miles.} \\ ACE &= 33^\circ\ 45'. \\ DCB &= 67^\circ\ 30'. \\ ABF &= 11^\circ\ 15'. \end{aligned}$$

$$\begin{aligned} ACB &= 180^\circ - (ACE + DCB) \\ &= 78^\circ\ 45'. \\ CBD &= 90^\circ - DCB = 22^\circ\ 30'. \\ ABC &= 90^\circ - (CBD + ABF) \\ &= 56^\circ\ 15'. \\ \therefore BAC &= 45^\circ. \end{aligned}$$

$$\frac{b}{a} = \frac{\sin B}{\sin A}.$$

$$\begin{aligned} \log a &= 1.25527 \\ \log \sin B &= 9.91985 \\ \operatorname{colog} \sin A &= \underline{0.15051} \\ \log b &= 1.32563 \\ b &= 21.166. \end{aligned}$$

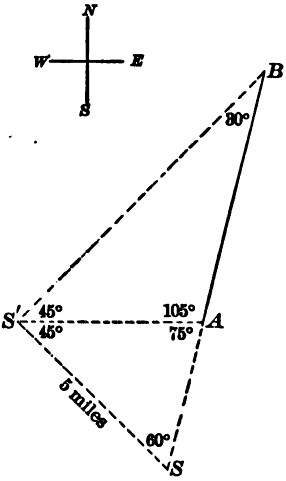
$$\frac{c}{a} = \frac{\sin C}{\sin A}.$$

$$\begin{aligned} \log a &= 1.25527 \\ \log \sin C &= 9.99157 \\ \operatorname{colog} \sin A &= \underline{0.15051} \\ \log c &= 1.39735 \\ c &= 24.966. \end{aligned}$$

Therefore, the required distances are 21.166 miles and 24.966 miles.

2. Two objects A and B were observed from a ship to be at the same instant in a line bearing N. 15° E. The ship then sailed northwest

5 miles, when it was found that *A* bore due east and *B* bore north-east. Find the distance from *A* to *B*.



Let *A* and *B* represent the objects, *S* and *S'* the first and second positions of the ship.

$$\frac{S'A}{SS'} = \frac{\sin ASS'}{\sin S'AS}.$$

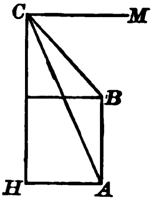
$$\begin{aligned}
\log SS' &= 0.69897\\
\log\sin ASS' &= 9.93753\\
\text{colog}\sin SAS' &= \underline{0.01506}\\
\log S'A &= 0.65156
\end{aligned}$$

$$\frac{AB}{S'A} = \frac{\sin BS'A}{\sin S'BA}.$$

$$\begin{aligned}
\log S'A &= 0.65156\\
\log\sin BS'A &= 9.84949\\
\text{colog}\sin S'BA &= \underline{0.30103}\\
\log AB &= 0.80208\\
AB &= 6.3399.
\end{aligned}$$

Therefore, the distance from *A* to *B* is 6.3399 miles.

3. A castle and a monument stand on the same horizontal plane. The angles of depression of the top and the bottom of the monument viewed from the top of the castle are 40° and 80°; the height of the castle is 140 feet. Find the height of the monument.



HC = height of castle.
AB = height of monument.

$$\begin{aligned}
MCB &= 40°.\\
HCA &= 10°.\\
HAC &= 80°.\\
HC &= 140 \text{ feet.}\\
AC &= \frac{140}{\sin A}.
\end{aligned}$$

$$\begin{aligned}
\log 140 &= 2.14613\\
\text{colog}\sin A &= \underline{0.00665}\\
\log AC &= 2.15278\\
HCA &= 10°,\\
MCB &= 40°.\\
\therefore ACB &= 40°,\\
CAB &= 10°.\\
\therefore ABC &= 130°.\\
AB &= \frac{AC\sin C}{\sin B}.
\end{aligned}$$

$$\begin{aligned}
\log AC &= 2.15278\\
\log\sin C &= 9.80807\\
\text{colog}\sin B &= \underline{0.11575}\\
\log AB &= 2.07660\\
AB &= 119.29.
\end{aligned}$$

Therefore, the height of the monument is 119.29 feet.

4. If the sun's altitude is 60°, what angle must a stick make with the horizon in order that its shadow in a horizontal plane may be the longest possible?

The shadow of the stick will be the longest when the stick is perpendicular to the rays of the sun.

Let BC represent the stick, and AC the horizontal plane.

$$B = 90^\circ.$$
$$A = 60^\circ.$$
$$\therefore C = 30^\circ.$$

5. If the sun's altitude is 30°, find the length of the longest shadow cast on a horizontal plane by a stick 10 feet in length.

Let a be a stick $\perp$ to rays of sun, and c be the longest shadow.

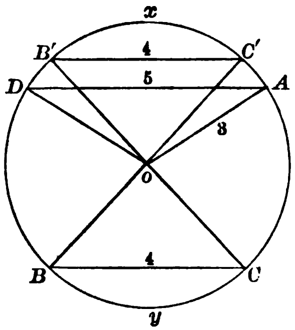
$$\frac{a}{c} = \sin A = \tfrac{1}{2}.$$
$$c = 2a = 20.$$

Therefore, the longest shadow is 20 feet.

6. In a circle with the radius 3 find the area of the part comprised between parallel chords whose lengths are 4 and 5. (Two solutions.)

In triangle BOC,

$$h = \sqrt{3^2 - 2^2}$$
$$= \sqrt{5}.$$
$$F = \tfrac{1}{2} \times \sqrt{5} \times 4$$
$$= 2\sqrt{56}$$
$$= 4.4722.$$



$$\sin \tfrac{1}{2} BOC = \tfrac{2}{3}.$$

$$\begin{aligned} \log 2 &= 0.30103 \\ \text{colog } 3 &= \underline{9.52288 - 10} \\ \log \sin \tfrac{1}{2} BOC &= 9.82391 \end{aligned}$$

$$\tfrac{1}{2} BOC = 41^\circ\ 48'\ 38''.$$
$$BOC = 83^\circ\ 37'\ 16''.$$

$$\text{area } \odot = \pi \times 3^2.$$

$$\begin{aligned} \log \pi &= 0.49715 \\ \log 3^2 &= \underline{0.95424} \\ \log \text{area } \odot &= 1.45139 \end{aligned}$$

$$\text{area } \odot = 28.274.$$

Area of sector BOC

$$= \frac{83^\circ\ 37'\ 16''}{360^\circ} \times 28.274$$
$$= \frac{301036}{1296000} \times 28.274$$
$$= \frac{75259}{324000} \times 28.274.$$

$$\begin{aligned} \log\ 75259 &= 4.87656 \\ \log 28.274 &= 1.45139 \\ \text{colog } 324000 &= \underline{4.48945 - 10} \\ \log \text{area} &= 0.81740 \end{aligned}$$

Area of sector BOC

$$= 6.5675.$$

Area of segment ByC

$$= 6.5675 - 4.4722$$
$$= 2.0953.$$

In triangle DOA,

$$\begin{aligned} h &= \sqrt{3^2 - 2.5^2} \\ &= 1.6583. \\ F &= \tfrac{1}{2} \times 1.6583 \times 5 \\ &= 4.1458. \\ \sin \tfrac{1}{2} DOA &= \tfrac{5}{6}. \\ \log 5 &= 0.69897 \\ \text{colog } 6 &= \underline{9.22185 - 10} \\ \log \sin \tfrac{1}{2} DOA &= 9.92082 \\ \tfrac{1}{2} DOA &= 56^\circ\ 26'\ 33''. \\ DOA &= 112^\circ\ 53'\ 7''. \end{aligned}$$

Area of sector DOA

$$= \frac{112^\circ\ 53'\ 7''}{360^\circ} \times 28.274$$

$$= \frac{406387}{1296000} \times 28.274.$$

$$\begin{aligned} \log 406387 &= 5.60894 \\ \log 28.274 &= 1.45139 \\ \text{colog } 1296000 &= \underline{3.88739 - 10} \\ \log \text{area} &= 0.94772 \end{aligned}$$

Area sector DOA

$$= 8.8658.$$

Area segment DxA

$$= 8.8658 - 4.1458 = 4.72.$$

Area $DACB$

$$\begin{aligned} &= \text{area} \odot - [ByC + DxA] \\ &= 28.274 - (2.0953 + 4.72) \\ &= 21.4587. \end{aligned}$$

Area $DAC'B'$

$$\begin{aligned} &= DxA - B'xC' \\ &= 4.72 - 2.0953 \\ &= 2.6247. \end{aligned}$$

EXERCISE XXIII. PAGE 90.

1. The angle of elevation of a tower is 48° 19′ 14″, and the distance of the base from the point of observation is 95 feet. Find the height of the tower, and the distance of its top from the point of observation.

Given $A = 48^\circ\,19'\,14''$, $b = 95$ feet; required a and c.

$$\begin{aligned} a &= b \tan A. \\ c &= b \sec A. \end{aligned}$$

$$\begin{aligned} \log b &= 1.97772 \\ \log \tan A &= \underline{10.05045} \\ \log a &= 2.02817 \\ a &= 106.70. \end{aligned}$$

$$\begin{aligned} \log b &= 1.97772 \\ \log \sec A &= \underline{0.17720} \\ \log c &= 2.15492 \\ c &= 142.86. \end{aligned}$$

Height of tower, 106.70 feet; distance of top from point of observation, 142.86 feet.

2. From a mountain 1000 feet high, the angle of depression of a ship is 77° 35′ 11″. Find the distance of the ship from the summit of the mountain.

Given $B = 12^\circ\ 24'\ 49''$, $a = 1000$ feet; required c.

$$\begin{aligned} c &= a \sec B. \\ \log a &= 3.00000 \\ \log \sec B &= \underline{0.01027} \\ \log c &= 3.01027 \\ c &= 1023.9. \end{aligned}$$

Required distance, 1023.9 feet.

3. A flagstaff 90 feet high, on a horizontal plane, casts a shadow of

117 feet. Find the altitude of the sun.

Given $a = 90$ feet, $b = 117$ feet; required A.

$$\tan A = \frac{a}{b}.$$

$$\begin{aligned} \log a &= 1.95424 \\ \operatorname{colog} b &= \underline{7.93181 - 10} \\ \log \tan A &= 9.88605 \\ A &= 37° \, 34' \, 5''. \end{aligned}$$

Altitude of sun, 37° 34′ 5″.

4. When the moon is setting at any place, the angle at the moon subtended by the earth's radius passing through that place is 57′ 3″. If the earth's radius is 3956.2 miles, what is the moon's distance from the earth's centre?

Let C represent the place, A the moon, and B the earth's centre. Then in the right triangle ABC, given $A = 57' \, 3''$, $a = 3956.2$ miles; required c.

$$c = a \csc A.$$

$$\begin{aligned} \log a &= 3.59728 \\ \log \csc A &= \underline{1.78004} \\ \log c &= 5.37732 \\ c &= 238{,}410. \end{aligned}$$

Moon's distance, 238,410 miles.

5. The angle at the earth's centre subtended by the sun's radius is 16′ 2″, and the sun's distance is 92,400,000 miles. Find the sun's diameter in miles.

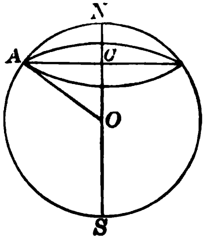
Let A represent the centre of the earth, B that of the sun, and C a point on the edge of the sun's disk. Then in the right triangle ABC, given $A = 16' \, 2''$, $c = 92{,}400{,}000$ miles; required $2a$.

$$a = c \sin A.$$

$$\begin{aligned} \log c &= 7.96567 \\ \log \sin A &= \underline{7.66874} \\ \log a &= 5.63441 \\ a &= 430{,}930. \\ 2a &= 861{,}860. \end{aligned}$$

Sun's diameter, 861,860 miles.

6. The latitude of Cambridge, Mass., is 42° 22′ 49″. What is the length of the radius of that parallel of latitude?



Let O be the centre of the earth, NS the axis, NAS the meridian of Cambridge, A the position of Cambridge, and C the centre of its parallel of latitude. Then, in the right triangle OAC, given $O = 90° - 42° \, 22' \, 49'' = 47° \, 37' \, 11''$, $OA = 3956.2$ miles; required AC.

$$AC = AO \sin O.$$

$$\begin{aligned} \log AO &= 3.59728 \\ \log \sin O &= \underline{9.86846} \\ \log AC &= 3.46574 \\ AC &= 2922.4. \end{aligned}$$

Radius of parallel of latitude, 2922.4 miles.

7. At what latitude is the circumference of the parallel of latitude half of that of the equator?

The radius of the parallel will be half of the radius of the earth.

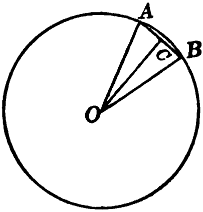
In the figure of Example 6, given $AC = \frac{1}{2} AO$; required $90° -$ angle O, *i.e.*, angle A.

$$\cos A = \frac{AC}{AO} = \tfrac{1}{2}.$$

$$\therefore A = 60°.$$

The required latitude is 60°.

8. In a circle with a radius of 6.7 is inscribed a regular polygon of thirteen sides. Find the length of one of its sides.



Let O be the centre of the circle, AB a side of the polygon, and C the middle point. Then in the right triangle OCB, given $O = \frac{360°}{26} = 13° 50' 46''$, $OB = 6.7$; required $AB = 2\,CB$.

$$CB = OB \sin BOC.$$

$$\begin{aligned} \log OB &= 0.82607 \\ \log \sin BOC &= \underline{9.37897} \\ \log CB &= 0.20504 \end{aligned}$$

$$CB = 1.6034.$$

$$AB = 3.2068.$$

Length of a side of the polygon, 3.2068.

9. A regular heptagon one side of which is 5.73 is inscribed in a circle. Find the radius of the circle.

In the figure of Example 8, given $BC = \frac{1}{2} \times 5.73 = 2.865$ and angle $BOC = \frac{360°}{14} = 25° 42' 51''$; required OB.

$$OB = BC \csc BOC.$$

$$\begin{aligned} \log BC &= 0.45712 \\ \log \csc BOC &= \underline{0.36263} \\ \log OB &= 0.81975 \end{aligned}$$

$$OB = 6.6031.$$

Radius of circle, 6.6031.

10. A tower 93.97 feet high is situated on the bank of a river. The angle of depression of an object on the opposite bank is 25° 12′ 54″. Find the breadth of the river.

Given $A = 90 - 25° \; 12' \; 54'' = 64° 47' 6''$, $b = 93.97$; required a.

$$a = b \tan A.$$

$$\begin{aligned} \log b &= 1.97299 \\ \log \tan A &= \underline{10.32708} \\ \log a &= 2.30007 \end{aligned}$$

$$a = 199.56.$$

Breadth of river, 199.56 feet.

11. From a tower 58 feet high the angles of depression of two objects situated in the same horizontal line with the base of the tower, and on the same side, are 30° 13′ 18″ and 45° 46′ 14″. Find the distance between these two objects.

(i) Given $A = 90° - 30° \; 13' \; 18'' = 59° 46' 42''$, $b = 58$; required a.

$$a = b \tan A.$$

$$\begin{aligned}\log b &= 1.76343\\ \log\tan A &= \underline{10.23469}\\ \log a &= 1.99812\\ a &= 99.568.\end{aligned}$$

(ii) Given $A' = 90° - 45° 46' 14''$ $= 44° 13' 46''$, $b = 58$; required a'.

$$a' = b \tan A'.$$

$$\begin{aligned}\log b &= 1.76343\\ \log\tan A' &= \underline{9.98832}\\ \log a' &= 1.75175\\ a' &= 56.461.\\ a - a' &= 43.107.\end{aligned}$$

Distance between the objects, 43.107 feet.

12. Standing directly in front of one corner of a flat-roofed house which is 150 feet in length, I observe that the horizontal angle which the length subtends has for its cosine $\sqrt{\frac{1}{5}}$, and that the vertical angle subtended by its height has for its sine $\frac{3}{\sqrt{34}}$. What is the height of the house?

Let a = distance of observer from house,
b = height of house,
B = horizontal angle subtended by length of house,
B' = vertical angle subtended by height of house.

Then $a = 150 \cot B.$
$b = a \tan B'$
$= 150 \cot B \tan B'.$

But $\cos B = \sqrt{\frac{1}{5}}$;
hence $\sin B = \sqrt{1 - \frac{1}{5}}$
$= \frac{2}{\sqrt{5}}.$

$$\therefore \cot B = \frac{\cos B}{\sin B} = \frac{1}{2}.$$

Also $\sin B' = \frac{3}{\sqrt{34}}.$

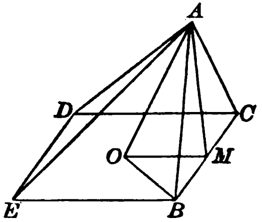
$$\therefore \cos B' = \frac{5}{\sqrt{34}}.$$

$$\therefore \tan B' = \frac{3}{5}.$$

Hence $b = 150 \times \frac{1}{2} \times \frac{3}{5} = 45.$

Height of house, 45 feet.

13. A regular pyramid, with a square base, has a lateral edge 150 feet long, and a side of its base is 200 feet. Find the inclination of the face of the pyramid to the base.



Let A be the vertex of the pyramid, $BCDE$ its base, O the centre of the base, and M the middle point of the side BC. Required the angle AMO.

In the right triangle AOB,

$$\begin{aligned}AB &= 150,\\ OB &= \tfrac{1}{2} BD = 100\sqrt{2}.\\ \therefore AO &= \sqrt{AB^2 - OB^2}\\ &= 50.\end{aligned}$$

In the right triangle AOM,

$$\tan OMA = \frac{AO}{OM} = \frac{50}{100} = 0.5.$$

$$OMA = 26° 34'.$$

Inclination of face of pyramid to base, 26° 34'.

14. From one edge of a ditch 36 feet wide the angle of elevation of a wall on the opposite edge is 62° 39′ 10″. Find the length of a ladder that will just reach from the point of observation to the top of the wall.

Given $b = 36$, $A = 62° \, 39' \, 10''$; required c.

$$c = b \sec A.$$

$$\begin{aligned} \log b &= 1.55630 \\ \log \sec A &= \underline{0.33783} \\ \log c &= 1.89413 \\ c &= 78.367. \end{aligned}$$

Length of ladder, 78.367 feet.

15. The top of a flagstaff has been broken off and touches the ground at a distance of 15 feet from the foot of the staff. If the length of the broken part is 39 feet, find the length of the whole staff.

Given $c = 39$, $b = 15$; required $c + a$.

$$\begin{aligned} a &= \sqrt{(c+b)(c-b)} \\ &= \sqrt{(39+15)(39-15)} \\ &= \sqrt{54 \times 24} \\ &= \sqrt{6^2 \times 9 \times 4} \\ &= 36. \end{aligned}$$

$$c + a = 39 + 36 = 75.$$

Whole length of flagstaff, 75 feet.

16. From a balloon, which is directly above one town, is observed the angle of depression of another town, 10° 14′ 9″. The towns being 8 miles apart, find the height of the balloon.

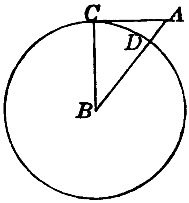
Given $A = 90° - 10° \, 14' \, 9'' = 79° \, 45' \, 51''$, $a = 8$; required b.

$$b = a \cot A.$$

$$\begin{aligned} \log a &= 0.90309 \\ \log \cot A &= \underline{9.25666} \\ \log b &= 0.15975 \\ b &= 1.4446. \end{aligned}$$

Height of balloon, 1.4446 miles.

17. From the top of a mountain 3 miles high the angle of depression of the most distant object which is visible on the earth's surface is found to be 2° 13′ 50″. Find the diameter of the earth.



Let A be the top of the mountain, C the object observed, B the centre of the earth. Then given $B = 90° - A = 2° \, 13' \, 50''$, $AD = 3$; required a.

$$\begin{aligned} BC &= AB \cos B. \\ a &= (a+3) \cos B. \\ \therefore a(1 - \cos B) &= 3 \cos B. \\ a &= \frac{3 \cos B}{1 - \cos B} \end{aligned}$$

By [16], $$= \frac{3 \cos B}{2 \sin^2 \frac{B}{2}}.$$

$$\begin{aligned} \log \tfrac{3}{2} &= 0.17609 \\ \log \cos B &= 9.99967 \\ \text{colog} \sin^2 \frac{B}{2} &= \underline{3.42154} \\ \log a &= 3.59730 \\ a &= 3956.4. \\ 2a &= 7912.8. \end{aligned}$$

Diameter of earth, 7912.8 miles.

18. A ladder 40 feet long reaches a window 33 feet high on one side of a street. Being turned over upon its foot, it reaches another window 21 feet high, on the opposite side of the street. Find the width of the street.

Width of the one part of the street

$$= \sqrt{40^2 - 33^2}$$
$$= \sqrt{511}$$
$$= 22.605.$$

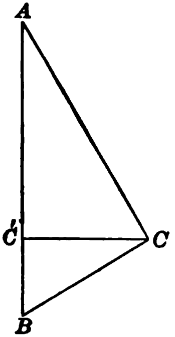
Width of other part

$$= \sqrt{40^2 - 21^2}$$
$$= \sqrt{1159}$$
$$= 34.044.$$

$$22.605 + 34.044 = 56.649.$$

Total width of the street, 56.649 feet.

19. The height of a house subtends a right angle at a window on the other side of the street; and the angle of elevation of the top of the house from the same point is 60°. The street is 30 feet wide. How high is the house?



Given $CC' = 30$, $ACC' = 60°$, $BCC' = 30°$; required AB.

Now $\quad CAB = 30°.$

$$\therefore AC = 2 \times CC' = 60.$$
$$\cos A = \frac{AC}{AB}.$$
$$\therefore AB = \frac{AC}{\cos A} = \frac{60}{\frac{1}{2}\sqrt{3}}$$
$$= 40\sqrt{3} = 69.282.$$

Height of house, 69.282 feet.

20. A lighthouse 54 feet high is situated on a rock. The angle of elevation of the top of the lighthouse, as observed from a ship, is 4° 52′, and the angle of elevation of the top of the rock is 4° 2′. Find the height of the rock and its distance from the ship.

Let $\quad h =$ height of rock.

$a =$ distance of ship.

Then
$$\frac{h + 54}{h} = \frac{\tan 4° 52'}{\tan 4° 2'}.$$
$$1 + \frac{54}{h} = \frac{\tan 4° 52'}{\tan 4° 2'}.$$
$$\frac{54}{h} = \frac{\tan 4° 52' - \tan 4° 2'}{\tan 4° 2'}.$$
$$h = 54 \frac{\tan 4° 2'}{\tan 4° 52' - \tan 4° 2'}$$
$$= 54 \frac{\cos 4° 52' \sin 4° 2'}{\sin (4° 52' - 4° 2')}$$
$$= 54 \frac{\cos 4° 52' \sin 4° 2'}{\sin 50'}.$$

$$\log 54 = 1.73239$$
$$\log \cos 4° 52' = 9.99843$$
$$\log \sin 4° 2' = 8.84718$$
$$\text{colog} \sin 50' = \underline{1.83732}$$
$$\log h = 2.41532$$

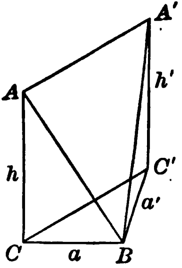
$$h = 260.21.$$

Also $\quad a = h \cot 4° 2'.$

$$\begin{aligned} \log h &= 2.41532 \\ \log \cot 4^\circ 2' &= \underline{11.15174} \\ \log a &= 3.56706 \\ a &= 3690.3. \end{aligned}$$

Height of rock, 260.21 feet; distance of ship, 3690.3 feet.

21. A man in a balloon observes the angle of depression of an object on the ground, bearing south, to be 35° 30′; the balloon drifts $2\frac{1}{2}$ miles east at the same height, when the angle of depression of the same object is 23° 14′. Find the height of the balloon.



Let A and A' be the first and second positions of the balloon, respectively, C and C' the points on the ground directly under A and A', and B the object observed.

Then

$$\begin{aligned} A &= 54^\circ 30', \\ A' &= 66^\circ 46', \\ CC' = AA' &= 2\tfrac{1}{2}. \\ a &= h \tan A, \\ a' &= h \tan A'. \\ a'^2 - a^2 &= (2\tfrac{1}{2})^2. \\ h^2 \tan^2 A' - h^2 \tan^2 A &= (2\tfrac{1}{2})^2. \end{aligned}$$

$$h^2 = \frac{(2\frac{1}{2})^2}{\tan^2 A' - \tan^2 A}.$$

$$h = \frac{2\frac{1}{2}}{\sqrt{\tan^2 A' - \tan^2 A}}.$$

But $\tan^2 A' - \tan^2 A$

$$\begin{aligned} &= (\tan A' + \tan A)(\tan A' - \tan A) \\ &= \frac{\sin(A' + A)}{\cos A' \cos A} \times \frac{\sin(A' - A)}{\cos A' \cos A} \\ &= \frac{\sin(A' + A)\sin(A' - A)}{\cos^2 A' \cos^2 A}. \end{aligned}$$

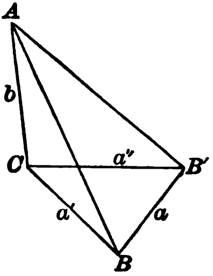
Hence

$$h = \frac{2\frac{1}{2} \cos A' \cos A}{\sqrt{\sin(A' + A)\sin(A' - A)}}.$$

$$\begin{aligned} \log 2\tfrac{1}{2} &= 0.39794 \\ \log \cos A' &= 9.59602 \\ \log \cos A &= 9.76395 \\ \operatorname{colog} \sqrt{\sin(A' + A)} &= 0.03408 \\ \operatorname{colog} \sqrt{\sin(A' - A)} &= \underline{0.33636} \\ \log h &= 0.12835 \\ h &= 1.3438. \end{aligned}$$

Height of balloon, 1.3438 miles.

22. A man standing south of a tower, on the same horizontal plane, observes its angle of elevation to be 54° 16′; he goes east 100 yards, and then finds its angle of elevation is 50° 8′. Find the height of the tower.



Let AC be the tower, B and B' the first and second positions of the observer.

Then

$$\begin{aligned} BB' &= 100. \\ a' &= b \cot ABC. \end{aligned}$$

$$a'' = b\cot AB'C.$$

$$a''^2 - a'^2 = a^2.$$

$$b^2(\cot^2 AB'C - \cot^2 ABC) = 100^2.$$

$$\begin{aligned} b &= \frac{100}{\sqrt{\cot^2 50^\circ\,8' - \cot^2 54^\circ\,16'}} \\ &= \frac{100\sin 54^\circ\,16'\sin 50^\circ\,8'}{\sqrt{\sin 104^\circ\,24'\sin 4^\circ\,8'}}. \end{aligned}$$

$$\begin{aligned} \log 100 &= 2.00000 \\ \log\sin 54^\circ\,16' &= 9.90942 \\ \log\sin 50^\circ\,8' &= 9.88510 \\ \text{colog}\sqrt{\sin 104^\circ\,24'} &= 0.00693 \\ \text{colog}\sqrt{\sin 4^\circ\,8'} &= \underline{0.57110} \\ \log b &= 2.37255 \\ b &= 235.81. \end{aligned}$$

Height of tower, 235.81 yards.

23. The angle of elevation of a tower at a place A south of it is 30°; and at a place B, west of A, and at a distance a from it, the angle of elevation is 18°. Show that the height of the tower is $\frac{a}{\sqrt{2+2\sqrt{5}}}$, the tangent of 18° being $\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$.

With the figure and notation of the last example,

$$b = \frac{a}{\sqrt{\cot^2 18^\circ - \cot^2 30^\circ}}.$$

$$\begin{aligned} \text{But } \cot^2 18^\circ &= \frac{10+2\sqrt{5}}{6-2\sqrt{5}} \\ &= \frac{(10+2\sqrt{5})(6+2\sqrt{5})}{6^2-(2\sqrt{5})^2} \\ &= 5+2\sqrt{5}, \end{aligned}$$

and $\cot^2 30^\circ = 3.$

Hence $$b = \frac{a}{\sqrt{2+2\sqrt{5}}}.$$

24. A pole is fixed on the top of a mound, and the angles of elevation of the top and bottom of the pole are 60° and 30°, respectively. Prove that the length of the pole is twice the height of the mound.

Let l = length of pole,
h = height of mound,
and a = horizontal distance of observer.

Then $h = a\tan 30^\circ.$

$$h + l = a\tan 60^\circ.$$

$$\begin{aligned} \frac{h+l}{h} &= \frac{\tan 60^\circ}{\tan 30^\circ} \\ &= \frac{\sqrt{3}}{\frac{1}{\sqrt{3}}} = 3. \end{aligned}$$

$$h + l = 3h.$$

$$\therefore l = 2h.$$

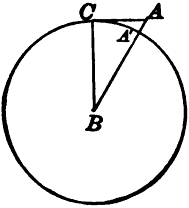
25. At a distance a from the foot of a tower, the angle of elevation A of the top of the tower is the complement of the angle of elevation of a flagstaff on top of it. Show that the length of the staff is $2a\cot 2A$.

Let h = height of tower,
and l = length of staff.

Then $h = a\tan A.$

$$\begin{aligned} h + l &= a\cot A. \\ l &= a(\cot A - \tan A) \\ &= a\frac{\cot^2 A - 1}{\cot A} \\ &= 2a\cot 2A. \end{aligned}$$

26. A line of true level is a line every point of which is equally distant from the centre of the earth. A line drawn tangent to a line of true level at any point is a line of apparent level. If at any point both these lines are drawn, and extended one mile, find the distance they are then apart.



Given $CA = 1$ mile, $BC =$ radius of the earth $= 3956.2$ miles; required $AA' = AB - CB$.

The required distance is much too small to be obtained by the usual process of solution. It is most easily found as follows:

$$\begin{aligned}\overline{AC}^2 &= \overline{AB}^2 - \overline{BC}^2\\ &= (AB - BC)(AB + BC).\end{aligned}$$

$$\therefore AB - BC = \frac{\overline{AC}^2}{AB + BC}.$$

Now, as AB differs very little from BC, and both are very large in comparison with $\overline{AC}^2$, we may assume as a close approximation that $AB = BC$. Then

$$\begin{aligned}AA' &= AB - BC\\ &= \frac{\overline{AC}^2}{2\,BC}\\ &= \frac{1}{7912.4}\text{ miles}\\ &= \frac{5280 \times 12}{7912.4}\text{ inches.}\end{aligned}$$

$$\begin{aligned}\log 5280 &= 3.72263\\ \log 12 &= 1.07918\\ \text{colog } 7912.4 &= \underline{6.10169 - 10}\\ \log AA' &= 0.90350\\ AA' &= 8.0076.\end{aligned}$$

The required distance is 8.0076 inches.

27. In Problem 1, page 90, determine the effect upon the computed height of the tower, of an error in either the angle of elevation or the measured distance.

With the notation of Problem 1, suppose that the error in the angle is e_1 and that in the measured distance is e_2. Then the formulas

$$a = b \tan A, \qquad c = b \sec A$$

become

$$a = (b + e_2)\tan(A + e_1), \qquad c = (b + e_2)\sec(A + e_1),$$

and the error in the computed value of a is

$$\begin{aligned}&(b + e_2)\tan(A + e_1) - b\tan A\\ &\quad= b\{\tan(A + e_1) - \tan A\} + e_2\tan(A + e_1)\\ &\quad= \frac{b \sin e_1}{\cos(A + e_1)\cos A} + e_2\tan(A + e_1),\end{aligned}$$

or, approximately, for small errors,

$$\frac{be_1}{\cos^2 A} + e_2 \tan A,$$

where e_1 is measured in radians.

The error in c is

$$b\{\sec(A+e_1)-\sec A\}+e_2\sec(A+e_1)$$

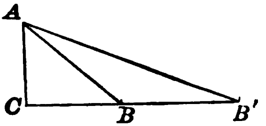
$$=\frac{b\{\cos A-\cos(A+e_1)\}}{\cos(A+e_1)\cos A}+e_2\sec(A+e_1)$$

By [23], $$=\frac{2b\sin(A+\frac{1}{2}e_1)\sin(\frac{1}{2}e_1)}{\cos(A+e_1)\cos A}+e_2\sec(A+e_1),$$

or, approximately, for small errors,

$$\frac{be^1\sin A}{\cos^2 A}+e_2\sec A=(be_1\tan A+e_2)\sec A.$$

28. To determine the height of an inaccessible object situated on a horizontal plane, by observing its angles of elevation at two points in the same line with its base, and measuring the distance between these two points.



Let AC be the object, B and B' the two points of observation. Then given the angles B' and ABC, and the side BB'; required AC.

$$AB=BB'\frac{\sin B'}{\sin BAB'}$$

$$=BB'\frac{\sin B'}{\sin(ABC-B')}.$$

$$AC=AB\sin ABC$$

$$=BB'\frac{\sin B'\sin ABC}{\sin(ABC-B')}.$$

29. The angle of elevation of an inaccessible tower situated on a horizontal plane is 63° 26′; at a point 500 feet farther from the base of the tower the angle of elevation of its top is 32° 14′. Find the height of the tower.

From the solution of Example 28,

$$AC=500\frac{\sin 32°\ 14'\sin 63°\ 26'}{\sin(63°\ 26'-32°\ 14')}$$

$$=500\frac{\sin 32°\ 14'\sin 63°\ 26'}{\sin 31°\ 12'}.$$

$$\log 500=2.69897$$
$$\log\sin 32°\ 14'=9.72703$$
$$\log\sin 63°\ 26'=9.95154$$
$$\text{colog}\sin 31°\ 12'=\underline{0.28565}$$
$$\log AC=2.66319$$

$$AC=460.46.$$

Height of the tower, 460.46 feet.

30. A tower is situated on the bank of a river. From the opposite bank the angle of elevation of the tower is 60° 13′, and from a point 40 feet more distant the angle of elevation is 50° 19′. Find the breadth of the river.

In the figure for the solution of Example 28,

$$CB=AB\cos ABC$$

$$=BB'\frac{\sin B'\cos ABC}{\sin(ABC-B')}.$$

Hence

$$CB=40\frac{\sin 50°\ 19'\cos 60°\ 13'}{\sin 9°\ 54'}.$$

$$\begin{aligned} \log 40 &= 1.60206 \\ \log\sin 50^\circ\, 19' &= 9.88626 \\ \log\cos 60^\circ\, 13' &= 9.69611 \\ \text{colog}\sin 9^\circ\, 54' &= \underline{0.76465} \\ \log CB &= 1.94908 \\ CB &= 88.936. \end{aligned}$$

Breadth of river, 88.936 feet.

31. A ship sailing north sees two lighthouses 8 miles apart, in a line due west; after an hour's sailing, one lighthouse bears S.W., and the other S.S.W. Find the ship's rate.

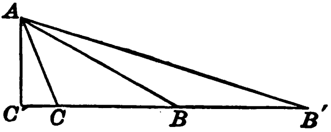
In the figure for the solution of Example 28, let B and B' be the lighthouses, C the original position of the ship, and A its final position. Then $CAB = 22^\circ\, 30'$ and $CAB' = 45^\circ$; hence $ABC = 67^\circ\, 30'$ and $B' = 45^\circ$.

$$\begin{aligned} AC &= 8\frac{\sin 45^\circ \sin 67^\circ\, 30'}{\sin 22^\circ\, 30'} \\ &= 8 \sin 45^\circ \cot 22^\circ\, 30'. \end{aligned}$$

$$\begin{aligned} \log 8 &= 0.90309 \\ \log\sin 45^\circ &= 9.84949 \\ \log\cot 22^\circ\, 30' &= \underline{10.38278} \\ \log AC &= 1.13536 \\ AC &= 13.657. \end{aligned}$$

Ship's rate, 13.657 miles per hour.

32. To determine the height of an accessible object situated on an inclined plane.



Let CBB' be the inclined plane, AC the object, B and B' two points of observation, AC' the perpendicular from A on CBB'. Then given CB, BB', and the angles ABC, B'; required AC.

From the solution of Example 28,

$$AC' = BB'\frac{\sin B' \sin ABC}{\sin (ABC - B')},$$

and $$C'B = BB'\frac{\sin B' \cos ABC}{\sin (ABC - B')}.$$

Then $C'C = C'B - CB$,

and $$AC = \sqrt{\overline{AC'}^2 + \overline{C'C}^2}.$$

33. At a distance of 40 feet from the foot of a tower on an inclined plane the tower subtends an angle of 41° 19'; at a point 60 feet farther away the angle subtended by the tower is 23° 45'. Find the height of the tower.

From the solution of Example 32,

$$AC' = 60\frac{\sin 23^\circ\, 45' \sin 41^\circ\, 19'}{\sin 17^\circ\, 34'}.$$

$$C'B = 60\frac{\sin 23^\circ\, 45' \cos 41^\circ\, 19'}{\sin 17^\circ\, 34'}.$$

$$\begin{aligned} \log 60 &= 1.77815 \\ \log\sin 23^\circ\, 45' &= 9.60503 \\ \log\sin 41^\circ\, 19' &= 9.81969 \\ \text{colog}\sin 17^\circ\, 34' &= \underline{0.52026} \\ \log AC' &= 1.72313 \\ AC' &= 52.860. \end{aligned}$$

$$\begin{aligned} \log 60 &= 1.77815 \\ \log\sin 23^\circ\, 45' &= 9.60503 \\ \log\cos 41^\circ\, 19' &= 9.87568 \\ \text{colog}\sin 17^\circ\, 34' &= \underline{0.52026} \\ \log C'B &= 1.77912 \\ C'B &= 60.134. \\ C'C &= 60.134 - 40 \\ &= 20.134. \end{aligned}$$

$$\tan ACC' = \frac{AC'}{C'C}.$$

$$AC = AC' \csc ACC'.$$

$$\begin{aligned} \log AC' &= 1.72313 \\ \text{colog}\, C'C &= \underline{8.69607 - 10} \\ \log \tan ACC' &= 0.41920 \\ ACC' &= 69^\circ\ 8'\ 55''. \\ \log AC' &= 1.72313 \\ \log \csc ACC' &= \underline{0.02941} \\ \log AC &= 1.75254 \\ AC &= 56.564. \end{aligned}$$

Height of tower, 56.564 feet.

34. A tower makes an angle of 113° 12′ with the inclined plane on which it stands; and at a distance of 89 feet from its base, measured down the plane, the angle subtended by the tower is 23° 27′. Find the height of the tower.

In the triangle ACB, given $CB = 89$ feet, $C = 113^\circ\ 12'$, $B = 23^\circ\ 27'$; required AC.

$$\begin{aligned} A &= 180^\circ - (B + C) \\ &= 43^\circ\ 21'. \end{aligned}$$

$$AC = CB \frac{\sin B}{\sin A}.$$

$$\begin{aligned} \log 89 &= 1.94939 \\ \log \sin 23^\circ\ 27' &= 9.59983 \\ \text{colog} \sin 43^\circ\ 21' &= \underline{0.16339} \\ \log AC &= 1.71261 \\ AC &= 51.595. \end{aligned}$$

Height of tower, 51.595 feet.

35. From the top of a house 42 feet high the angle of elevation of the top of a pole is 14° 13′; at the bottom of the house it is 23° 19′. Find the height of the pole.

Let A be the top of the pole, B and B' the top and bottom of the house, and C the foot of the perpendicular from A on BB'; required $B'C$.

From the solutions of Examples 28 and 30,

$$\begin{aligned} CB &= BB' \frac{\sin AB'C \cos ABC}{\sin (ABC - AB'C)} \\ &= 42 \frac{\sin 66^\circ\ 41' \cos 75^\circ\ 47'}{\sin 9^\circ\ 6'}. \end{aligned}$$

$$\begin{aligned} \log 42 &= 1.62325 \\ \log \sin 66^\circ\ 41' &= 9.96300 \\ \log \cos 75^\circ\ 47' &= 9.39021 \\ \text{colog} \sin 9^\circ\ 6' &= \underline{0.80091} \\ \log CB &= 1.77737 \\ CB &= 59.892. \\ B'C &= CB + BB' \\ &= 59.892 + 42 \\ &= 101.892. \end{aligned}$$

Height of pole, 101.892 feet.

36. The sides of a triangle are 17, 21, 28. Prove that the length of a line bisecting the greatest side and drawn from the opposite angle is 13.

Let $a = 28$, $b = 21$, $c = 17$,

Then [26]

$$17^2 = 28^2 + 21^2 - 2 \times 28 \times 21 \cos C;$$

to prove that

$$13^2 = 14^2 + 21^2 - 2 \times 14 \times 21 \cos C.$$

Subtract the first equation from twice the second,

$$\begin{aligned} 2 \times 13^2 - 17^2 &= 2 \times 14^2 - 28^2 + 21^2 \\ &= 21^2 - 2 \times 14^2 \\ &= 7^2 (3^2 - 2^3) = 7^2. \\ 2 \times 169 - 289 &= 49, \\ 49 &= 49. \end{aligned}$$

37. A privateer 10 miles S.W. of a harbor sees a ship sail from it in a direction S. 80° E. at a rate of

9 miles an hour. In what direction, and at what rate, must the privateer sail in order to come up with the ship in $1\frac{1}{2}$ hours?

Let A be the harbor, B the original position of the privateer, and C the point where the vessels are to meet. Then $A = 125°$, $b = 13\frac{1}{2}$, $c = 10$; required B and $\frac{a}{1\frac{1}{2}}$.

$$\tan\tfrac{1}{2}(B - C) = \frac{b - c}{b + c}\tan\tfrac{1}{2}(B + C)$$
$$= \frac{3.5}{23.5}\tan 27° \, 30'$$
$$= \tfrac{7}{47}\tan 27° \, 30'.$$

$$\begin{aligned} \log 7 &= 0.84510 \\ \text{colog}\, 47 &= 8.32790 - 10 \\ \log\tan 27° \, 30' &= \underline{9.71648} \\ \log\tan\tfrac{1}{2}(B - C) &= 8.88948 \\ \tfrac{1}{2}(B - C) &= 4° \, 26'. \\ B - C &= 8° \, 52'. \\ B + C &= 55°. \\ \therefore B &= 31° \, 56'. \end{aligned}$$

$$a = b\frac{\sin A}{\sin B}$$
$$= 13.5\frac{\sin 125°}{\sin 31° \, 56'}.$$

$$\begin{aligned} \log 13.5 &= 1.13033 \\ \log\sin 125° &= 9.91336 \\ \text{colog}\sin 31° \, 56' &= \underline{0.27660} \\ \log a &= 1.32029 \\ a &= 20.907. \end{aligned}$$

$$\frac{a}{1\frac{1}{2}} = 13.938.$$

Privateer's course, 31° 56′ E. of N.E., or N. 76° 56′ E.; rate 13.938 miles per hour.

38. A person goes 70 yards up a slope of 1 in $3\frac{1}{2}$ from the edge of a river, and observes the angle of depression of an object on the opposite bank to be $2\frac{1}{4}°$. Find the breadth of the river.

Let A and B be the original and final positions of the observer, and C the object observed. Then given $c = 70$, $C = 2\frac{1}{4}°$, $A = 180° - \tan^{-1}\frac{1}{3\frac{1}{2}}$; required b.

$$\begin{aligned} A &= 180° - \tan^{-1}\tfrac{2}{7} \\ &= 180° - \tan^{-1} 0.2857 \\ &= 180° - 15° \, 56' \, 40'' \\ &= 164° \, 3' \, 20''. \\ B &= 180° - (A + C) \\ &= 13° \, 41' \, 40''. \end{aligned}$$

$$b = c\frac{\sin B}{\sin C}.$$

$$\begin{aligned} \log 70 &= 1.84510 \\ \log\sin 13° \, 41' \, 40'' &= 9.37428 \\ \text{colog}\sin 2° \, 15' &= \underline{1.40605} \\ \log b &= 2.62543 \\ b &= 422.11. \end{aligned}$$

Breadth of river, 422.11 yards.

39. The length of a lake subtends, at a certain point, an angle of 46° 24′, and the distances from this point to the two extremities of the lake are 346 and 290 feet. Find the length of the lake.

Given $A = 46° \, 24'$, $b = 346$, $c = 290$; required a.

$$\tan\tfrac{1}{2}(B - C) = \frac{b - c}{b + c}\tan\tfrac{1}{2}(B + C)$$
$$= \frac{56}{636}\tan 66° \, 48'.$$

$$\begin{aligned} \log 56 &= 1.74819 \\ \text{colog}\, 636 &= 7.19654 - 10 \\ \log\tan 66° \, 48' &= \underline{10.36795} \\ \log\tan\tfrac{1}{2}(B - C) &= 9.31268 \end{aligned}$$

$$\tfrac{1}{2}(B-C) = 11^\circ\ 36'\ 33''.$$
$$B-C = 23^\circ\ 13'\ 6''.$$
$$B+C = 133^\circ\ 36'.$$
$$\therefore B = 78^\circ\ 24'\ 33''.$$
$$a = b\frac{\sin A}{\sin B} = 346\frac{\sin 46^\circ\ 24'}{\sin 78^\circ\ 24'\ 33''}.$$

$\log 346 = 2.53908$
$\log\sin 46^\circ\ 24' = 9.85984$
$\text{colog}\sin 78^\circ\ 24'\ 33'' = 0.00895$
$\log a = 2.40787$
$a = 255.78.$

Length of lake, 255.78 feet.

40. Two ships are a mile apart. The angular distance of the first ship from a fort on shore, as observed from the second ship, is 35° 14′ 10″; the angular distance of the second ship from the fort, observed from the first ship, is 42° 11′ 53″. Find the distance in feet from each ship to the fort.

Given $B = 35^\circ\ 14'\ 10''$, $C = 42^\circ\ 11'\ 53''$, $a = 5280$; required b and c.

$$A = 180 - (B+C)$$
$$= 102^\circ\ 33'\ 57''.$$
$$b = a\frac{\sin B}{\sin A}.$$

$\log 5280 = 3.72263$
$\log\sin 35^\circ\ 14'\ 10'' = 9.76114$
$\text{colog}\sin 102^\circ\ 33'\ 57'' = 0.01053$
$\log b = 3.49430$
$b = 3121.1.$

$$c = a\frac{\sin C}{\sin A}.$$

$\log 5280 = 3.72263$
$\log\sin 42^\circ\ 11'\ 53'' = 9.82717$
$\text{colog}\sin 102^\circ\ 33'\ 57'' = 0.01053$
$\log c = 3.56033$
$c = 3633.5.$

Distance of first ship from fort, 3121.1 feet; of second ship from fort, 3633.5 feet.

41. Along the bank of a river is drawn a base line of 500 feet. The angular distance of one end of this line from an object on the opposite side of the river, as observed from the other end of the line, is 53°; that of the second extremity from the same object, observed at the first, is 79° 12′. Find the breadth of the river.

Given $B = 53^\circ$, $C = 79^\circ\ 12'$, $a = 500$; required p, the perpendicular from A on a.

$$A = 180^\circ - (B+C)$$
$$= 47^\circ\ 48'.$$
$$b = a\frac{\sin B}{\sin A}.$$
$$p = b\sin C = a\frac{\sin B\sin C}{\sin A}$$
$$= 500\frac{\sin 53^\circ \sin 79^\circ\ 12'}{\sin 47^\circ\ 48'}.$$

$\log 500 = 2.69897$
$\log\sin 53^\circ = 9.90235$
$\log\sin 79^\circ\ 12' = 9.99224$
$\text{colog}\sin 47^\circ\ 48' = 0.13030$
$\log p = 2.72386$
$p = 529.49.$

Breadth of river, 529.49 feet.

42. A vertical tower stands on a declivity inclined 15° to the horizon. A man ascends the declivity 80 feet from the base of the tower, and finds the angle then subtended by the tower to be 30°. Find the height of the tower.

Let A and B be the top and bottom of the tower, and C the position of observation. Then given $a = 80$, $B = 75°$, $C = 30°$; required c.

$$A = 180° - (B + C) = 75°.$$

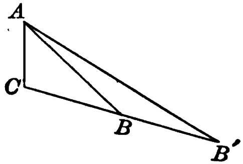
$$c = \frac{a \sin C}{\sin A}$$

$$= \frac{80 \sin 30°}{\sin 75°}.$$

$$\log 80 = 1.90309$$
$$\log \sin 30° = 9.69897$$
$$\text{colog} \sin 75° = \underline{0.01506}$$
$$\log c = 1.61712$$
$$c = 41.411.$$

Height of tower, 41.411 feet.

43. The angle subtended by a tower on an inclined plane is, at a certain point, 42° 17′; 325 feet farther down, it is 21° 47′. The inclination of the plane is 8° 53′. Find the height of the tower.



$$AB = BB' \frac{\sin B'}{\sin BAB'}$$

$$= BB' \frac{\sin B'}{\sin (B - B')}.$$

$$AC = \frac{AB \sin B}{\sin C}$$

$$= BB' \frac{\sin B \sin B'}{\sin C \sin (B - B')}$$

$$= 325 \frac{\sin 42° 17' \sin 21° 47'}{\sin 98° 53' \sin 20° 30'}.$$

$$\log 325 = 2.51188$$
$$\log \sin 42° 17' = 9.82788$$
$$\log \sin 21° 47' = 9.56949$$
$$\text{colog} \sin 98° 53' = 0.00524$$
$$\text{colog} \sin 20° 30' = \underline{0.45567}$$
$$\log AC = 2.37016$$
$$AC = 234.51.$$

Height of tower, 234.51 feet.

44. A cape bears north by east, as seen from a ship. The ship sails northwest 30 miles, and then the cape bears east. How far is it from the second point of observation?

Let A be the cape, B and C the first and second positions of the ship. Then given $B = 56° \ 15'$, $C = 45°$, $a = 30$; required b.

$$A = 180° - (B + C) = 78° 45'.$$

$$b = \frac{a \sin B}{\sin A} = \frac{30 \sin 56° 15'}{\sin 78° 45'}.$$

$$\log 30 = 1.47712$$
$$\log \sin 56° 15' = 9.91985$$
$$\text{colog} \sin 78° 45' = \underline{0.00843}$$
$$\log b = 1.40540$$
$$b = 25.433.$$

Distance of cape from second point of observation, 25.433 miles.

45. Two observers, stationed on *opposite* sides of a cloud, observe its angles of elevation to be 44° 56′ and 36° 4′. Their distance from each other is 700 feet. What is the height of the cloud?

Given $A = 44° \ 56'$, $B = 36° \ 4'$, $c = 700$; required the perpendicular p from C on c.

$$C = 180° - (A + B) = 99°.$$

$$b = c \frac{\sin B}{\sin c}.$$

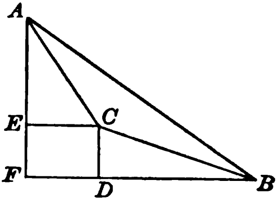
$$p = b \sin A = c \frac{\sin B \sin A}{\sin C}$$

$$= 700 \frac{\sin 36° \, 4' \sin 44° \, 56'}{\sin 99°}.$$

$$\begin{aligned} \log 700 &= 2.84510 \\ \log \sin 36° \, 4' &= 9.76991 \\ \log \sin 44° \, 56' &= 9.84898 \\ \text{colog} \sin 99° &= \underline{0.00538} \\ \log p &= 2.46937 \\ p &= 294.69. \end{aligned}$$

Height of cloud, 294.69 feet.

46. **From a point B at the foot of a mountain, the angle of elevation of the top A is 60°. After ascending the mountain one mile, at an inclination of 30° to the horizon, and reaching a point C, the angle ACB is found to be 135°. Find the height of the mountain in feet.**



$$\begin{aligned} CD &= CB \sin CBD \\ &= 5280 \times \tfrac{1}{2} = 2640. \end{aligned}$$

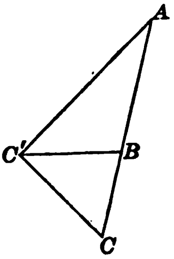
$$AC = \frac{CB \sin CBA}{\sin CAB}.$$

$$\begin{aligned} AE &= AC \sin ECA \\ &= \frac{CB \sin CBA \sin ECA}{\sin CAB} \\ &= \frac{5280 \sin 30° \sin 75°}{\sin 15°} \\ &= \frac{5280 \times \frac{1}{2} \cos 15°}{\sin 15°} \\ &= 2640 \cot 15°. \end{aligned}$$

$$\begin{aligned} \log 2640 &= 3.42160 \\ \log \cot 15° &= \underline{10.57195} \\ \log AE &= 3.99355 \\ AE &= 9852.6. \\ AF &= AE + CD \\ &= 12{,}492.6. \end{aligned}$$

Height of the mountain, 12,492.6 feet.

47. **From a ship two rocks are seen in the same right line with the ship, bearing N. 15° E. After the ship has sailed northwest 5 miles, the first rock bears east, and the second northeast. Find the distance between the rocks.**



Let A and B be the two rocks, C and C' the first and second positions of the ship. Then given $C = 60°$, $CC'B = 45°$, $CC'A = 90°$, $CC' = 5$; required AB.

$$AC = CC' \sec C = 5 \times 2 = 10.$$

$$BC = CC' \frac{\sin BC'C}{\sin CBC'} = 5 \frac{\sin 45°}{\sin 75°}.$$

$$\begin{aligned} \log 5 &= 0.69897 \\ \log \sin 45° &= 9.84949 \\ \text{colog} \sin 75° &= \underline{0.01506} \\ \log BC &= 0.56352 \\ BC &= 3.6603. \end{aligned}$$

$$AB = AC - BC$$
$$= 6.3397.$$

Distance between rocks, 6.3397 miles.

48. From a window on a level with the bottom of a steeple the angle of elevation of the steeple is 40°, and from a second window 18 feet higher the angle of elevation is 37° 30′. Find the height of the steeple.

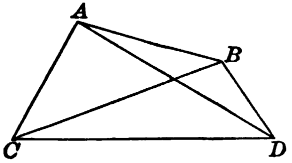
Let A and B be the windows, and C the top of the steeple. Then given $c = 18$, $A = 50°$, $B = 127° 30'$; required height of steeple.

$$h = b \sin 40°.$$
$$C = 180° - (A + B) = 2° 30'.$$
$$b = c\frac{\sin B}{\sin C} = 18\frac{\sin 127° 30'}{\sin 2° 30'}.$$
$$h = 18\frac{\sin 127° 30' \sin 40°}{\sin 2° 30'}.$$

$$\begin{aligned} \log 18 &= 1.25527 \\ \log\sin 127° 30' &= 9.89947 \\ \log\sin 40° &= 9.80807 \\ \operatorname{colog}\sin 2° 30' &= \underline{1.36032} \\ \log h &= 2.32313 \\ h &= 210.44. \end{aligned}$$

Height of steeple, 210.44 feet.

49. To determine the distance between two inaccessible objects by observing angles at the extremities of a line of known length.



Let A and B be the inaccessible objects, C and D the extremities of the given line. Then, given CD, ACD, BCD, ADC, and BDC; required AB.

$$AC = CD\frac{\sin ADC}{\sin CAD}.$$
$$BC = CD\frac{\sin BDC}{\sin CBD}.$$

Then, in the triangle CAB, two sides and the included angle are known, and the third side can be computed as usual.

50. Wishing to determine the distance between a church A and a tower B, on the opposite side of a river, I measure a line CD along the river (C being nearly opposite A), and observe the angles ACB, 58° 20′; ACD, 95° 20′; ADB, 53° 30′; BDC, 98° 45′. CD is 600 feet. What is the distance required?

From the solution of Example 49,

$$AC = CD\frac{\sin ADC}{\sin CAD}$$
$$= 600\frac{\sin 45° 15'}{\sin 39° 25'}.$$
$$BC = CD\frac{\sin BDC}{\sin CBD}$$
$$= 600\frac{\sin 98° 45'}{\sin 44° 15'}.$$

$$\begin{aligned} \log 600 &= 2.77815 \\ \log\sin 45° 15' &= 9.85137 \\ \operatorname{colog}\sin 39° 25' &= \underline{0.19726} \\ \log AC &= 2.82678 \\ AC &= 671.09. \end{aligned}$$

$$\begin{aligned} \log 600 &= 2.77815 \\ \log\sin 98° 45' &= 9.99492 \\ \operatorname{colog}\sin 44° 15' &= \underline{0.15627} \\ \log BC &= 2.92934 \end{aligned}$$

$$BC = 849.84.$$

$$\begin{aligned}\tan \tfrac{1}{2}(CAB - CBA) &= \frac{BC - AC}{BC + AC} \tan \tfrac{1}{2}(CAB + CBA)\\ &= \frac{178.75}{1520.93} \tan 60^\circ\, 50'.\end{aligned}$$

$$\begin{aligned}\log 178.75 &= 2.25224\\ \operatorname{colog} 1520.93 &= 6.81789-10\\ \log \tan 60^\circ\, 50' &= \underline{10.25327}\\ \log \tan \tfrac{1}{2}(CAB - CBA) &= 9.32340\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}(CAB - CBA) &= 11^\circ\, 53'\, 28''\\ \tfrac{1}{2}(CAB + CBA) &= \underline{60^\circ\, 50'}\\ CAB &= 72^\circ\, 43'\, 28''\end{aligned}$$

$$\begin{aligned}AB &= BC \frac{\sin ACB}{\sin CAB}\\ &= 849.84 \frac{\sin 58^\circ\, 20'}{\sin 72^\circ\, 43'\, 28''}.\end{aligned}$$

$$\begin{aligned}\log 849.84 &= 2.92934\\ \log \sin 58^\circ\, 20' &= 9.92999\\ \operatorname{colog} \sin 72^\circ\, 43'\, 28'' &= \underline{0.02005}\\ \log AB &= 2.87938\end{aligned}$$

$$AB = 757.50.$$

Required distance, 757.50 feet.

51. Wishing to find the height of a summit A, I measure a horizontal base line CD, 440 yards. At C, the angle of elevation of A is 37° 18′, and the horizontal angle between D and the summit is 76° 18′; at D the horizontal angle between C and the summit is 67° 14′. Find the height.

Let A' be the point directly under A, in the same horizontal plane with CD. Then, in the triangle $A'CD$,

$$\begin{aligned}A'C &= CD \frac{\sin D}{\sin A'}\\ &= 440 \frac{\sin 67^\circ\, 14'}{\sin 36^\circ\, 28'}.\end{aligned}$$

$$\begin{aligned}AA' &= A'C \tan ACA'\\ &= 440 \frac{\sin 67^\circ\, 14'}{\sin 36^\circ\, 28'} \tan 37^\circ\, 18'.\end{aligned}$$

$$\begin{aligned}\log 440 &= 2.64345\\ \log \sin 67^\circ\, 14' &= 9.96477\\ \log \tan 37^\circ\, 18' &= 9.88184\\ \operatorname{colog} \sin 36^\circ\, 28' &= \underline{0.22595}\\ \log AA' &= 2.71601\\ AA' &= 520.01.\end{aligned}$$

Height, 520.01 yards.

52. A balloon is observed from two stations 3000 feet apart. At the first station the horizontal angle of the balloon and the other station is 75° 25′, and the angle of elevation of the balloon is 18°. The horizontal angle of the first station and the balloon, measured at the second station, is 64° 30′. Find the height of the balloon.

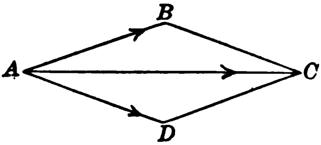
Let B be the first station, C the second, A the position of the balloon, and A' the point directly under A, in the same horizontal plane as BC. Then

$$\begin{aligned}AA' &= A'B \tan A'BA\\ &= BC \frac{\sin A'CB}{\sin BA'C} \tan A'BA\\ &= 3000 \frac{\sin 64^\circ\, 30'}{\sin 40^\circ\, 5'} \tan 18^\circ\end{aligned}$$

$$\begin{aligned}\log 3000 &= 3.47712\\ \log \sin 64^\circ\, 30' &= 9.95549\\ \log \tan 18^\circ &= 9.51178\\ \operatorname{colog} \sin 40^\circ\, 5' &= \underline{0.19118}\\ \log AA' &= 3.13557\\ AA' &= 1366.4.\end{aligned}$$

Height of balloon, 1366.4 feet.

53. Two forces, one of 410 pounds, and the other of 320 pounds, make an angle of 51° 37′. Find the intensity and the direction of their resultant.



Let AB and AD represent the forces, and AC their resultant. Then, in the triangle ABC, given $c = 410$, $a = 320$, $B = 180° - 51° 37' = 128° 23'$; required b and A.

$$\begin{aligned} \tan \tfrac{1}{2}(C - A) &= \frac{c-a}{c+a} \tan \tfrac{1}{2}(C + A) \\ &= \frac{90}{730} \tan 25° 48' 30''. \end{aligned}$$

$$\begin{aligned} \log 90 &= 1.95424 \\ \log \tan 25° 48' 30'' &= 9.68448 \\ \text{colog } 730 &= \underline{7.13668 - 10} \\ \tan \tfrac{1}{2}(C - A) &= 8.77540 \\ \tfrac{1}{2}(C - A) &= 3° 24' 43'' \\ \tfrac{1}{2}(C + A) &= \underline{25° 48' 30''} \\ A &= 22° 23' 47'' \end{aligned}$$

$$\begin{aligned} b &= a \frac{\sin B}{\sin A} \\ &= 320 \frac{\sin 128° 23'}{\sin 22° 23' 47''}. \end{aligned}$$

$$\begin{aligned} \log 320 &= 2.50515 \\ \log \sin 128° 23' &= 9.89425 \\ \text{colog} \sin 22° 23' 47'' &= \underline{0.41906} \\ \log b &= 2.81846 \\ b &= 658.36. \end{aligned}$$

Intensity of resultant, 658.36 pounds; angle between resultant and first force, 22° 23′ 47″.

54. An unknown force, combined with one of 128 pounds, produces a resultant of 200 pounds, and this resultant makes an angle of 18° 24′ with the known force. Find the intensity and direction of the unknown force.

In the figure for the solution of Example 53, given, in the triangle ABC, $c = 128$, $A = 18° 24'$, $b = 200$; required a and B.

$$\begin{aligned} \tan \tfrac{1}{2}(B - C) &= \frac{b-c}{b+c} \tan \tfrac{1}{2}(B + C) \\ &= \frac{72}{328} \tan 80° 48'. \end{aligned}$$

$$\begin{aligned} \log 72 &= 1.85733 \\ \log \tan 80° 48' &= 10.79058 \\ \text{colog } 328 &= \underline{7.48413 - 10} \\ \log \tan \tfrac{1}{2}(B - C) &= 10.13204 \\ \tfrac{1}{2}(B - C) &= 53° 34' 44'' \\ \tfrac{1}{2}(B + C) &= \underline{80° 48'} \\ B &= 134° 22' 44'' \\ 180° - B &= 45° 37' 16''. \end{aligned}$$

$$a = \frac{b \sin A}{\sin B} = 200 \frac{\sin 18° 24'}{\sin 134° 22' 44''}.$$

$$\begin{aligned} \log 200 &= 2.30103 \\ \log \sin 18° 24' &= 9.49920 \\ \text{colog} \sin 134° 22' 44'' &= \underline{0.14586} \\ \log a &= 1.94609 \\ a &= 88.326. \end{aligned}$$

Intensity of unknown force, 88.326 pounds; angle between known and unknown forces, 45° 37′ 16″.

55. At two stations, the height of a kite subtends the same angle A. The angle which the line joining one station and the kite subtends at the other station is B; and the distance between the two stations is a. Show that the height of the kite is $\frac{1}{2} a \sin A \sec B$.

Let C be the position of the kite, D and E the stations, and C' the point directly under C in the same horizontal plane with DE.

Since the elevation of the kite is the same at D and E, the triangle CDE is isosceles, and

$$CD = CE = \tfrac{1}{2} a \sec B.$$

Also
$$CC' = CD \sin A$$
$$= \tfrac{1}{2} a \sin A \sec B.$$

56. Two towers on a horizontal plane are 120 feet apart. A person standing successively at their bases observes that the angle of elevation of one is double that of the other; but, when he is halfway between them, the angles of elevation are complementary. Prove that the heights of the towers are 90 and 40 feet.

Let A and B be the tops of the towers, A' and B' their bases, and C the point halfway between them. Then the triangles $AA'C$ and $BB'C$ are similar, and

$$\frac{AA'}{B'C} = \frac{A'C}{BB'}.$$

$$AA' \times BB' = B'C \times A'C$$
$$= 3600.$$

Also
$$AB'A' = 2\,BA'B'.$$

$$\therefore \tan AB'A' = \frac{2 \tan BA'B'}{1 - \tan^2 BA'B'},$$

or
$$\frac{AA'}{120} = \frac{2\,\dfrac{BB'}{120}}{1 - \dfrac{\overline{BB'}^2}{120^2}}$$

$$= \frac{240\,BB'}{120^2 - \overline{BB'}^2}.$$

$$AA'(120^2 - \overline{BB'}^2) = 120 \times 240\,BB'.$$

$$\frac{3600}{BB'}(120^2 - \overline{BB'}^2) = 120 \times 240\,BB'.$$

$$120^2 - \overline{BB'}^2 = 8\,\overline{BB'}^2.$$

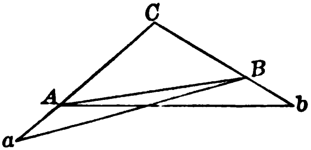
$$\overline{BB'}^2 = 40^2.$$

$$BB' = 40.$$

$$AA' = 90.$$

Heights of towers, 90 feet and 40 feet.

57. To find the distance of an inaccessible point C from either of two points A and B, having no instruments to measure angles. Prolong CA to a, and CB to b, and join AB, Ab, and Ba. Measure AB, 500; aA, 100; aB, 560; bB, 100; and Ab, 550. Compute the distances AC and BC.



In the triangle aAB,

$$s = \tfrac{1}{2}(500 + 100 + 560) = 580.$$

$$\tan \tfrac{1}{2} aAB = \sqrt{\frac{80 \times 480}{580 \times 20}} = \sqrt{\frac{96}{29}}.$$

$$\log 96 = 1.98227$$
$$\text{colog } 29 = 8.53760 - 10$$
$$2\,)\,\underline{0.51987}$$
$$\log \tan \tfrac{1}{2} aAB = 10.25993$$
$$\tfrac{1}{2} aAB = 61° 12' 20''.$$
$$aAB = 122° 24' 40''.$$
$$CAB = 57° 35' 20''.$$

In the triangle bAB,

$$s = \tfrac{1}{2}(500 + 550 + 100) = 575.$$

$$\tan \tfrac{1}{2} bBA = \sqrt{\frac{75 \times 475}{575 \times 25}} = \sqrt{\frac{57}{23}}.$$

$$\begin{aligned}
\log 57 &= 1.75587 \\
\operatorname{colog} 23 &= \underline{8.63827 - 10} \\
&\;\; 2\,\overline{)\,0.39414} \\
\log\tan \tfrac{1}{2} bBA &= 10.19707 \\
\tfrac{1}{2} bBA &= 57^\circ\, 34'\, 30''. \\
bBA &= 115^\circ\ \ 9'. \\
CBA &= 64^\circ\, 51'.
\end{aligned}$$

In the triangle ABC,

$$\begin{aligned}
A &= 57^\circ\, 35'\, 20'', \\
B &= 64^\circ\, 51', \\
C &= 57^\circ\, 33'\, 40''.
\end{aligned}$$

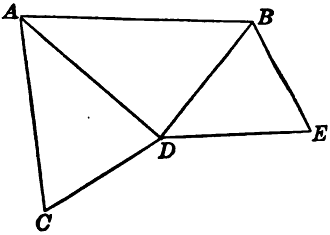
$$\begin{aligned}
BC &= AB\frac{\sin A}{\sin C} \\
&= 500\,\frac{\sin 57^\circ\, 35'\, 20''}{\sin 57^\circ\, 33'\, 40''}. \\
AC &= AB\frac{\sin B}{\sin C} \\
&= 500\,\frac{\sin 64^\circ\, 51'}{\sin 57^\circ\, 33'\, 40''}.
\end{aligned}$$

$$\begin{aligned}
\log 500 &= 2.69897 \\
\log\sin 57^\circ\, 35'\, 20'' &= 9.92646 \\
\operatorname{colog}\sin 57^\circ 33' 40'' &= \underline{0.07368} \\
\log BC &= 2.69911 \\
BC &= 500.16.
\end{aligned}$$

$$\begin{aligned}
\log 500 &= 2.69897 \\
\log\sin 64^\circ\, 51' &= 9.95674 \\
\operatorname{colog}\sin 57^\circ 33' 40'' &= \underline{0.07368} \\
\log AC &= 2.72939 \\
AC &= 536.28.
\end{aligned}$$

Distances of C from A and B, 536.28 feet; 500.16 feet.

58. Two inaccessible points A and B are visible from D, but no other point can be found whence both are visible. Take some point C whence A and D can be seen, and measure CD, 200 feet; ADC, 89°; ACD, 50° 30′. Then take some point E whence D and B are visible, and measure DE, 200 feet; BDE, 54° 30′; BED, 88° 30′. At D measure ADB, 72° 30′. Compute the distance AB.



$$\begin{aligned}
AD &= CD\frac{\sin ACD}{\sin CAD} \\
&= 200\,\frac{\sin 50^\circ\, 30'}{\sin 40^\circ\, 30'}.
\end{aligned}$$

$$\begin{aligned}
\log 200 &= 2.30103 \\
\log\sin 50^\circ\, 30' &= 9.88741 \\
\operatorname{colog}\sin 40^\circ\, 30' &= \underline{0.18746} \\
\log AD &= 2.37590 \\
AD &= 237.63.
\end{aligned}$$

$$\begin{aligned}
BD &= DE\frac{\sin BED}{\sin DBE} \\
&= 200\,\frac{\sin 88^\circ\, 30'}{\sin 37^\circ}.
\end{aligned}$$

$$\begin{aligned}
\log 200 &= 2.30103 \\
\log\sin 88^\circ\, 30' &= 9.99985 \\
\operatorname{colog}\sin 37^\circ &= \underline{0.22054} \\
\log BD &= 2.52142 \\
BD &= 332.22.
\end{aligned}$$

$$\begin{aligned}
&\tan \tfrac{1}{2}(DAB - DBA) \\
&\quad = \frac{BD - AD}{BD + AD}\tan \tfrac{1}{2}(DAB + DBA) \\
&\quad = \frac{94.59}{569.85}\tan 53^\circ\, 45'.
\end{aligned}$$

$$\begin{aligned}
\log 94.59 &= 1.97585\\
\text{colog}\, 569.85 &= 7.24424\\
\log \tan 53^\circ\, 45' &= \underline{10.13476}\\
\log \tan \tfrac{1}{2}(DAB - DBA) &= 9.35485
\end{aligned}$$

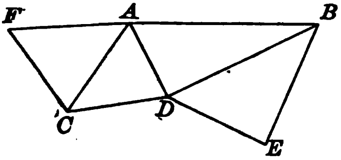
$$\begin{aligned}
\tfrac{1}{2}(DAB - DBA) &= 12^\circ\, 45'\, 21''\\
\tfrac{1}{2}(DAB + DBA) &= \underline{53^\circ\, 45'}\\
DAB &= 66^\circ\, 30'\, 21''
\end{aligned}$$

$$\begin{aligned}
AB &= BD\frac{\sin ADB}{\sin DAB}\\
&= 332.22\,\frac{\sin 72^\circ\, 30'}{\sin 66^\circ\, 30'\, 21''}.
\end{aligned}$$

$$\begin{aligned}
\log 332.22 &= 2.52142\\
\log \sin 72^\circ\, 30' &= 9.97942\\
\text{colog} \sin 66^\circ\, 30'\, 21'' &= \underline{0.03758}\\
\log AB &= 2.53842\\
AB &= 345.48.
\end{aligned}$$

Distance AB, 345.48 feet.

59. To compute the horizontal distance between two inaccessible points A and B, when no point can be found whence both can be seen. Take two points C and D, distant 200 yards, so that A can be seen from C, and B from D. From C measure CF, 200 yards to F, whence A can be seen; and from D, measure DE, 200 yards to E, whence B can be seen. Measure AFC, 83°; ACD, 53° 30′; ACF, 54° 31′; BDE, 54° 30′; BDC, 156° 25′; DEB, 88° 30′.



$$\begin{aligned}
AC &= CF\frac{\sin AFC}{\sin CAF}\\
&= 200\,\frac{\sin 83^\circ}{\sin 42^\circ\, 29'}.
\end{aligned}$$

$$\begin{aligned}
\log 200 &= 2.30103\\
\log \sin 83^\circ &= 9.99675\\
\text{colog} \sin 42^\circ\, 29' &= \underline{0.17045}\\
\log AC &= 2.46823\\
AC &= 293.92.
\end{aligned}$$

$$\begin{aligned}
BD &= DE\frac{\sin BED}{\sin DBE}\\
&= 200\,\frac{\sin 88^\circ\, 30'}{\sin 37^\circ}.
\end{aligned}$$

$$\begin{aligned}
\log 200 &= 2.30103\\
\log \sin 88^\circ\, 30' &= 9.99985\\
\text{colog} \sin 37^\circ &= \underline{0.22054}\\
\log BD &= 2.52142\\
BD &= 332.22.
\end{aligned}$$

$$\begin{aligned}
&\tan \tfrac{1}{2}(ADC - CAD)\\
&= \frac{AC - CD}{AC + CD}\tan \tfrac{1}{2}(ADC + CAD)\\
&= \frac{93.92}{493.92}\tan 63^\circ\, 15'.
\end{aligned}$$

$$\begin{aligned}
\log 93.92 &= 1.97276\\
\text{colog}\, 493.92 &= 7.30634 - 10\\
\log \tan 63^\circ\, 15' &= \underline{10.29753}\\
\log \tan \tfrac{1}{2}(ADC - CAD) &= 9.57663.
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}(ADC - CAD) &= 20^\circ\, 40'\, 8''\\
\tfrac{1}{2}(ADC + CAD) &= \underline{63^\circ\, 15'}\\
ADC &= 83^\circ\, 55'\, 8''
\end{aligned}$$

$$\begin{aligned}
AD &= AC\frac{\sin ACD}{\sin ADC}\\
&= 293.92\,\frac{\sin 53^\circ\, 30'}{\sin 83^\circ\, 55'\, 8''}
\end{aligned}$$

$$\begin{aligned}
\log 293.92 &= 2.46823\\
\log \sin 53^\circ\, 30' &= 9.90518\\
\text{colog} \sin 83^\circ\, 55'\, 8'' &= \underline{0.00245}\\
\log AD &= 2.37586\\
AD &= 237.61.
\end{aligned}$$

$$\begin{aligned} BDA &= BDC - ADC \\ &= 156° \, 25' - 83° \, 55' \, 8'' \\ &= 72° \, 29' \, 52''. \end{aligned}$$

$$\begin{aligned} &\tan \tfrac{1}{2}(DAB - DBA) \\ &= \frac{BD - AD}{BD + AD} \tan \tfrac{1}{2}(DAB + DBA) \\ &= \frac{94.61}{569.83} \tan 53° \, 45' \, 4''. \end{aligned}$$

$$\begin{aligned} \log 94.61 &= 1.97594 \\ \text{colog } 569.83 &= 7.24426 - 10 \\ \log \tan 53° \, 45' \, 4'' &= \underline{10.13478} \\ \log \tan \tfrac{1}{2}(DAB - DBA) & \\ &= 9.35498 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(DAB - DBA) &= 12° \, 45' \, 35'' \\ \tfrac{1}{2}(DAB + DBA) &= \underline{53° \, 45' \, \ 4''} \\ DAB &= 66° \, 30' \, 39'' \end{aligned}$$

$$\begin{aligned} AB &= BD \frac{\sin ADB}{\sin BAD} \\ &= 332.22 \frac{\sin 72° \, 29' \, 52''}{\sin 66° \, 30' \, 39''}. \end{aligned}$$

$$\begin{aligned} \log 332.22 &= 2.52142 \\ \log \sin 72° \, 29' \, 52'' &= 9.97941 \\ \text{colog} \sin 66° \, 30' \, 39'' &= \underline{0.03757} \\ \log AB &= 2.53840 \\ AB &= 345.46. \end{aligned}$$

Distance AB, 345.46 yards.

60. A column in the north temperate zone is east-southeast of an observer, and at noon the extremity of its shadow is northeast of him. The shadow is 80 feet in length, and the elevation of the column, at the observer's station, is 45°. Find the height of the column.

Let A be the observer's position, B the extremity of the shadow, and C the base of the column. Then given $A = 67° \, 30'$, $C = 67° \, 30'$, $a = 80$; required b.

$$\begin{aligned} b &= a \frac{\sin B}{\sin A} \\ &= 80 \frac{\sin 45°}{\sin 67° \, 30'}. \end{aligned}$$

$$\begin{aligned} \log 80 &= 1.90309 \\ \log \sin 45° &= 9.84949 \\ \text{colog} \sin 67° \, 30' &= \underline{0.03438} \\ \log b &= 1.78696 \\ b &= 61.23. \end{aligned}$$

Let B' be the top of the column. Then $\triangle \, AB'C$ is isosceles since $A = B' = 45°$.

Therefore, the height of the column is 61.23 feet.

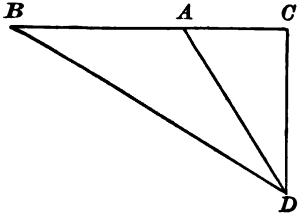
61. From the top of a hill the angles of depression of two objects situated in the horizontal plane of the base of the hill are 45° and 30°; and the horizontal angle between the two objects is 30°. Show that the height of the hill is equal to the distance between the objects.

Let A be the top of the hill, A' the point directly under A in the horizontal plane of the base of the hill, B and C the objects observed.

Then

$$\begin{aligned} A'B &= A'A. \\ A'C &= A'A \tan 60° = \sqrt{3} \, A'A. \\ \overline{BC}^2 &= \overline{A'B}^2 + \overline{A'C}^2 \\ &\quad - 2 \, A'B \times A'C \cos BA'C \\ &= \overline{A'A}^2 + 3 \, \overline{A'A}^2 \\ &\quad - 2 \, A'A \times \sqrt{3} \, A'A \times \tfrac{1}{2}\sqrt{3} \\ &= \overline{A'A}^2 + 3 \, \overline{A'A}^2 - 3 \, \overline{A'A}^2 \\ &= \overline{A'A}^2. \\ BC &= A'A. \end{aligned}$$

62. Wishing to know the breadth of a river from A to B, I take AC, 100 yards in the prolongation of BA, and then take CD, 200 yards at right angles to AC. The angle BDA is 37° 18′ 30″. Find AB.



$$\tan ADC = \frac{AC}{CD} = \frac{100}{200} = \frac{1}{2}.$$

$$\log \tan ADC = 9.69897.$$
$$ADC = 26° 33' 54''.$$
$$BDC = ADB + ADC$$
$$= 63° 52' 24''.$$
$$BC = CD \tan BDC$$
$$= 200 \tan 63° 52' 24''.$$

$$\begin{aligned} \log 200 &= 2.30103 \\ \log \tan 63° 52' 24'' &= \underline{10.30939} \\ \log BC &= 2.61042 \end{aligned}$$

$$BC = 407.77.$$
$$AB = BC - AC$$
$$= 307.77.$$
$$AB = 307.77 \text{ yards.}$$

63. The sum of the sides of a triangle is 100. The angle at A is double that at B, and the angle at B is double that at C. Determine the sides.

$$B = 2C.$$
$$A = 2B = 4C.$$
$$A + B + C = 7C = 180°.$$
$$\therefore C = 25° 42' 51\tfrac{3}{7}''.$$
$$B = 51° 25' 42\tfrac{6}{7}''.$$
$$A = 102° 51' 25\tfrac{5}{7}''.$$
$$\frac{a}{c} = \frac{\sin A}{\sin C}.$$

$$\begin{aligned} \log \sin A &= 9.98897 \\ \text{colog} \sin C &= \underline{0.36263} \\ \log \frac{a}{c} &= 0.35160 \end{aligned}$$

$$\frac{a}{c} = 2.247.$$
$$a = 2.247\,c.$$
$$\frac{b}{c} = \frac{\sin B}{\sin C}.$$

$$\begin{aligned} \log \sin B &= 9.89311 \\ \text{colog} \sin C &= \underline{0.36263} \\ \log \frac{b}{c} &= 0.25574 \end{aligned}$$

$$\frac{b}{c} = 1.802.$$
$$b = 1.802\,c.$$
$$a + b + c = (2.247 + 1.802 + 1)\,c$$
$$= 5.049\,c.$$

$$\begin{aligned} \therefore c = \frac{100}{5.049} &= 19.806 \\ a = 2.247\,c &= 44.504 \\ b = 1.802\,c &= \underline{35.690} \\ a + b + c &= 100.000 \end{aligned}$$

The sides are 19.8, 35.7, 44.5.

64. If $\sin^2 A + 5 \cos^2 A = 3$, find A.

$$\sin^2 A + 5 \cos^2 A = 3.$$
$$\sin^2 A + 5 - 5 \sin^2 A = 3.$$
$$4 \sin^2 A = 2.$$
$$\sin^2 A = \tfrac{1}{2}.$$
$$\sin A = \pm \sqrt{\tfrac{1}{2}}.$$
$$\therefore A = \pm 45°, \pm 135°.$$

65. If $\sin^2 A = m \cos A - n$, find $\cos A$.

$$\sin^2 A = m \cos A - n.$$
$$1 - \cos^2 A = m \cos A - n.$$
$$\cos^2 A + m \cos A = n + 1.$$
$$4 \cos^2 A + (\;) + m^2 = m^2 + 4(n + 1).$$

$$2\cos A + m = \pm\sqrt{m^2 + 4(n+1)}.$$

$$\therefore \cos A = \tfrac{1}{2}\left[-m \pm \sqrt{m^2 + 4(n+1)}\right].$$

66. Given $\sin A = m \sin B$, and $\tan A = n \tan B$; find $\sin A$ and $\cos B$.

$$\tan A = n \tan B.$$

$$\frac{\sin A}{\cos A} = n\frac{\sin B}{\cos B}.$$

$$\frac{m \sin B}{\cos A} = \frac{n \sin B}{\cos B}.$$

$$\cos A = \frac{m}{n}\cos B.$$

$$\cos^2 A = \frac{m^2}{n^2}\cos^2 B.$$

$$\sin^2 A = m^2 \sin^2 B.$$

$$1 = \frac{m^2}{n^2}\cos^2 B + m^2 \sin^2 B.$$

$$\frac{m^2}{n^2}\cos^2 B + m^2(1 - \cos^2 B) = 1.$$

$$\cos^2 B = \frac{1 - m^2}{\dfrac{m^2}{n^2} - m^2} = \frac{(1 - m^2)n^2}{(1 - n^2)m^2}.$$

$$\cos B = \frac{n}{m}\sqrt{\frac{1 - m^2}{1 - n^2}}.$$

$$\cos^2 A = \frac{m^2}{n^2}\cos^2 B = \frac{1 - m^2}{1 - n^2}.$$

$$\sin^2 A = 1 - \frac{1 - m^2}{1 - n^2} = \frac{m^2 - n^2}{1 - n^2}.$$

$$\sin A = \sqrt{\frac{m^2 - n^2}{1 - n^2}}.$$

67. If $\tan^2 A + 4\sin^2 A = 6$, find A.

$$\tan^2 A + 4\sin^2 A = 6.$$

$$\frac{\sin^2 A}{1 - \sin^2 A} + 4\sin^2 A = 6.$$

$$\sin^2 A + 4\sin^2 A - 4\sin^4 A = 6 - 6\sin^2 A.$$

$$4\sin^4 A - 11\sin^2 A + 6 = 0.$$

$$(4\sin^2 A - 3)(\sin^2 A - 2) = 0.$$

$$\sin^2 A = \tfrac{3}{4} \text{ or } 2.$$

$$\sin A = \pm\tfrac{1}{2}\sqrt{3}.$$

$$\therefore A = \pm 60^\circ,\ \pm 120^\circ.$$

68. If $\sin A = \sin 2A$, find A.

$$\sin A = \sin 2A = 2\sin A \cos A.$$

$$\therefore \sin A(1 - 2\cos A) = 0.$$

$$\therefore \sin A = 0,$$

or
$$1 - 2\cos A = 0.$$

$$\therefore \cos A = \tfrac{1}{2}.$$

$$A = 0^\circ, 180^\circ, \pm 60^\circ.$$

69. If $\tan 2A = 3\tan A$, find A.

$$\tan 2A = 3\tan A.$$

$$\frac{2\tan A}{1 - \tan^2 A} = 3\tan A.$$

$$2\tan A = 3\tan A - 3\tan^3 A.$$

$$3\tan^3 A - \tan A = 0.$$

$$\tan A(3\tan^2 A - 1) = 0.$$

$$\tan A = 0,$$

or
$$3\tan^2 A - 1 = 0.$$

$$\therefore \tan A = \pm\tfrac{1}{3}\sqrt{3}.$$

$$A = 0^\circ,\ 180^\circ,\ 30^\circ,\ 150^\circ,\ 210^\circ,\ 330^\circ.$$

70. Prove that $\tan 50^\circ + \cot 50^\circ = 2\sec 10^\circ$.

$$\tan 50^\circ + \cot 50^\circ = \tan 50^\circ + \frac{1}{\tan 50^\circ}$$

$$= \frac{\tan^2 50^\circ + 1}{\tan 50^\circ}$$

$$= \frac{\sec^2 50^\circ}{\tan 50^\circ}$$

$$= \frac{1}{\sin 50^\circ \cos 50^\circ}$$

$$= \frac{2}{2\sin 50^\circ \cos 50^\circ}$$

$$= \frac{2}{\sin 100^\circ}$$

$$= \frac{2}{\cos 10^\circ}$$

$$= 2\sec 10^\circ.$$

71. Given a regular polygon of n sides, and calling one of them a, find expressions for the radii of the inscribed and the circumscribed circles in terms of n and a.

If P, H, D are the sides of a regular inscribed pentagon, hexagon, and decagon, prove $P^2 = H^2 + D^2$.

(i) Angle subtended by each side a at the centre of the circle is $\frac{360^\circ}{n}$. Hence, if R is the radius of the circumscribed circle, and r that of the inscribed circle,

$$\frac{\frac{1}{2}a}{R} = \sin\frac{180^\circ}{n},$$

$$\frac{\frac{1}{2}a}{r} = \tan\frac{180^\circ}{n}.$$

$$\therefore R = \frac{a}{2}\csc\frac{180^\circ}{n},$$

$$r = \frac{a}{2}\cot\frac{180^\circ}{n}.$$

(ii) Let $R = 1$; then

$$P = 2\sin 36^\circ.$$

$$H = 2\sin 30^\circ = 1.$$

$$D = 2\sin 18^\circ.$$

To prove $P^2 = H^2 + D^2$,

or $4\sin^2 36^\circ = 1 + 4\sin^2 18^\circ$.

Now $\sin 36^\circ = \cos 54^\circ$,

or $\sin(2\times 18^\circ) = \cos(3\times 18^\circ)$.

By [12] and Prob. 19, Ex. XIV,

$$2\sin 18^\circ\cos 18^\circ = 4\cos^3 18^\circ - 3\cos 18^\circ.$$

$$\begin{aligned} 2\sin 18^\circ &= 4\cos^2 18^\circ - 3 \\ &= 4 - 4\sin^2 18^\circ - 3 \\ &= 1 - 4\sin^2 18^\circ. \end{aligned}$$

$$\begin{aligned} \therefore 4\sin^2 18^\circ &= 1 - 2\sin 18^\circ \\ &= 1 - 2\cos 72^\circ. \end{aligned}$$

$$\begin{aligned} 1 + 4\sin^2 18^\circ &= 2 - 2\cos 72^\circ \\ &= 2(1 - \cos 72^\circ) \end{aligned}$$

By [16], $\quad = 4\sin^2 36^\circ$.

72. Obtain the formula for the area of a triangle, given two sides b, c and the included angle A.

Let p be the length of the perpendicular from B on b. Then

$$F = \tfrac{1}{2}pb.$$

But $\quad p = c\sin A.$

$$\begin{aligned} \therefore F &= \tfrac{1}{2}c\sin A \times b \\ &= \tfrac{1}{2}bc\sin A. \end{aligned}$$

73. Obtain the formula for the area of a triangle, given two angles A, B, and the included side c.

$$a = c\frac{\sin A}{\sin C}.$$

$$b = c\frac{\sin B}{\sin C}.$$

$$\begin{aligned} F &= \tfrac{1}{2}ab\sin C \\ &= \tfrac{1}{2}c^2\frac{\sin A\sin B}{\sin C} \\ &= \tfrac{1}{2}c^2\frac{\sin A\sin B}{\sin(A+B)}. \end{aligned}$$

74. Obtain the formula for the area of a triangle, given the three sides.

$$F = \tfrac{1}{2}ac\sin B.$$

By [12],

$$\sin B = 2\sin\tfrac{1}{2}B\cos\tfrac{1}{2}B.$$

By [28],

$$\sin\tfrac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{ac}}.$$

By [29],

$$\cos\tfrac{1}{2}B = \sqrt{\frac{s(s-b)}{ac}}.$$

$$\therefore \sin B = \frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)}.$$

$$\therefore F = \sqrt{s(s-a)(s-b)(s-c)}.$$

75. If a is the side of an equilateral triangle, show that its area is $\frac{a^2\sqrt{3}}{4}$.

$$\begin{aligned} F &= \tfrac{1}{2} bc \sin A \\ &= \tfrac{1}{2} a^2 \sin 60^\circ \\ &= \tfrac{1}{2} a^2 \times \tfrac{1}{2}\sqrt{3} \\ &= \frac{a^2\sqrt{3}}{4}. \end{aligned}$$

76. Two consecutive sides of a rectangle are 52.25 chains and 38.24 chains. Find the area.

Area = 52.25 × 38.24.

$$\begin{aligned} \log 52.25 &= 1.71809 \\ \log 38.24 &= \underline{1.58252} \\ \log \text{area} &= 3.30061 \end{aligned}$$

Area = 1998.

1998 sq. ch. = 199 A. 8 sq. ch.

77. Two sides of a parallelogram are 59.8 chains and 37.05 chains, and the included angle is 72° 10′. Find the area.

Area = 59.8 × 37.05 sin 72° 10′.

$$\begin{aligned} \log 59.8 &= 1.77670 \\ \log 37.05 &= 1.56879 \\ \log \sin 72^\circ 10' &= \underline{9.97861} \\ \log \text{area} &= 3.32410 \end{aligned}$$

Area = 2109.1.

2109.1 sq. ch. = 210 A. 9.1 sq. ch.

78. Two sides of a parallelogram are 15.36 chains and 11.46 chains, and the included angle is 47° 30′. Find the area.

Area = 15.36 × 11.46 sin 47° 30′.

$$\begin{aligned} \log 15.36 &= 1.18639 \\ \log 11.46 &= 1.05918 \\ \log \sin 47^\circ 30' &= \underline{9.86763} \\ \log \text{area} &= 2.11320 \end{aligned}$$

Area = 129.78.

129.78 sq. ch. = 12 A. 9.78 sq. ch.

79. Two sides of a triangle are 12.38 chains and 6.78 chains, and the included angle is 46° 24′. Find the area.

$$\text{Area} = \tfrac{1}{2} \times 12.38 \times 6.78 \sin 46^\circ 24'.$$

$$\begin{aligned} \log 6.19 &= 0.79169 \\ \log 6.78 &= 0.83123 \\ \log \sin 46^\circ 24' &= \underline{9.85984} \\ \log \text{area} &= 1.48276 \end{aligned}$$

Area = 30.392.

30.392 sq. ch. = 3 A. 0.392 sq. ch.

80. Two sides of a triangle are 18.37 chains and 13.44 chains, and they form a right angle. Find the area.

$$\text{Area} = \tfrac{1}{2} \times 18.37 \times 13.44.$$

$$\begin{aligned} \text{colog } 2 &= 9.69897 - 10 \\ \log 18.37 &= 1.26411 \\ \log 13.44 &= \underline{1.12840} \\ \log \text{area} &= 2.39251 \end{aligned}$$

Area = 123.45.

123.45 sq. ch. = 12 A. 3.45 sq. ch.

81. Two angles of a triangle are 76° 54′ and 57° 33′ 12″, and the included side is 9 chains. Find the area.

From [34],

$$\text{Area} = \frac{9^2 \sin 76^\circ 54' \sin 57^\circ 33' 12''}{2 \sin 134^\circ 27' 12''}.$$

$$\begin{aligned} \log 81 &= 1.90849 \\ \log \sin 76^\circ 54' &= 9.98855 \\ \log \sin 57^\circ 33' 12'' &= 9.92629 \\ \text{colog } 2 &= 9.69897 - 10 \\ \text{colog } \sin 134^\circ 27' 12'' &= \underline{0.14641} \\ \log \text{area} &= 1.66871 \end{aligned}$$

Area = 46.634.

46.634 sq. ch. = 4 A. 6.634 sq. ch.

82. Two sides of a triangle are 19.74 chains and 17.34 chains. The first bears N. 82° 30′ W.; the second, S. 24° 15′ E. Find the area.

Included angle = 121° 45′.

$$\text{Area} = \tfrac{1}{2} \times 19.74 \times 17.34 \sin 121^\circ 45'.$$

$$\begin{aligned}
\text{colog } 2 &= 9.69897 - 10\\
\log 19.74 &= 1.29535\\
\log 17.34 &= 1.23905\\
\log \sin 121^\circ 45' &= \underline{9.92960}\\
\log \text{area} &= 2.16297\\
\text{Area} &= 145.54.
\end{aligned}$$

145.54 sq. ch. = 14 A. 5.54 sq. ch.

83. The three sides of a triangle are 49 chains, 50.25 chains, and 25.69 chains. Find the area.

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}.$$

$$\begin{aligned}
s &= \tfrac{1}{2}(49 + 50.25 + 25.69)\\
&= 62.47.\\
s - a &= 13.47.\\
s - b &= 12.22.\\
s - c &= 36.78.
\end{aligned}$$

$$\begin{aligned}
\log 62.47 &= 1.79567\\
\log 13.47 &= 1.12937\\
\log 12.22 &= 1.08707\\
\log 36.78 &= \underline{1.56561}\\
&\, 2\overline{)5.57772}\\
\log \text{area} &= 2.78886\\
\text{Area} &= 614.97.
\end{aligned}$$

614.97 sq. ch. = 61 A. 4.97 sq. ch.

84. The three sides of a triangle are 10.64 chains, 12.28 chains, and 9 chains. Find the area.

$$\begin{aligned}
s &= \tfrac{1}{2}(10.64 + 12.28 + 9)\\
&= 15.96.\\
s - a &= 5.32.\\
s - b &= 3.68.\\
s - c &= 6.96.
\end{aligned}$$

$$\begin{aligned}
\log 15.96 &= 1.20303\\
\log 5.32 &= 0.72591\\
\log 3.68 &= 0.56585\\
\log 6.96 &= \underline{0.84261}\\
&\, 2\overline{)3.33740}\\
\log \text{area} &= 1.66870\\
\text{Area} &= 46.633.
\end{aligned}$$

46.633 sq. ch. = 4 A. 6.633 sq. ch.

85. The sides of a triangular field, of which the area is 14 acres, are in the ratio of 3, 5, 7. Find the sides.

Let the sides, measured in chains, be $3x$, $5x$, $7x$.

$$14 \text{ A.} = 140 \text{ sq. ch.}$$

Then

$$\begin{aligned}
s &= \tfrac{1}{2}(3x + 5x + 7x)\\
&= 7.5x.\\
s - a &= 4.5x.\\
s - b &= 2.5x.\\
s - c &= 0.5x.
\end{aligned}$$

$$\begin{aligned}
140 &= \sqrt{7.5x \times 4.5x \times 2.5x \times 0.5x}\\
&= \frac{x^2}{4}\sqrt{15 \times 9 \times 5}\\
&= \frac{15x^2}{4}\sqrt{3}.
\end{aligned}$$

$$\therefore x^2 = \frac{4 \times 140}{15\sqrt{3}} = \frac{112}{3\sqrt{3}}.$$

$$\begin{aligned}
\log 112 &= 2.04922\\
\text{colog } 3\sqrt{3} &= \underline{9.28432 - 10}\\
&\, 2\overline{)1.33354}\\
\log x &= 0.66677\\
x &= 4.6427.\\
3x &= 13.9281.\\
5x &= 23.2135.\\
7x &= 32.4989.
\end{aligned}$$

Sides are 13.93 chains, 23.21 chains, 32.50 chains.

86. In the quadrilateral *ABCD* we have *AB*, 17.22 chains; *AD*, 7.45 chains; CD, 14.10 chains; BC, 5.25 chains; and the diagonal AC, 15.04 chains. Find the area.

In the triangle ABC,

$$\begin{aligned} s &= \tfrac{1}{2}(17.22 + 5.25 + 15.04) \\ &= 18.755. \end{aligned}$$

$$\begin{aligned} s - a &= 1.535. \\ s - b &= 13.505. \\ s - c &= 3.715. \end{aligned}$$

$$\begin{aligned} \log 18.755 &= 1.27312 \\ \log 1.535 &= 0.18611 \\ \log 13.505 &= 1.13049 \\ \log 3.715 &= \underline{0.56996} \\ 2 &\overline{)3.15968} \\ \log \text{area} &= 1.57984 \end{aligned}$$

$$\text{Area} = 38.005.$$

In the triangle ACD,

$$\begin{aligned} s &= \tfrac{1}{2}(15.04 + 14.10 + 7.45) \\ &= 18.295. \end{aligned}$$

$$\begin{aligned} s - a &= 3.255. \\ s - b &= 4.195. \\ s - c &= 10.845. \end{aligned}$$

$$\begin{aligned} \log 18.295 &= 1.26233 \\ \log 3.255 &= 0.51255 \\ \log 4.195 &= 0.62273 \\ \log 10.845 &= \underline{1.03523} \\ 2 &\overline{)3.43284} \\ \log \text{area} &= 1.71642 \end{aligned}$$

$$\begin{aligned} \text{Area} &= 52.050. \\ \text{Area } ABC &= 38.005 \\ \text{Area } ACD &= \underline{52.050} \\ \text{Area } ABCD &= 90.055 \end{aligned}$$

90.055 sq. ch. = 9 A. 0.055 sq. ch.

87. The diagonals of a quadrilateral are a and b, and they intersect at an angle D. Show that the area of the quadrilateral is $\frac{1}{2}ab\sin D$.

Let the parts into which the diagonals are divided by their intersection be a_1, a_2, and b_1, b_2, so that $a = a_1 + a_2$ and $b = b_1 + b_2$. Then the areas of the four triangles into which the diagonals divide the quadrilateral are

$$\begin{aligned} &\tfrac{1}{2}a_1b_1 \sin D, \qquad \tfrac{1}{2}a_2b_1 \sin D, \\ &\tfrac{1}{2}a_1b_2 \sin D, \qquad \tfrac{1}{2}a_2b_2 \sin D. \end{aligned}$$

The area of the quadrilateral is therefore

$$\begin{aligned} \tfrac{1}{2}a_1(b_1 + b_2)\sin D + \tfrac{1}{2}a_2(b_1 + b_2)\sin D \\ = \tfrac{1}{2}(a_1 + a_2)(b_1 + b_2)\sin D \\ = \tfrac{1}{2}ab\sin D. \end{aligned}$$

88. The diagonals of a quadrilateral are 34 and 56, intersecting at an angle of 67°. Find the area.

$$\text{Area} = \tfrac{1}{2} \times 34 \times 56 \times \sin 67°.$$

$$\begin{aligned} \log 17 &= 1.23045 \\ \log 56 &= 1.74819 \\ \log \sin 67° &= \underline{9.96403} \\ \log \text{area} &= 2.94267 \end{aligned}$$

$$\text{Area} = 876.34.$$

89. The diagonals of a quadrilateral are 75 and 49, intersecting at an angle of 42°. Find the area.

$$\begin{aligned} \text{colog } 2 &= 9.69897 - 10 \\ \log 75 &= 1.87506 \\ \log 49 &= 1.69020 \\ \log \sin 42° &= \underline{9.82551} \\ \log \text{area} &= 3.08974 \end{aligned}$$

$$\text{Area} = 1229.5.$$

90. Show that the area of a regular polygon of n sides, of which one is a, is $\dfrac{na^2}{4}\cot\dfrac{180°}{n}$.

Lines joining the vertices to the centre divide the polygon into n equal isosceles triangles, the bases of which are a, and the vertical angles $\frac{360°}{n}$. The altitude of each triangle is

$$h = \frac{a}{2}\cot\frac{180°}{n};$$

and the area of each is

$$\frac{1}{2}ah = \frac{a^2}{4}\cot\frac{180°}{n}.$$

Hence, the area of the polygon is

$$\frac{na^2}{4}\cot\frac{180°}{n}.$$

91. One side of a regular pentagon is 25. Find the area.

$$\begin{aligned}\text{Area} &= \frac{5 \times 25^2}{4}\cot\frac{180°}{5}\\ &= 781.25 \cot 36°.\end{aligned}$$

$$\begin{aligned}\log 781.25 &= 2.89279\\ \log\cot 36° &= \underline{10.13874}\\ \log \text{area} &= 3.03153\\ \text{Area} &= 1075.3.\end{aligned}$$

92. One side of a regular hexagon is 32. Find the area.

$$\begin{aligned}\text{Area} &= \frac{6 \times 32^2}{4}\cot\frac{180°}{6}\\ &= 1536 \cot 30°.\end{aligned}$$

$$\begin{aligned}\log 1536 &= 3.18639\\ \log\cot 30° &= \underline{10.23856}\\ \log \text{area} &= 3.42495\\ \text{Area} &= 2660.4.\end{aligned}$$

93. One side of a regular decagon is 46. Find the area.

$$\begin{aligned}\text{Area} &= \frac{10 \times 46^2}{4}\cot\frac{180°}{10}\\ &= 5290 \cot 18°.\end{aligned}$$

$$\begin{aligned}\log 5290 &= 3.72346\\ \log\cot 18° &= \underline{10.48822}\\ \log \text{area} &= 4.21168\\ \text{Area} &= 16{,}281.\end{aligned}$$

94. Find the area of a circle whose circumference is 74 feet.

$$\begin{aligned}2\pi r &= 74.\\ \therefore r &= \frac{37}{\pi}.\\ \text{Area} &= \pi r^2 = \frac{37^2}{\pi}.\end{aligned}$$

$$\begin{aligned}\log 37^2 &= 3.13640\\ \text{colog}\ \pi &= \underline{9.50285 - 10}\\ \log \text{area} &= 2.63925\\ \text{Area} &= 435.76.\\ \text{Area} &= 435.76 \text{ sq. ft.}\end{aligned}$$

95. Find the area of a circle whose radius is 125 feet.

$$\text{Area} = \pi \times 125^2.$$

$$\begin{aligned}\log 125^2 &= 4.19382\\ \log \pi &= \underline{0.49715}\\ \log \text{area} &= 4.69097\\ \text{Area} &= 49{,}088.\\ \text{Area} &= 49{,}088 \text{ sq. ft.}\end{aligned}$$

96. In a circle with a diameter of 125 feet find the area of a sector with an arc of 22°.

Area of sector : area of circle = 22 : 360.

$$\begin{aligned}\therefore \text{area of sector} &= \tfrac{22}{360}\pi\left(\tfrac{125}{2}\right)^2\\ &= \frac{11 \times 125^2}{720}\pi.\end{aligned}$$

$$\begin{aligned}\log 11 &= 1.04139\\ \log 125^2 &= 4.19382\\ \text{colog}\ 720 &= 7.14267 - 10\\ \log \pi &= \underline{0.49715}\\ \log \text{area} &= 2.87503\\ \text{Area} &= 749.95.\\ \text{Area} &= 749.95 \text{ sq. ft.}\end{aligned}$$

97. In a circle with a radius of 44 feet find the area of sector with an arc of 25°.

$$\text{Area} = \tfrac{25}{360}\,\pi\, 44^2$$
$$= \frac{1210\,\pi}{9}.$$

$$\log 1210 = 3.08279$$
$$\log \pi = 0.49715$$
$$\text{colog } 9 = \underline{9.04576 - 10}$$
$$\log \text{area} = 2.62570$$

$$\text{Area} = 422.38.$$
$$\text{Area} = 422.38 \text{ sq. ft.}$$

98. In a circle with a diameter of 50 feet find the area of a segment with an arc of 280°.

Area of segment = area of sector with same arc + area of triangle with two sides equal to radius, and included angle of 80°.

$$\text{Area of sector} = \tfrac{280}{360}\,\pi\, 25^2$$
$$= \frac{4375\,\pi}{9}.$$

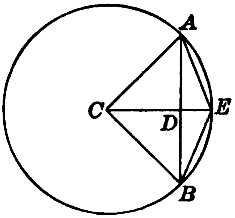
$$\log 4375 = 3.64098$$
$$\log \pi = 0.49715$$
$$\text{colog } 9 = \underline{9.04576 - 10}$$
$$\log \text{area} = 3.18389$$

$$\text{Area of sector} = 1527.2.$$
$$\text{Area of triangle} = \tfrac{1}{2} \times 25^2 \sin 80^\circ$$
$$= 312.5 \sin 80^\circ.$$

$$\log 312.5 = 2.49485$$
$$\log \sin 80^\circ = \underline{9.99335}$$
$$\log \text{area} = 2.48820$$

$$\text{Area of triangle} = 307.75.$$
$$\text{Area of segment} = 1834.95 \text{ sq. ft.}$$

99. Find the area of a segment (less than a semicircle) of which the chord is 20, and the distance of the chord from the middle point of the smaller arc is 2.



$$\tan AED = \tfrac{10}{2} = 5.$$
$$\log \tan AED = 10.69897.$$
$$AED = 78^\circ\, 41'\, 24''.$$
$$ACD = 180^\circ - 2\,AED$$
$$= 22^\circ\, 37'\, 12''.$$
$$AC = AD \csc ACD$$
$$= 10 \csc 22^\circ 37' 12''.$$

$$\log 10 = 1.00000$$
$$\log \csc 22^\circ 37' 12'' = \underline{0.41497}$$
$$\log AC = 1.41497$$

$$AC = 26.$$
$$ACB = 45^\circ\, 14'\, 24''$$
$$= 45.24^\circ.$$

Area of sector CAB

$$= \frac{ACB}{360^\circ}\,\pi\, \overline{AC}^2$$
$$= \frac{45.24}{360}\,\pi\, 26^2$$
$$= \tfrac{377}{3000}\,\pi\, 26^2.$$

$$\log 377 = 2.57634$$
$$\log \pi = 0.49715$$
$$\log 26^2 = 2.82994$$
$$\text{colog } 3000 = \underline{6.52288 - 10}$$
$$\log \text{area} = 2.42631$$

$$\text{Area of sector} = 266.88.$$

Area of triangle CAB

$$= AD \times CD$$
$$= 10\,(26 - 2)$$
$$= 240.$$

$$\text{Area of segment} = 26.88.$$

100. If r is the radius of a circle, the area of a regular circumscribed polygon of n sides is $nr^2 \tan \frac{180°}{n}$.

The area of a regular inscribed polygon is $\frac{n}{2} r^2 \sin \frac{360°}{n}$.

Lines drawn from the vertices to the centre divide the polygon into n equal isosceles triangles, the bases of which are the sides of the polygon and the vertical angles $\frac{360°}{n}$.

In the circumscribed polygon, each side $= 2r \tan \frac{180°}{n}$, and the altitude of each triangle is r. Hence, the area of each triangle is $r^2 \tan \frac{180°}{n}$, and the area of the polygon $nr^2 \tan \frac{180°}{n}$.

In the inscribed polygon, each side $= 2r \sin \frac{180°}{n}$, and the altitude of each triangle is $r \cos \frac{180°}{n}$. Hence, the area of each triangle is $r^2 \sin \frac{180°}{n} \cos \frac{180°}{n} = \frac{r^2}{2} \sin \frac{360°}{n}$, and the area of the polygon is $\frac{nr^2}{2} \sin \frac{360°}{n}$.

101. If a is a side of a regular polygon of n sides, the area of the inscribed circle is $\frac{\pi a^2}{4} \cot^2 \frac{180°}{n}$.

The area of the circumscribed circle is $\frac{\pi a^2}{4} \csc^2 \frac{180°}{n}$.

If r is the radius of the inscribed circle,

$$a = 2r \tan \frac{180°}{n}.$$

$$\therefore r = \frac{a}{2} \cot \frac{180°}{n}.$$

$$\pi r^2 = \frac{\pi a^2}{4} \cot^2 \frac{180°}{n}.$$

If R is the radius of the circumscribed circle,

$$a = 2R \sin \frac{180°}{n}.$$

$$\therefore R = \frac{a}{2} \csc \frac{180°}{n}.$$

$$\pi R^2 = \frac{\pi a^2}{4} \csc^2 \frac{180°}{n}.$$

102. The area of a regular polygon inscribed in a circle is to that of the circumscribed regular polygon of the same number of sides as 3 to 4. Find the number of sides.

$$\frac{n}{2} r^2 \sin \frac{360°}{n} : nr^2 \tan \frac{180°}{n} = 3 : 4.$$

$$2nr^2 \sin \frac{360°}{n} = 3nr^2 \tan \frac{180°}{n}.$$

$$2 \sin \frac{360°}{n} = 3 \tan \frac{180°}{n}.$$

$$4 \sin \frac{180°}{n} \cos \frac{180°}{n} = 3 \frac{\sin \frac{180°}{n}}{\cos \frac{180°}{n}}.$$

$$4 \cos^2 \frac{180°}{n} = 3.$$

$$\cos \frac{180°}{n} = \tfrac{1}{2}\sqrt{3}.$$

$$\frac{180°}{n} = 30°.$$

$$n = 6.$$

103. The area of a regular polygon inscribed in a circle is the geometric mean between the areas of an inscribed and a circumscribed regular polygon of half the number of sides.

$$\begin{aligned}\log 207 &= 2.31597\\ \log \cot 64^\circ\, 41'\, 15'' &= \underline{9.67483}\\ \log \text{diff. lat.} &= 1.99080\\ \text{Diff. lat.} &= 97.904'\\ &= \ \ 1^\circ\, 37'\, 54''.\\ 13^\circ\, 17' - 1^\circ\, 37'\, 54'' &= 11^\circ\, 39'\, 6''.\end{aligned}$$

Distance, 228.98 miles; latitude reached, 11° 39′ 6″ S.

112. A ship sails on a course between S. and E. 244 miles, leaving latitude 2° 52′ S., and reaching latitude 5° 8′ S. Find the course and the departure.

$$\begin{aligned}\text{Diff. lat.} &= 2^\circ\, 16' = 136 \text{ miles.}\\ \text{Dist.} &= 244 \text{ miles.}\\ \cos \text{course} &= \frac{136}{244}.\\ \text{Depart.} &= \sqrt{244^2 - 136^2}\\ &= \sqrt{(244+136)(244-136)}\\ &= \sqrt{380 \times 108}.\end{aligned}$$

$$\begin{aligned}\log 136 &= 2.13354\\ \operatorname{colog} 244 &= \underline{7.61261 - 10}\\ \log \cos \text{course} &= 9.74615\\ \text{Course} &= 56^\circ\, 7'\, 32''.\\ \log 380 &= 2.57978\\ \log 108 &= \underline{2.03342}\\ &\ 2\,\overline{)\,4.61320}\\ \log \text{depart.} &= 2.30660\\ \text{Depart.} &= 202.58.\end{aligned}$$

Course, S. 56° 7′ 32″ E.; departure, 202.58 miles.

113. A ship sails from latitude 32° 18′ N., on a course between N. and W., a distance of 344 miles, and a departure of 103 miles. Find the course, and the latitude reached.

$$\begin{aligned}\text{Dist.} &= 344 \text{ miles.}\\ \text{Depart.} &= 103 \text{ miles.}\\ \sin \text{course} &= \frac{103}{344}.\\ \text{Diff. lat.} &= \sqrt{344^2 - 103^2}\\ &= \sqrt{(344+103)(344-103)}\\ &= \sqrt{447 \times 241}.\end{aligned}$$

$$\begin{aligned}\log 103 &= 2.01284\\ \operatorname{colog} 344 &= \underline{7.46344 - 10}\\ \log \sin \text{course} &= 9.47628\\ \text{Course} &= 17^\circ\, 25'\, 22''.\\ \log 447 &= 2.65031\\ \log 241 &= \underline{2.38202}\\ &\ 2\,\overline{)\,5.03233}\\ \log \text{diff. lat.} &= 2.51616\\ \text{Diff. lat.} &= 328.22'\\ &= \ \ 5^\circ\, 28'\, 13''.\\ 32^\circ\, 18' + 5^\circ\, 28'\, 13'' &= 37^\circ\, 46'\, 13''.\end{aligned}$$

Course, N. 17° 25′ 22″ W.; latitude reached, 37° 46′ 13″ N.

114. A ship sails on a course between S. and E., making a difference of latitude 136 miles, and a departure 203 miles. Find the distance, and the course.

$$\begin{aligned}\text{Diff. lat.} &= 136 \text{ miles.}\\ \text{Depart.} &= 203 \text{ miles.}\\ \tan \text{course} &= \frac{203}{136}.\\ \log 203 &= \ \ 2.30750\\ \operatorname{colog} 136 &= \ \ \underline{7.86646 - 10}\\ \log \tan \text{course} &= 10.17396\\ \text{Course} &= 56^\circ\, 10'\, 49''.\\ \text{Dist.} &= 203 \csc 56^\circ\, 10'\, 49''.\\ \log 203 &= 2.30750\\ \log \csc 56^\circ\, 10'\, 49'' &= \underline{0.08051}\\ \log \text{dist.} &= 2.38801\\ \text{Dist.} &= 244.35.\end{aligned}$$

Course, S. 56° 10′ 49″ E.; distance, 244.35 miles.

115. **A ship sails due north 15 *statute* miles an hour for one day. What is the distance in a straight line from the point left to the point reached? (Take earth's radius, 3962.8 statute miles.)**

Distance sailed in one day

$$
\begin{aligned}
&= 24 \times 15 \text{ miles} = 360 \text{ miles} \\
&= \frac{360}{2\pi \times 3962.8} \times 360^\circ \\
&= \frac{64800^\circ}{3962.8\,\pi}.
\end{aligned}
$$

$$
\begin{aligned}
\log 64800 &= 4.81158 \\
\text{colog } 3962.8 &= 6.40200 - 10 \\
\text{colog } \pi &= \underline{9.50285 - 10} \\
\log \text{dist.} &= 0.71643
\end{aligned}
$$

$$
\begin{aligned}
\text{Distance sailed } &= 5.2051^\circ \\
&= 5^\circ\ 12'\ 18''.
\end{aligned}
$$

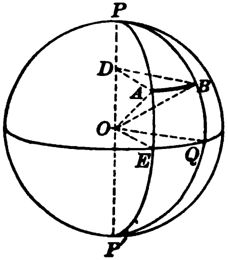
Chord of arc sailed

$$
\begin{aligned}
&= 2 \times 3962.8 \sin 2^\circ\ 36'\ 9'' \\
&= 7925.6 \sin 2^\circ\ 36'\ 9''.
\end{aligned}
$$

$$
\begin{aligned}
\log 7925.6 &= 3.89903 \\
\log \sin 2^\circ\ 36'\ 9'' &= \underline{8.65712} \\
\log \text{chord} &= 2.55615 \\
\text{Chord} &= 359.87.
\end{aligned}
$$

Required distance, 359.87 miles.

116. Given the departure between any two meridians at any latitude; find the difference of longitude of any point on one meridian from any point on the other.



In rt. △ ODA, $\angle AOD = 90^\circ - \text{lat.}$ Hence,

$$\frac{DA}{OA} = \sin(90^\circ - \text{lat.}) = \cos \text{lat.}$$

The △ DAB and OEQ are similar. Therefore,

$$\frac{DA}{OE} = \frac{AB}{EQ}, \text{ or } \frac{DA}{OA} = \frac{AB}{EQ}.$$

Substituting, $\cos \text{lat.} = \dfrac{AB}{EQ}.$

Therefore,

$$EQ = \frac{AB}{\cos \text{lat.}} = AB \times \sec \text{lat.}$$

That is,

Diff. long. = depart. × sec lat.

117. A ship in latitude 42° 16′ N., longitude 72° 16′ W., sails due east a distance of 149 miles. What is the position of the point reached?

Diff. long. = 149 sec 42° 16′.

$$
\begin{aligned}
\log 149 &= 2.17319 \\
\log \sec 42^\circ\ 16' &= \underline{0.13076} \\
\log.\ \text{diff. long.} &= 2.30395
\end{aligned}
$$

Diff. long. = 201.35′ = 3° 21′ 21″.

Longitude of position reached, 68° 54′ 39″ W.

118. A ship in latitude 44° 49′ S., longitude 119° 42′ E., sails due west until it reaches longitude 117° 16′ E. Find the distance made.

Diff. long. = 2° 26′ = 146′.
Depart. = 146 cos 44° 49′.

$$
\begin{aligned}
\log 146 &= 2.16435 \\
\log \cos 44^\circ\ 49' &= \underline{9.85087} \\
\log \text{depart.} &= 2.01522
\end{aligned}
$$

Depart. = 103.57.

Distance made, 103.57 miles.

119. A ship leaves latitude 31° 14′ N., longitude 42° 19′ W., and sails E.N.E. 325 miles. Find the position reached.

$$\text{Course} = 67^{\circ}\ 30'.$$

$$\text{Diff. lat.} = 325 \cos 67^{\circ}\ 30'.$$

$$\begin{aligned} \log 325 &= 2.51188 \\ \log \cos 67^{\circ}\ 30' &= \underline{9.58284} \\ \log \text{diff. lat.} &= 2.09472 \end{aligned}$$

$$\begin{aligned} \text{Diff. lat.} &= 124.37' = 2^{\circ}\ 4'\ 22''. \\ \text{Mid. lat.} &= 32^{\circ}\ 16'\ 11''. \\ \text{Depart.} &= 325 \sin 67^{\circ}\ 30'. \end{aligned}$$

Diff. long.

$$= 325 \sin 67^{\circ}\ 30' \sec 32^{\circ}\ 16'\ 11''.$$

$$\begin{aligned} \log 325 &= 2.51188 \\ \log \sin 67^{\circ}\ 30' &= 9.96562 \\ \log \sec 32^{\circ}\ 16'\ 11'' &= \underline{0.07286} \\ \log.\ \text{diff. long.} &= 2.55036 \end{aligned}$$

$$\text{Diff. long.} = 355.11' = 5^{\circ}\ 55'\ 7''.$$

Latitude of position reached, 33° 18′ 22″ N.; longitude, 36° 23′ 53″ W.

120. Find the bearing and distance of Cape Cod from Havana. (Cape Cod, 42° 2′ N., 70° 3′ W.; Havana, 23° 9′ N., 82° 22′ W.)

$$\begin{aligned} \text{Diff. long.} &= 12^{\circ}\ 19' = 739'. \\ \text{Diff. lat.} &= 18^{\circ}\ 53' = 1133'. \\ \text{Mid. lat.} &= 32^{\circ}\ 35'\ 30''. \end{aligned}$$

$$\begin{aligned} \text{Depart.} &= \text{diff. long.} \times \cos \text{mid. lat.} \\ &= 739 \cos 32^{\circ}\ 35'\ 30''. \end{aligned}$$

$$\begin{aligned} \tan \text{course} &= \frac{\text{depart.}}{\text{diff. lat.}} \\ &= \frac{739 \cos 32^{\circ}\ 35'\ 30''}{1133}. \end{aligned}$$

$$\begin{aligned} \log 739 &= 2.86864 \\ \log \cos 32^{\circ}\ 35'\ 30'' &= 9.92559 \\ \text{colog}\ 1133 &= \underline{6.94577 - 10} \\ \log \tan \text{course} &= 9.74000 \end{aligned}$$

$$\text{Course} = 28^{\circ}\ 47'\ 26''.$$

$$\begin{aligned} \text{Dist.} &= \text{diff. lat.} \times \sec \text{course} \\ &= 1133 \sec 28^{\circ}\ 47'\ 26''. \end{aligned}$$

$$\begin{aligned} \log 1133 &= 3.05423 \\ \log \sec 28^{\circ}\ 47'\ 26'' &= \underline{0.05730} \\ \log \text{dist.} &= 3.11153 \end{aligned}$$

$$\text{Dist.} = 1292.8.$$

Bearing, N. 28° 47′ 26″ E.; distance, 1292.8 miles.

121. Leaving latitude 49° 57′ N., longitude 15° 16′ W., a ship sails between S. and W. till the departure is 194 miles, and the latitude is 47° 18′ N. Find the course, distance, and longitude reached.

$$\begin{aligned} \text{Diff. lat.} &= 2^{\circ}\ 39'' = 159 \text{ miles.} \\ \text{Mid. lat.} &= 48^{\circ}\ 37'\ 30''. \\ \text{Depart.} &= 194 \text{ miles.} \\ \text{Diff. long.} &= 194 \sec 48^{\circ}\ 37'\ 30''. \end{aligned}$$

$$\begin{aligned} \log 194 &= 2.28780 \\ \log \sec 48^{\circ}\ 37'\ 30'' &= \underline{0.17981} \\ \log \text{diff. long.} &= 2.46761 \end{aligned}$$

$$\begin{aligned} \text{Diff. long.} &= 293.50' \\ &= 4^{\circ}\ 53'\ 30''. \end{aligned}$$

$$\tan \text{course} = \frac{194}{159}.$$

$$\begin{aligned} \log 194 &= 2.28780 \\ \text{colog}\ 159 &= \underline{7.79860 - 10} \\ \log \tan \text{course} &= 10.08640 \end{aligned}$$

$$\begin{aligned} \text{Course} &= 50^{\circ}\ 39'\ 44''. \\ \text{Dist.} &= 159 \sec 50^{\circ}\ 39'\ 44''. \end{aligned}$$

$$\begin{aligned} \log 159 &= 2.20140 \\ \log \sec 50^{\circ}\ 39'\ 44'' &= \underline{0.19799} \\ \log \text{dist.} &= 2.39939 \end{aligned}$$

$$\text{Dist.} = 250.84.$$

Course, S. 50° 39′ 44″ W.; distance, 250.84 miles; longitude reached, 20° 9′ 30″ W.

122. Leaving latitude 42° 30′ N., longitude 58° 51′ W., a ship sails S.E. by S. 300 miles. Find the position reached.

$$\begin{aligned}\text{Course} &= 33^\circ\ 45'.\\ \text{Diff. lat.} &= 300 \cos 33^\circ 45'.\end{aligned}$$

$$\begin{aligned}\log 300 &= 2.47712\\ \log \cos 33^\circ\ 45' &= \underline{9.91985}\\ \log \text{diff. lat.} &= 2.39697\end{aligned}$$

$$\begin{aligned}\text{Diff. lat.} &= 249.44'\\ &= 4^\circ\ 9'\ 26''.\\ \text{Mid. lat.} &= 40^\circ\ 25'\ 17''.\\ \text{Depart.} &= 300 \sin 33^\circ 45'.\end{aligned}$$

$$\begin{aligned}&\text{Diff. long.}\\ &= 300 \sin 33^\circ\ 45' \sec 40^\circ 25' 17''.\end{aligned}$$

$$\begin{aligned}\log 300 &= 2.47712\\ \log \sin 33^\circ\ 45' &= 9.74474\\ \log \sec 40^\circ\ 25'\ 17'' &= \underline{0.11845}\\ \log \text{diff. long.} &= 2.34031\end{aligned}$$

$$\begin{aligned}\text{Diff. long.} &= 218.93'\\ &= 3^\circ\ 38'\ 56''.\end{aligned}$$

Latitude of position reached, 38° 20′ 34″ N.; longitude, 55° 12′ 4″ W.

123. Leaving latitude 49° 57′ N., longitude 30° W., a ship sails S. 39° W., and reaches latitude 47° 44′ N. Find the distance, and longitude reached.

$$\begin{aligned}\text{Course} &= 39^\circ.\\ \text{Diff. lat.} &= 2^\circ\ 13' = 133 \text{ miles.}\\ \text{Mid. lat.} &= 48^\circ\ 50'\ 30''.\\ \text{Dist.} &= 133 \sec 39^\circ.\end{aligned}$$

$$\begin{aligned}\log 133 &= 2.12385\\ \log \sec 39^\circ &= \underline{0.10950}\\ \log \text{dist.} &= 2.23335\end{aligned}$$

$$\begin{aligned}\text{Dist.} &= 171.14.\\ \text{Depart.} &= 133 \tan 39^\circ.\end{aligned}$$

$$\begin{aligned}&\text{Diff. long.}\\ &= 133 \tan 39^\circ \sec 48^\circ\ 50'\ 30''.\end{aligned}$$

$$\begin{aligned}\log 133 &= 2.12385\\ \log \tan 39^\circ &= 9.90837\\ \log \sec 48^\circ\ 50'\ 30'' &= \underline{0.18168}\\ \log \text{diff. long.} &= 2.21390\end{aligned}$$

$$\begin{aligned}\text{Diff. long.} &= 163.64'\\ &= 2^\circ\ 43'\ 38''.\end{aligned}$$

Distance, 171.14 miles; longitude reached, 32° 43′ 38″ W.

124. Leaving latitude 37° N., longitude 32° 16′ W., a ship sails between N. and W. 300 miles, and reaches latitude 41° N. Find the course, and longitude reached.

$$\begin{aligned}\text{Diff. lat.} &= 4^\circ = 240 \text{ miles.}\\ \text{Mid. lat.} &= 39^\circ.\\ \text{Dist.} &= 300.\end{aligned}$$

$$\cos \text{course} = \frac{240}{300}.$$

$$\begin{aligned}\log 240 &= 2.38021\\ \text{colog } 300 &= \underline{7.52288 - 10}\\ \log \cos \text{course} &= 9.90309\end{aligned}$$

$$\begin{aligned}\text{Course} &= 36^\circ\ 52'\ 12''.\\ \text{Depart.} &= \sqrt{300^2 - 240^2}\\ &= \sqrt{60 \times 540}\\ &= 180.\\ \text{Diff. long.} &= 180 \sec 39^\circ.\end{aligned}$$

$$\begin{aligned}\log 180 &= 2.25527\\ \log \sec 39^\circ &= \underline{0.10950}\\ \log \text{diff. long.} &= 2.36477\end{aligned}$$

$$\begin{aligned}\text{Diff. long.} &= 231.62'\\ &= 3^\circ\ 51'\ 37''.\end{aligned}$$

Course, N. 36° 52′ 12″ W.; longitude reached, 36° 7′ 37″ W.

125. Leaving latitude 50° 10′ S., longitude 30° E., a ship sails E.S.E., making a departure of 160 miles. Find the distance, and position reached.

$$\begin{aligned}\text{Course} &= 67^\circ\ 30'.\\ \text{Depart.} &= 160 \text{ miles.}\\ \text{Dist.} &= 160 \csc 67^\circ\ 30'.\\ \text{Diff. lat.} &= 160 \cot 67^\circ\ 30'.\end{aligned}$$

$$\begin{aligned}\log 160 &= 2.20412\\ \log \csc 67^\circ\ 30' &= \underline{0.03438}\\ \log \text{dist.} &= 2.23850\end{aligned}$$

$$\text{Dist.} = 173.18.$$

$$\begin{aligned}\log 160 &= 2.20412\\ \log \cot 67^\circ\ 30' &= \underline{9.61722}\\ \log \text{diff. lat.} &= 1.82134\end{aligned}$$

$$\begin{aligned}\text{Diff. lat.} &= 66.273'\\ &= 1^\circ\ 6'\ 16''.\\ \text{Lat. reached} &= 51^\circ\ 16'\ 16''.\\ \text{Mid. lat.} &= 50^\circ\ 43'\ 8''.\\ \text{Diff. long.} &= 160 \sec 50^\circ 43'\ 8''.\end{aligned}$$

$$\begin{aligned}\log 160 &= 2.20412\\ \log \sec 50^\circ\ 43'\ 8'' &= \underline{0.19851}\\ \log \text{diff. long.} &= 2.40263\end{aligned}$$

$$\begin{aligned}\text{Diff. long.} &= 252.71'\\ &= 4^\circ\ 12'\ 43''.\end{aligned}$$

Distance, 173.18 miles; latitude of position reached, 51° 16′ 16″ S.; longitude, 34° 12′ 43″ E.

126. Leaving latitude 49° 30′ N., longitude 25° W., a ship sails between S. and E. 215 miles, making a departure of 167 miles. Find the course, and position reached.

$$\sin \text{course} = \frac{167}{215}.$$

$$\begin{aligned}\log 167 &= 2.22272\\ \text{colog } 215 &= \underline{7.66756 - 10}\\ \log \sin \text{course} &= 9.89028\end{aligned}$$

$$\begin{aligned}\text{Course} &= 50^\circ\ 57'\ 48''.\\ \text{Diff. lat.} &= \sqrt{215^2 - 167^2}\\ &= \sqrt{48 \times 382}.\end{aligned}$$

$$\begin{aligned}\log 48 &= 1.68124\\ \log 382 &= \underline{2.58206}\\ 2\,&\overline{)4.26330}\\ \log \text{diff. lat.} &= 2.13165\end{aligned}$$

$$\begin{aligned}\text{Diff. lat.} &= 135.41'\\ &= 2^\circ\ 15'\ 25''.\end{aligned}$$

$$\begin{aligned}\text{Mid. lat.} &= 48^\circ\ 22'\ 18''.\\ \text{Diff. long.} &= 167 \sec 48^\circ\ 22'\ 18''.\end{aligned}$$

$$\begin{aligned}\log 167 &= 2.22272\\ \log \sec 48^\circ\ 22'\ 18'' &= \underline{0.17764}\\ \log \text{diff. long.} &= 2.40036\end{aligned}$$

$$\begin{aligned}\text{Diff. long.} &= 251.39'\\ &= 4^\circ\ 11'\ 23''.\end{aligned}$$

Course, S. 50° 57′ 48″ E.; latitude of position reached, 47° 14′ 35″ N.; longitude, 20° 48′ 37″ W.

127. Leaving latitude 43° S., longitude 21° W., a ship sails 273 miles, and reaches latitude 40° 17′ S. What are the *two* courses and longitudes which will satisfy the data?

The two courses make equal angles with the meridian on opposite sides.

$$\begin{aligned}\text{Diff. lat.} &= 2^\circ\ 43' = 163 \text{ miles.}\\ \text{Dist.} &= 273 \text{ miles.}\end{aligned}$$

$$\cos \text{course} = \frac{163}{273}.$$

$$\begin{aligned}\log 163 &= 2.21219\\ \text{colog } 273 &= \underline{7.56384 - 10}\\ \log \cos \text{course} &= 9.77603\end{aligned}$$

$$\begin{aligned}\text{Course} &= 53^\circ\ 20'\ 21''.\\ \text{Depart.} &= \sqrt{273^2 - 163^2}\\ &= \sqrt{110 \times 436}.\\ \text{Mid. lat.} &= 41^\circ\ 38'\ 30''.\end{aligned}$$

Diff. long.

$$= \sqrt{110 \times 436} \sec 41^\circ\ 38'\ 30''.$$

$$\begin{aligned}\log\sqrt{110} &= 1.02069\\ \log\sqrt{436} &= 1.31975\\ \log\sec 41^\circ\ 38'\ 30'' &= \underline{0.12650}\\ \log \text{diff. long.} &= 2.46694\end{aligned}$$

$$\begin{aligned}\text{Diff. long.} &= 293.05'\\ &= 4^\circ\ 53'\ 3''.\end{aligned}$$

(i) Course, N. 53° 20′ 21″ E.; longitude of position reached, 16° 6′ 57′ W.

(ii) Course, N. 53° 20′ 21″ W.; longitude of position reached, 25° 53′ 3″ W.

128. Leaving latitude 17° N., longitude 119° E., a ship sails 219 miles, making a departure of 162 miles. What four sets of answers do we get?

The four courses all make the same angle with the meridian.

$$\sin \text{course} = \frac{162}{219}.$$

$$\begin{aligned}\log 162 &= 2.20952\\ \text{colog}\,219 &= \underline{7.65956 - 10}\\ \log\sin\text{course} &= 9.86908\end{aligned}$$

$$\text{Course} = 47^\circ\ 42'\ 33''.$$

$$\begin{aligned}\text{Diff. lat.} &= \sqrt{219^2 - 162^2}\\ &= \sqrt{57 \times 381}.\end{aligned}$$

$$\begin{aligned}\log 57 &= 1.75587\\ \log 381 &= \underline{2.58092}\\ 2\,\big)\,&\underline{4.33679}\\ \log\text{diff. lat.} &= 2.16840\end{aligned}$$

$$\begin{aligned}\text{Diff. lat.} &= 147.37'\\ &= 2^\circ\ 27'\ 22''.\end{aligned}$$

(i) Mid. lat. = 18° 13′ 41″.

Diff. long. = 162 sec 18° 13′ 41″.

$$\begin{aligned}\log 162 &= 2.20952\\ \log\sec 18^\circ\ 13'\ 41'' &= \underline{0.02236}\\ \log\text{diff. long.} &= 2.23188\end{aligned}$$

$$\begin{aligned}\text{Diff. long.} &= 170.56'\\ &= 2^\circ\ 50'\ 34''.\end{aligned}$$

(ii) Mid. lat. = 15° 46′ 19″.

Diff. long. = 162 sec 15° 46′ 19″.

$$\begin{aligned}\log 162 &= 2.20952\\ \log\sec 15^\circ\ 46'\ 19'' &= \underline{0.01667}\\ \log\text{diff. long.} &= 2.22619\end{aligned}$$

$$\begin{aligned}\text{Diff. long.} &= 168.34'\\ &= 2^\circ\ 48'\ 20''.\end{aligned}$$

(i) Course, N. 47° 42′ 33″ E.; latitude of position reached, 19° 27′ 22″ N., longitude, 121° 50′ 34″ E.

Course, N. 47° 42′ 33″ W.; latitude of position reached, 19° 27′ 22″ N., longitude, 116° 9′ 26″ E.

(ii) Course, S. 47° 42′ 33″ E.; latitude of position reached, 14° 32′ 38″ N., longitude, 121° 48′ 20″ E.

Course, S. 47° 42′ 33″ W.; latitude of position reached, 14° 32′ 38″ N., longitude, 116° 11′ 40″ E.

129. A ship in latitude 30° sails due east 360 statute miles. What is the shortest distance from the point left to the point reached?

Solve the same problem for latitudes 45°, 60°.

By Prob. 116, Ex. XXIII, radius of parallel

$$= 3962.8 \cos \text{lat}.$$

Arc sailed, in degrees,

$$= \frac{360 \times 360}{2\pi \times 3962.8 \cos \text{lat.}}.$$

$$\begin{aligned}\log 360^2 &= 5.11260\\ \text{colog}\,2\pi &= 9.20182 - 10\\ \text{colog}\,3962.8 &= \underline{6.40200 - 10}\\ &\ 0.71642\end{aligned}$$

Arc sailed, in degrees,

$$= 5.205 \sec \text{lat.}$$

Arc sailed, in minutes,
= 312.3 sec lat.

Chord of arc
= 2 rad. of parallel sin (½ arc)
= 2 × 3962.8 cos lat. sin (156.15′ sec lat.)
= 7925.6 cos lat. sin (156.15′ sec lat.).

(i) lat. = 30°.

log 156.15 = 2.19354
log sec 30° = 0.06247
log ½ arc = 2.25601

½ arc = 180.30′
= 3° 0′ 18″.

log 7925.6 = 3.89903
log cos 30° = 9.93753
log sin 3° 0′ 18″ = 8.71952
log chord = 2.55608

Chord = 359.82.

(ii) lat. = 45°.

log 156.15 = 2.19354
log sec 45° = 0.15051
log ½ arc = 2.34405

½ arc = 220.82′
= 3° 40′ 49″.

log 7925.6 = 3.89903
log cos 45° = 9.84949
log sin 3° 40′ 49″ = 8.80746
log chord = 2.55598

Chord = 359.73.

(iii) lat. = 60°.
sec lat. = 2.
½ arc = 312.30′
= 5° 12′ 18″.

log 7925.6 = 3.89903
log sin 5° 12′ 18″ = 8.95770
log cos 60° = 9.69897
log chord = 2.55570

Chord = 359.50.

Shortest distance, in lat. 30°, 359.82 miles; in lat. 45°, 359.73 miles; in lat. 60°, 359.50 miles; in general 7925.6 cos lat. × sin (156.15′ sec lat.).

130. Leaving latitude 37° 16′ S., longitude 18° 42′ W., a ship sails N.E. 104 miles, then N.N.W. 60 miles, then W. by S. 216 miles. Find the position reached, and its bearing and distance from the point left.

First course = 45°.
Diff. lat. = 104 cos 45°.
Depart. = 104 sin 45°.

log 104 = 2.01703
log cos 45° = 9.84949
log diff. lat. = 1.86652

Diff. lat. = 73.54 N.
Depart. = 73.54 E.

Second course = 22° 30′.
Diff. lat. = 60 cos 22° 30′.
Depart. = 60 sin 22° 30′.

log 60 = 1.77815
log. cos 22° 30′ = 9.96562
log diff. lat. = 1.74377

Diff. lat. = 55.434 N.

log 60 = 1.77815
log sin 22° 30′ = 9.58284
log depart. = 1.36099

Depart. = 22.961 W.

Third course = 78° 45′.
Diff. lat. = 216 cos 78° 45′.
Depart. = 216 sin 78° 45′.

log 216 = 2.33445
log cos 78° 45′ = 9.29024
log diff. lat. = 1.62469

Diff. lat. = 42.14 S.

$\log 216 = 2.33445$

$\log \sin 78° 45' = \underline{9.99157}$

$\log \text{depart.} = 2.32602$

$\text{Depart.} = 211.85 \text{ W.}$

$\text{Total diff. lat.} = 86.834' \text{ N.}$

$= 1° 26' 50'' \text{ N.}$

$\text{Lat. reached} = 35° 49' 10'' \text{ S.}$

$\text{Mid. lat.} = 36° 32' 35''.$

$\text{Total depart.} = 161.271 \text{ W.}$

$\text{Diff. long.} = 161.271 \sec 36° 32' 35''.$

$\log 161.271 = 2.20755$

$\log \sec 36° 32' 35'' = \underline{0.09506}$

$\log \text{diff. long.} = 2.30261$

$\text{Diff. long.} = 200.73'$

$= 3° 20' 44''.$

$\text{Long. reached} = 22° 2' 44'' \text{ W.}$

$$\tan \text{course} = \frac{161.271}{86.834}.$$

$\log 161.271 = 2.20755$

$\text{colog } 86.834 = \underline{8.06131 - 10}$

$\log \tan \text{course} = 10.26886$

$\text{Course} = 61° 42'.$

$\text{Dist.} = 86.834 \sec 61° 42'.$

$\log 86.834 = 1.93869$

$\log \sec 61° 42' = \underline{0.32414}$

$\log \text{dist.} = 2.26283$

$\text{Dist.} = 183.16.$

Course, N. 61° 42′ W.; distance, 183.16 miles; latitude reached, 35° 49′ 10″ S.; longitude, 22° 2′ 44″ W.

131. A ship leaves Cape Cod (Example 120), and sails S.E. by S. 114 miles, N. by E. 94 miles, W.N.W. 42 miles. Solve as in Example 130.

$\text{First course} = 33° 45'.$

$\text{Diff. lat.} = 114 \cos 33° 45'.$

$\text{Depart.} = 114 \sin 33° 45'.$

$\log 114 = 2.05690$

$\log \cos 33° 45' = \underline{9.91985}$

$\log \text{diff. lat.} = 1.97675$

$\text{Diff. lat.} = 94.787 \text{ S.}$

$\log 114 = 2.05690$

$\log \sin 33° 45' = \underline{9.74474}$

$\log \text{depart.} = 1.80164$

$\text{Depart.} = 63.334 \text{ E.}$

$\text{Second course} = 11° 15'.$

$\text{Diff. lat.} = 94 \cos 11° 15'.$

$\text{Depart.} = 94 \sin 11° 15'.$

$\log 94 = 1.97313$

$\log \cos 11° 15' = \underline{9.99157}$

$\log \text{diff. lat.} = 1.96470$

$\text{Diff. lat.} = 92.194 \text{ N.}$

$\log 94 = 1.97313$

$\log \sin 11° 15' = \underline{9.29024}$

$\log \text{depart.} = 1.26337$

$\text{Depart.} = 18.339 \text{ E.}$

$\text{Third course} = 67° 30'.$

$\text{Diff. lat.} = 42 \cos 67° 30'.$

$\text{Depart.} = 42 \sin 67° 30'.$

$\log 42 = 1.62325$

$\log \cos 67° 30' = \underline{9.58284}$

$\log \text{diff. lat.} = 1.20609$

$\text{Diff. lat.} = 16.073 \text{ N.}$

$\log 42 = 1.62325$

$\log \sin 67° 30' = \underline{9.96562}$

$\log \text{depart.} = 1.58887$

$\text{Depart.} = 38.804 \text{ W.}$

$\text{Total diff. lat.} = 13.48' \text{ N.}$

$= 13' 29'' \text{ N.}$

$\text{Lat. of C. Cod} = 42° 2'.$

$\text{Lat. reached} = 42° 15' 29'' \text{ N.}$

$\text{Mid. lat.} = 42° 8' 44''.$

$\text{Total depart.} = 42.869 \text{ E.}$

$\text{Diff. long.} = 42.869 \sec 42° 8' 44''.$

$\log 42.869 = 1.63214$

$\log \sec 42° 8' 44'' = \underline{0.12992}$

$\log \text{diff. long.} = 1.76206$

$\text{Diff. long.} = 57.817' \text{ E.}$

$= 57' 49'' \text{ E.}$

$$\begin{aligned}
\text{Long. of Cape Cod} &= 70^\circ\ 3'\ \text{W.}\\
\text{Long. reached} &= 69^\circ\ 5'\ 11''\ \text{W.}\\
\text{tan course} &= \frac{42.869}{13.48}.\\
\log 42.869 &= 1.63214\\
\text{colog } 13.48 &= \underline{8.87031-10}\\
\log \tan \text{course} &= 10.50245\\
\text{Course} &= 72^\circ\ 32'\ 40''.\\
\text{Dist.} &= 13.48 \sec 72^\circ\ 32'\ 40''.\\
\log 13.48 &= 1.12969\\
\log \sec 72^\circ\ 32'\ 40'' &= \underline{0.52293}\\
\log \text{dist.} &= 1.65262\\
\text{Dist.} &= 44.939.
\end{aligned}$$

Course N. 72° 32′ 40″ E.; distance, 44.939 miles; latitude reached, 42° 15′ 29″ N.; longitude, 69° 5′ 11″ W.

132. A ship leaves Cape of Good Hope (latitude 34° 22′ S., longitude 18° 30′ E.) and sails N.W. 126 miles, N. by E. 84 miles, W.S.W. 217 miles. Solve as in Example 130.

$$\begin{aligned}
\text{First course} &= 45^\circ.\\
\text{Diff. lat.} &= 126 \cos 45^\circ.\\
\text{Depart.} &= 126 \sin 45^\circ.\\
\log 126 &= 2.10037\\
\log \cos 45^\circ &= \underline{9.84949}\\
\log \text{diff. lat.} &= 1.94986\\
\text{Diff. lat.} &= 89.096\ \text{N.}\\
\text{Depart.} &= 89.096\ \text{W.}\\
\text{Second course} &= 11^\circ\ 15'.\\
\text{Diff. lat.} &= 84 \cos 11^\circ\ 15'.\\
\text{Depart.} &= 84 \sin 11^\circ\ 15'.\\
\log 84 &= 1.92428\\
\log \cos 11^\circ\ 15' &= \underline{9.99157}\\
\log \text{diff. lat.} &= 1.91585\\
\text{Diff. lat.} &= 82.386\ \text{N.}\\
\log 84 &= 1.92428\\
\log \sin 11^\circ\ 15' &= \underline{9.29024}\\
\log \text{depart.} &= 1.21452
\end{aligned}$$

$$\begin{aligned}
\text{Depart.} &= 16.388\ \text{E.}\\
\text{Third course} &= 67^\circ\ 30'.\\
\text{Diff. lat.} &= 217 \cos 67^\circ 30'.\\
\text{Depart.} &= 217 \sin 67^\circ 30'.\\
\log 217 &= 2.33646\\
\log \cos 67^\circ\ 30' &= \underline{9.58284}\\
\log \text{diff. lat.} &= 1.91930\\
\text{Diff. lat.} &= 83.042\ \text{S.}\\
\log 217 &= 2.33646\\
\log \sin 67^\circ\ 30' &= \underline{9.96562}\\
\log \text{depart.} &= 2.30208\\
\text{Depart.} &= 200.49\ \text{W.}\\
\text{Total diff. lat.} &= 88.440'\ \text{N.}\\
&= \ 1^\circ\ 28'\ 26''\ \text{N.}\\
\text{Lat. reached} &= 32^\circ\ 53'\ 34''\ \text{S.}\\
\text{Mid. lat.} &= 33^\circ\ 37'\ 47''.\\
\text{Total depart.} &= 273.198\ \text{W.}\\
\text{Diff. long.} &= 273.198 \sec 33^\circ\ 37'\ 47''.\\
\log 273.198 &= 2.43648\\
\log \sec 33^\circ\ 37'\ 47'' &= \underline{0.07954}\\
\log \text{diff. long.} &= 2.51602\\
\text{Diff. long.} &= 328.11'\\
&= 5^\circ\ 28'\ 7''.\\
\text{Long. reached} &= 13^\circ\ 1'\ 53''\ \text{E.}\\
\text{tan course} &= \frac{273.198}{88.44}.\\
\log 273.198 &= 2.43648\\
\text{colog } 88.44 &= \underline{8.05335-10}\\
\log \tan \text{course} &= 10.48983\\
\text{Course} &= 72^\circ\ 3'\ 43''.\\
\text{Dist.} &= 88.44 \sec 72^\circ\ 3'\ 43''.\\
\log 88.44 &= 1.94665\\
\log \sec 72^\circ\ 3'\ 43'' &= \underline{0.51147}\\
\log \text{dist.} &= 2.45812\\
\text{Dist.} &= 287.16.
\end{aligned}$$

Course, N. 72° 3′ 43″ W.; distance, 287.16 miles; latitude reached, 32° 53′ 34″ S., longitude, 13° 1′ 53″ E.

EXERCISE XXIV. PAGE 107.

1. Prove that $\sin x + \cos x = \sqrt{2}\cos(x - \frac{1}{4}\pi)$.

By [9],
$$\begin{aligned}\cos(x - \tfrac{1}{4}\pi) &= \cos x \cos\tfrac{1}{4}\pi + \sin x \sin\tfrac{1}{4}\pi\\ &= \tfrac{1}{2}\sqrt{2}\cos x + \tfrac{1}{2}\sqrt{2}\sin x\\ &= \tfrac{1}{2}\sqrt{2}(\cos x + \sin x).\\ \therefore \sin x + \cos x &= \sqrt{2}\cos(x - \tfrac{1}{4}\pi).\end{aligned}$$

2. Prove that $\sin x - \cos x = -\sqrt{2}\cos(x + \frac{1}{4}\pi)$.

By [5],
$$\begin{aligned}\cos(x + \tfrac{1}{4}\pi) &= \cos x \cos\tfrac{1}{4}\pi - \sin x \sin\tfrac{1}{4}\pi\\ &= \tfrac{1}{2}\sqrt{2}\cos x - \tfrac{1}{2}\sqrt{2}\sin x\\ &= \tfrac{1}{2}\sqrt{2}(\cos x - \sin x).\\ \therefore \sin x - \cos x &= -\sqrt{2}\cos(x + \tfrac{1}{4}\pi).\end{aligned}$$

3. Prove that $\sin x + \sqrt{3}\cos x = 2\sin(x + \frac{1}{3}\pi)$.

By [4],
$$\begin{aligned}\sin(x + \tfrac{1}{3}\pi) &= \sin x \cos\tfrac{1}{3}\pi + \cos x \sin\tfrac{1}{3}\pi\\ &= \tfrac{1}{2}\sin x + \tfrac{1}{2}\sqrt{3}\cos x\\ &= \tfrac{1}{2}(\sin x + \sqrt{3}\cos x).\\ \therefore \sin x + \sqrt{3}\cos x &= 2\sin(x + \tfrac{1}{3}\pi).\end{aligned}$$

4. Prove that $\sin(x + \frac{1}{3}\pi) + \sin(x - \frac{1}{3}\pi) = \sin x$.

By [4],
$$\begin{aligned}\sin(x + \tfrac{1}{3}\pi) &= \sin x \cos\tfrac{1}{3}\pi + \cos x \sin\tfrac{1}{3}\pi\\ &= \tfrac{1}{2}\sin x + \tfrac{1}{2}\sqrt{3}\cos x.\end{aligned}$$

By [8],
$$\sin(x - \tfrac{1}{3}\pi) = \tfrac{1}{2}\sin x - \tfrac{1}{2}\sqrt{3}\cos x.$$
$$\therefore \sin(x + \tfrac{1}{3}\pi) + \sin(x - \tfrac{1}{3}\pi) = \sin x.$$

5. Prove that $\cos(x + \frac{1}{6}\pi) + \cos(x - \frac{1}{6}\pi) = \sqrt{3}\cos x$.

By [5],
$$\cos(x + \tfrac{1}{6}\pi) = \tfrac{1}{2}\sqrt{3}\cos x - \tfrac{1}{2}\sin x.$$

By [9],
$$\cos(x - \tfrac{1}{6}\pi) = \tfrac{1}{2}\sqrt{3}\cos x + \tfrac{1}{2}\sin x.$$
$$\therefore \cos(x + \tfrac{1}{6}\pi) + \cos(x - \tfrac{1}{6}\pi) = \sqrt{3}\cos x.$$

6. Prove that $\tan x + \sec x = \tan(\frac{1}{2}x + \frac{1}{4}\pi)$.

$$\begin{aligned}\tan x + \sec x &= \frac{\sin x}{\cos x} + \frac{1}{\cos x}\\ &= \frac{\sin x + 1}{\cos x}\\ &= \frac{1 - \cos(x + \frac{1}{2}\pi)}{\sin(x + \frac{1}{2}\pi)}\end{aligned}$$

By [16] and [12],
$$= \frac{2\sin^2\frac{1}{2}(x + \frac{1}{2}\pi)}{2\sin\frac{1}{2}(x + \frac{1}{2}\pi)\cos\frac{1}{2}(x + \frac{1}{2}\pi)}$$

$$= \frac{\sin\left(\tfrac{1}{2}x + \tfrac{1}{4}\pi\right)}{\cos\left(\tfrac{1}{2}x + \tfrac{1}{4}\pi\right)}$$
$$= \tan\left(\tfrac{1}{2}x + \tfrac{1}{4}\pi\right).$$

7. Prove that $\tan x + \sec x = \dfrac{1}{\sec x - \tan x}$.

By Prob. 2, Ex. V, $\sec^2 x = 1 + \tan^2 x.$

$$\sec^2 x - \tan^2 x = 1.$$
$$(\sec x + \tan x)(\sec x - \tan x) = 1.$$
$$\therefore \sec x + \tan x = \frac{1}{\sec x - \tan x}.$$

8. Prove that $\dfrac{1 - \tan x}{1 + \tan x} = \dfrac{\cot x - 1}{\cot x + 1}$.

$$\frac{\cot x - 1}{\cot x + 1} = \frac{\dfrac{1}{\tan x} - 1}{\dfrac{1}{\tan x} + 1} = \frac{1 - \tan x}{1 + \tan x}.$$

9. Prove that $\dfrac{\sin x}{1 + \cos x} + \dfrac{1 + \cos x}{\sin x} = 2 \csc x$.

$$\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x} = \frac{\sin^2 x + (1 + \cos x)^2}{\sin x\,(1 + \cos x)}$$
$$= \frac{\sin^2 x + \cos^2 x + 2\cos x + 1}{\sin x\,(1 + \cos x)}$$
$$= \frac{1 + 2\cos x + 1}{\sin x\,(1 + \cos x)}$$
$$= \frac{2\,(1 + \cos x)}{\sin x\,(1 + \cos x)}$$
$$= \frac{2}{\sin x}$$
$$= 2 \csc x.$$

10. Prove that $\tan x + \cot x = 2 \csc 2x$.

$$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$
$$= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x}$$
$$= \frac{1}{\sin x \cos x}$$

$$= \frac{2}{2 \sin x \cos x}$$

By [12],

$$= \frac{2}{\sin 2x}$$

$$= 2 \csc 2x.$$

11. Prove that $\cot x - \tan x = 2 \cot 2x$.

$$\cot x - \tan x = \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} = \frac{\cos^2 x - \sin^2 x}{\sin x \cos x} = \frac{2 \cos 2x}{\sin 2x} = 2 \cot 2x.$$

12. Prove that $1 + \tan x \tan 2x = \sec 2x$.

$$\begin{aligned} 1 + \tan x \tan 2x &= 1 + \frac{\sin x \sin 2x}{\cos x \cos 2x} \\ &= 1 + \frac{2 \sin^2 x \cos x}{\cos x (1 - 2 \sin^2 x)} \\ &= 1 + \frac{2 \sin^2 x}{1 - 2 \sin^2 x} \\ &= \frac{1}{1 - 2 \sin^2 x} \\ &= \frac{1}{\cos 2x} \\ &= \sec 2x. \end{aligned}$$

13. Prove that $\sec 2x = \frac{\sec^2 x}{2 - \sec^2 x}$.

$$\sec 2x = \frac{1}{\cos 2x} = \frac{1}{2 \cos^2 x - 1} = \frac{\frac{1}{\cos^2 x}}{2 - \frac{1}{\cos^2 x}} = \frac{\sec^2 x}{2 - \sec^2 x}.$$

14. Prove that $2 \sec 2x = \sec (x + 45°) \sec (x - 45°)$.

$$\begin{aligned} 2 \sec 2x &= \frac{2}{\cos 2x} \\ &= \frac{2}{\cos^2 x - \sin^2 x} \\ &= \frac{2}{(\cos x - \sin x)(\cos x + \sin x)} \end{aligned}$$

By Probs. 1 and 2, Ex. XXIV;

$$\begin{aligned} &= \frac{2}{2 \cos (x + 45°) \cos (x - 45°)} \\ &= \sec (x + 45°) \sec (x - 45°). \end{aligned}$$

15. Prove that $\tan 2x + \sec 2x = \dfrac{\cos x + \sin x}{\cos x - \sin x}$.

$$\tan 2x + \sec 2x = \frac{\sin 2x}{\cos 2x} + \frac{1}{\cos 2x}$$

$$= \frac{\sin 2x + 1}{\cos 2x}$$

$$= \frac{1 - \cos(2x + 90^\circ)}{\sin(2x + 90^\circ)}$$

By [16] and [12],

$$= \frac{2\sin^2(x + 45^\circ)}{2\sin(x + 45^\circ)\cos(x + 45^\circ)}$$

$$= \frac{\sin(x + 45^\circ)}{\cos(x + 45^\circ)}$$

By [4] and [5],

$$= \frac{\sin x \cos 45^\circ + \cos x \sin 45^\circ}{\cos x \cos 45^\circ - \sin x \sin 45^\circ}$$

$$= \frac{\frac{1}{2}\sqrt{2}\sin x + \frac{1}{2}\sqrt{2}\cos x}{\frac{1}{2}\sqrt{2}\cos x - \frac{1}{2}\sqrt{2}\sin x}$$

$$= \frac{\cos x + \sin x}{\cos x - \sin x}.$$

16. Prove that $\sin 2x = \dfrac{2\tan x}{1 + \tan^2 x}$.

$$\frac{2\tan x}{1 + \tan^2 x} = \frac{2\tan x}{\sec^2 x} = 2\tan x \cos^2 x = 2\sin x \cos x = \sin 2x.$$

17. Prove that $2\sin x + \sin 2x = \dfrac{2\sin^3 x}{1 - \cos x}$.

$$2\sin x + \sin 2x = 2\sin x + 2\sin x \cos x$$

$$= 2\sin x(1 + \cos x).$$

But

$$1 - \cos^2 x = \sin^2 x.$$

$$\therefore 1 + \cos x = \frac{\sin^2 x}{1 - \cos x}.$$

$$\therefore 2\sin x + \sin 2x = 2\sin x \frac{\sin^2 x}{1 - \cos x}$$

$$= \frac{2\sin^3 x}{1 - \cos x}.$$

18. Prove that $\sin 3x = \dfrac{\sin^2 2x - \sin^2 x}{\sin x}$.

By [20], $\sin 2x + \sin x = 2\sin \frac{3}{2}x \cos \frac{1}{2}x.$

By [21], $\sin 2x - \sin x = 2\cos \frac{3}{2}x \sin \frac{1}{2}x.$

$$\therefore \sin^2 2x - \sin^2 x = 2\sin\tfrac{3}{2}x\cos\tfrac{3}{2}x \times 2\sin\tfrac{1}{2}x\cos\tfrac{1}{2}x$$

By [12],
$$= \sin 3x \sin x.$$

$$\therefore \sin 3x = \frac{\sin^2 2x - \sin^2 x}{\sin x}.$$

19. Prove that $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$.

$$\tan 3x = \tan(2x + x)$$

By [6],
$$= \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$

By [14],
$$= \frac{\dfrac{2\tan x}{1 - \tan^2 x} + \tan x}{1 - \dfrac{2\tan x}{1 - \tan^2 x}\tan x}$$

$$= \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}.$$

20. Prove that $\dfrac{\tan 2x + \tan x}{\tan 2x - \tan x} = \dfrac{\sin 3x}{\sin x}$.

By [24],
$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan\frac{1}{2}(A + B)}{\tan\frac{1}{2}(A - B)}.$$

By composition and division,
$$\frac{\sin A}{\sin B} = \frac{\tan\frac{1}{2}(A + B) + \tan\frac{1}{2}(A - B)}{\tan\frac{1}{2}(A + B) - \tan\frac{1}{2}(A - B)}.$$

Put $3x$ for A, x for B,
$$\frac{\sin 3x}{\sin x} = \frac{\tan 2x + \tan x}{\tan 2x - \tan x}.$$

21. Prove that $\sin(x + y) + \cos(x - y) = 2\sin(x + \frac{1}{4}\pi)\sin(y + \frac{1}{4}\pi)$.

By [4], $\sin(x + y) = \sin x\cos y + \cos x\sin y.$

By [9], $\cos(x - y) = \cos x\cos y + \sin x\sin y.$

$$\therefore \sin(x + y) + \cos(x - y) = (\sin x + \cos x)\cos y + (\cos x + \sin x)\sin y$$
$$= (\sin x + \cos x)(\sin y + \cos y).$$

But
$$\sin x + \cos x = \sqrt{2}\left(\tfrac{1}{2}\sqrt{2}\sin x + \tfrac{1}{2}\sqrt{2}\cos x\right)$$
$$= \sqrt{2}\sin(x + \tfrac{1}{4}\pi).$$

Similarly, $\sin y + \cos y = \sqrt{2}\sin(y + \frac{1}{4}\pi).$

$$\therefore \sin(x + y) + \cos(x - y) = 2\sin(x + \tfrac{1}{4}\pi)\sin(y + \tfrac{1}{4}\pi).$$

22. Prove that $\sin(x + y) - \cos(x - y) = -2\sin(x - \frac{1}{4}\pi)\sin(y - \frac{1}{4}\pi)$.

By [4], $\sin(x + y) = \sin x\cos y + \cos x\sin y.$

By [9], $\cos(x - y) = \cos x\cos y + \sin x\sin y.$

$$\therefore \sin(x + y) - \cos(x - y) = (\sin x - \cos x)\cos y + (\cos x - \sin x)\sin y$$
$$= (\sin x - \cos x)(\cos y - \sin y)$$
$$= -2\sin(x - \tfrac{1}{4}\pi)\sin(y - \tfrac{1}{4}\pi).$$

23. Prove that $\tan x + \tan y = \dfrac{\sin(x+y)}{\cos x \cos y}$.

$$\tan x + \tan y = \frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}$$
$$= \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y}$$

By [4],
$$= \frac{\sin(x+y)}{\cos x \cos y}.$$

24. Prove that $\tan(x+y) = \dfrac{\sin 2x + \sin 2y}{\cos 2x + \cos 2y}$.

By [20], $\sin 2x + \sin 2y = 2\sin(x+y)\cos(x-y)$.

By [22], $\cos 2x + \cos 2y = 2\cos(x+y)\cos(x-y)$.

$$\therefore \frac{\sin 2x + \sin 2y}{\cos 2x + \cos 2y} = \frac{2\sin(x+y)\cos(x-y)}{2\cos(x+y)\cos(x-y)}$$
$$= \tan(x+y).$$

25. Prove that $\dfrac{\sin x + \cos y}{\sin x - \cos y} = \dfrac{\tan[\frac{1}{2}(x+y) + 45^\circ]}{\tan[\frac{1}{2}(x-y) - 45^\circ]}$.

$$\sin x + \cos y = \sin x + \sin(y + 90^\circ)$$

By [20],
$$= 2\sin\tfrac{1}{2}(x + y + 90^\circ)\cos\tfrac{1}{2}(x - y - 90^\circ).$$

By [21],
$$\sin x - \cos y = 2\cos\tfrac{1}{2}(x + y + 90^\circ)\sin\tfrac{1}{2}(x - y - 90^\circ).$$

$$\therefore \frac{\sin x + \cos y}{\sin x - \cos y} = \frac{\tan\frac{1}{2}(x + y + 90^\circ)}{\tan\frac{1}{2}(x - y - 90^\circ)}$$
$$= \frac{\tan[\frac{1}{2}(x+y) + 45^\circ]}{\tan[\frac{1}{2}(x-y) - 45^\circ]}.$$

26. Prove that $\sin 2x + \sin 4x = 2\sin 3x \cos x$.

By [20], $\sin 2x + \sin 4x = 2\sin 3x \cos x$.

27. Prove that

$$\sin 4x = 4\sin x \cos x - 8\sin^3 x \cos x = 8\cos^3 x \sin x - 4\cos x \sin x.$$

By [12], $\sin 4x = 2\sin 2x \cos 2x$

By [12] and [13], $= 4\sin x \cos x(1 - 2\sin^2 x)$

$= 4\sin x \cos x - 8\sin^3 x \cos x.$

Again, $\sin 4x = 4\sin x \cos x(2\cos^2 x - 1)$

$= 8\cos^3 x \sin x - 4\sin x \cos x.$

28. Prove that $\cos 4x = 1 - 8\cos^2 x + 8\cos^4 x = 1 - 8\sin^2 x + 8\sin^4 x$.

By [13], $\cos 4x = 2\cos^2 2x - 1$

By [13], $= 2(2\cos^2 x - 1)^2 - 1$

$= 8\cos^4 x - 8\cos^2 x + 2 - 1$

$= 1 - 8\cos^2 x + 8\cos^4 x.$

Again, $\cos 4x = 1 - 2\sin^2 2x$

By [12], $= 1 - 2(4\sin^2 x\cos^2 x)$

$= 1 - 8\sin^2 x(1 - \sin^2 x)$

$= 1 - 8\sin^2 x + 8\sin^4 x.$

29. Prove that $\cos 2x + \cos 4x = 2\cos 3x\cos x.$

By [22], $\cos 2x + \cos 4x = 2\cos 3x\cos x.$

30. Prove that $\sin 3x - \sin x = 2\cos 2x\sin x.$

By [21], $\sin 3x - \sin x = 2\cos 2x\sin x.$

31. Prove that $\sin^3 x\sin 3x + \cos^3 x\cos 3x = \cos^3 2x.$

$\sin^3 x\sin 3x = \sin x\sin^2 x\sin 3x$

$= \sin x(1 - \cos^2 x)\sin 3x$

$= \sin x\sin 3x - \sin x\cos^2 x\sin 3x.$

$\cos^3 x\cos 3x = \cos x\cos^2 x\cos 3x$

$= \cos x(1 - \sin^2 x)\cos 3x$

$= \cos x\cos 3x - \cos x\sin^2 x\cos 3x.$

$\therefore \sin^3 x\sin 3x + \cos^3 x\cos 3x$

$= \sin x\sin 3x + \cos x\cos 3x$
$\quad - \sin x\cos^2 x\sin 3x - \cos x\sin^2 x\cos 3x$

By [9], $= \cos(3x - x)$
$\quad - \sin x\cos x(\cos x\sin 3x + \sin x\cos 3x)$

By [4], $= \cos 2x - \sin x\cos x\sin(3x + x)$

By [12], $= \cos 2x - \frac{1}{2}\sin 2x\sin 4x$

By [12], $= \cos 2x - \sin^2 2x\cos 2x$

$= \cos 2x(1 - \sin^2 2x)$

By [13], $= \cos^3 2x.$

32. Prove that $\cos^4 x - \sin^4 x = \cos 2x.$

$\cos^4 x - \sin^4 x = (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)$

$= 1 \times \cos 2x$

$= \cos 2x.$

33. Prove that $\cos^4 x + \sin^4 x = 1 - \frac{1}{2}\sin^2 2x.$

$\cos^4 x + \sin^4 x = (\cos^2 x + \sin^2 x)^2 - 2\sin^2 x\cos^2 x$

$= 1 - 2\sin^2 x\cos^2 x$

$= 1 - \frac{1}{2}\sin^2 2x.$

34. Prove that $\cos^6 x - \sin^6 x = (1 - \sin^2 x \cos^2 x) \cos 2x$.

$$\cos^6 x - \sin^6 x = (\cos^2 x - \sin^2 x)(\cos^4 x + \cos^2 x \sin^2 x + \sin^4 x)$$

By [13],
$$= \cos 2x \left[(\cos^2 x + \sin^2 x)^2 - \cos^2 x \sin^2 x\right]$$
$$= \cos 2x (1 - \cos^2 x \sin^2 x).$$

35. Prove that $\cos^6 x + \sin^6 x = 1 - 3 \sin^2 x \cos^2 x$.

$$\cos^6 x + \sin^6 x = (\cos^2 x + \sin^2 x)(\cos^4 x - \cos^2 x \sin^2 x + \sin^4 x)$$
$$= \cos^4 x - \cos^2 x \sin^2 x + \sin^4 x$$
$$= (\cos^2 x + \sin^2 x)^2 - 3 \cos^2 x \sin^2 x$$
$$= 1 - 3 \cos^2 x \sin^2 x.$$

36. Prove that $\dfrac{\sin 3x + \sin 5x}{\cos 3x - \cos 5x} = \cot x$.

By [20], $\sin 3x + \sin 5x = 2 \sin 4x \cos x$.

By [23], $\cos 3x - \cos 5x = 2 \sin 4x \sin x$.

$$\therefore \frac{\sin 3x + \sin 5x}{\cos 3x - \cos 5x} = \frac{\cos x}{\sin x} = \cot x.$$

37. Prove that $\dfrac{\sin 3x + \sin 5x}{\sin x + \sin 3x} = 2 \cos 2x$.

By [20], $\sin 3x + \sin 5x = 2 \sin 4x \cos x$.

By [20], $\sin x + \sin 3x = 2 \sin 2x \cos x$.

$$\therefore \frac{\sin 3x + \sin 5x}{\sin x + \sin 3x} = \frac{\sin 4x}{\sin 2x}$$

By [12],
$$= \frac{2 \sin 2x \cos 2x}{\sin 2x}$$
$$= 2 \cos 2x.$$

38. Prove that $\csc x - 2 \cot 2x \cos x = 2 \sin x$.

$$\csc x - 2 \cot 2x \cos x = \csc x - 2 \frac{\cos 2x}{\sin 2x} \cos x$$

By [12],
$$= \csc x - \frac{2 \cos 2x \cos x}{2 \sin x \cos x}$$
$$= \frac{1}{\sin x} - \frac{\cos 2x}{\sin x}$$
$$= \frac{1 - \cos 2x}{\sin x}$$

By [13],
$$= \frac{2 \sin^2 x}{\sin x}$$
$$= 2 \sin x.$$

39. Prove that $(\sin 2x - \sin 2y)\tan(x+y) = 2(\sin^2 x - \sin^2 y)$.

By [21], $\sin 2x - \sin 2y = 2\cos(x+y)\sin(x-y)$.

$\therefore (\sin 2x - \sin 2y)\tan(x+y) = 2\sin(x+y)\sin(x-y)$.

By [20], $\sin x + \sin y = 2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$.

By [20], $\sin x - \sin y = 2\cos\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)$.

$\therefore \sin^2 x - \sin^2 y = 4\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)\cos\frac{1}{2}(x-y)$

By [12], $= \sin(x+y)\sin(x-y)$.

$$\begin{aligned}\therefore 2(\sin^2 x - \sin^2 y) &= 2\sin(x+y)\sin(x-y)\\ &= (\sin 2x - \sin 2y)\tan(x+y).\end{aligned}$$

40. Prove that $(1 + \cot x + \tan x)(\sin x - \cos x) = \dfrac{\sec x}{\csc^2 x} - \dfrac{\csc x}{\sec^2 x}$.

$$\begin{aligned}(1+\cot x+\tan x)(\sin x-\cos x) &= \sin x - \cos x + \cos x - \frac{\cos^2 x}{\sin x} + \frac{\sin^2 x}{\cos x} - \sin x\\ &= \frac{\sin^2 x}{\cos x} - \frac{\cos^2 x}{\sin x}\\ &= \frac{\sec x}{\csc^2 x} - \frac{\csc x}{\sec^2 x}.\end{aligned}$$

41. Prove that $\sin x + \sin 3x + \sin 5x = \dfrac{\sin^2 3x}{\sin x}$.

By [20], $\sin x + \sin 5x = 2\sin 3x\cos 2x$.

$$\begin{aligned}\therefore \sin x + \sin 3x + \sin 5x &= \sin 3x + 2\sin 3x\cos 2x\\ &= \sin 3x(1 + 2\cos 2x).\end{aligned}$$

By [21], $\sin 3x - \sin x = 2\cos 2x\sin x$.

$$\frac{\sin 3x}{\sin x} - 1 = 2\cos 2x.$$

$$1 + 2\cos 2x = \frac{\sin 3x}{\sin x}.$$

$$\begin{aligned}\therefore \sin x + \sin 3x + \sin 5x &= \sin 3x\frac{\sin 3x}{\sin x}\\ &= \frac{\sin^2 3x}{\sin x}.\end{aligned}$$

42. Prove that $\dfrac{3\cos x + \cos 3x}{3\sin x - \sin 3x} = \cot^3 x$.

$3\cos x + \cos 3x = 2\cos x + (\cos x + \cos 3x)$

By [22], $= 2\cos x + 2\cos x\cos 2x$

$= 2\cos x(1 + \cos 2x)$

By [17], $= 4\cos^3 x$.

$3\sin x - \sin 3x = 2\sin x - (\sin 3x - \sin x)$

By [21],
$$= 2\sin x - 2\sin x\cos 2x$$
$$= 2\sin x(1 - \cos 2x)$$
By [16],
$$= 4\sin^3 x.$$
$$\therefore \frac{3\cos x + \cos 3x}{3\sin x - \sin 3x} = \frac{4\cos^3 x}{4\sin^3 x}$$
$$= \cot^3 x.$$

43. Prove that $\sin 3x = 4\sin x\sin(60^\circ + x)\sin(60^\circ - x)$.

By [4], $\sin(60^\circ + x) = \frac{1}{2}\sqrt{3}\cos x + \frac{1}{2}\sin x.$

By [8], $\sin(60^\circ - x) = \frac{1}{2}\sqrt{3}\cos x - \frac{1}{2}\sin x.$

$$\therefore \sin(60^\circ + x)\sin(60^\circ - x) = \frac{3}{4}\cos^2 x - \frac{1}{4}\sin^2 x$$
$$= \frac{3(1 - \sin^2 x) - \sin^2 x}{4}$$
$$= \frac{3 - 4\sin^2 x}{4}.$$
$$4\sin x\sin(60^\circ + x)\sin(60^\circ - x) = \sin x(3 - 4\sin^2 x)$$
$$= 3\sin x - 4\sin^3 x$$
By Prob. 18, Ex. XIV,
$$= \sin 3x.$$

44. Prove that $\sin 4x = 2\sin x\cos 3x + \sin 2x$.

By [21], $\sin 4x - \sin 2x = 2\cos 3x\sin x.$
$$\therefore \sin 4x = 2\cos 3x\sin x + \sin 2x.$$

45. Prove that $\sin x + \sin(x - \frac{2}{3}\pi) + \sin(\frac{1}{3}\pi - x) = 0$.

By [20], $\sin(x - \frac{2}{3}\pi) + \sin(\frac{1}{3}\pi - x) = 2\sin(-\frac{1}{6}\pi)\cos(x - \frac{1}{2}\pi)$
$$= -2\sin\tfrac{1}{6}\pi\sin x$$
$$= -\sin x.$$
$$\therefore \sin x + \sin(x - \tfrac{2}{3}\pi) + \sin(\tfrac{1}{3}\pi - x) = 0.$$

46. Prove that $\cos x\sin(y - z) + \cos y\sin(z - x) + \cos z\sin(x - y) = 0$.

By [8], $\cos x\sin(y - z) = \cos x\sin y\cos z - \cos x\cos y\sin z.$
$$\cos y\sin(z - x) = \cos y\sin z\cos x - \cos y\cos z\sin x.$$
$$\cos z\sin(x - y) = \cos z\sin x\cos y - \cos z\cos x\sin y.$$
$$\therefore \cos x\sin(y - z) + \cos y\sin(z - x) + \cos z\sin(x - y) = 0.$$

47. Prove that
$$\cos(x + y)\sin y - \cos(x + z)\sin z = \sin(x + y)\cos y - \sin(x + z)\cos z.$$
By [4] and [5],
$$\sin(x + y)\cos y - \cos(x + y)\sin y = \sin x(\cos^2 y + \sin^2 y) = \sin x.$$
$$\sin(x + z)\cos z - \cos(x + z)\sin z = \sin x(\cos^2 z + \sin^2 z) = \sin x.$$
$$\therefore \sin(x + y)\cos y - \cos(x + y)\sin y = \sin(x + z)\cos z - \cos(x + z)\sin z.$$
$$\therefore \cos(x + y)\sin y - \cos(x + z)\sin z = \sin(x + y)\cos y - \sin(x + z)\cos z.$$

48. Prove that

$$\cos(x+y+z)+\cos(x+y-z)+\cos(x-y+z)+\cos(y+z-x)$$
$$=4\cos x\cos y\cos z.$$

By [22], $\cos[(x+y)+z]+\cos[(x+y)-z]=2\cos(x+y)\cos z.$

$\cos[z+(x-y)]+\cos[z-(x-y)]=2\cos z\cos(x-y).$

$$\begin{aligned}\therefore \cos(x+y+z)+\cos(x+y-z)+\cos(x-y+z)+\cos(y+z-x)\\ &=2\cos(x+y)\cos z+2\cos(x-y)\cos z\\ &=2\cos z[\cos(x+y)+\cos(x-y)]\\ &=2\cos z(2\cos x\cos y)\\ &=4\cos x\cos y\cos z.\end{aligned}$$

49. Prove that $\sin(x+y)\cos(x-y)+\sin(y+z)\cos(y-z)$
$+\sin(z+x)\cos(z-x)=\sin 2x+\sin 2y+\sin 2z$

By [20], $\sin(x+y)\cos(x-y)=\frac{1}{2}(\sin 2x+\sin 2y).$

$\sin(y+z)\cos(y-z)=\frac{1}{2}(\sin 2y+\sin 2z).$

$\sin(z+x)\cos(z-x)=\frac{1}{2}(\sin 2z+\sin 2x).$

$$\therefore \sin(x+y)\cos(x-y)+\sin(y+z)\cos(y-z)+\sin(z+x)\cos(z-x)$$
$$=\sin 2x+\sin 2y+\sin 2z.$$

50. Prove that $\dfrac{\sin 75^\circ+\sin 15^\circ}{\sin 75^\circ-\sin 15^\circ}=\tan 60^\circ.$

By [20], $\sin 75^\circ+\sin 15^\circ=2\sin 45^\circ\cos 30^\circ.$

By [21], $\sin 75^\circ-\sin 15^\circ=2\cos 45^\circ\sin 30^\circ.$

$$\begin{aligned}\therefore \frac{\sin 75^\circ+\sin 15^\circ}{\sin 75^\circ-\sin 15^\circ}&=\frac{2\sin 45^\circ\cos 30^\circ}{2\cos 45^\circ\sin 30^\circ}\\ &=\tan 45^\circ\cot 30^\circ\\ &=\tan 60^\circ.\end{aligned}$$

51. Prove that $\cos 20^\circ+\cos 100^\circ+\cos 140^\circ=0.$

By [22], $\cos 20^\circ+\cos 100^\circ=2\cos 60^\circ\cos 40^\circ$
$=\cos 40^\circ.$

Also, $\cos 140^\circ=\cos(180^\circ-40^\circ)$
$=-\cos 40^\circ.$

$\therefore \cos 20^\circ+\cos 100^\circ+\cos 140^\circ=0.$

52. Prove that $\cos 36^\circ+\sin 36^\circ=\sqrt{2}\cos 9^\circ.$

By Prob. 1, Ex. XXIV, $\cos 36^\circ+\sin 36^\circ=\sqrt{2}\cos(36^\circ-\frac{1}{4}\pi)$

$$\begin{aligned}&=\sqrt{2}\cos(-9^\circ)\\ &=\sqrt{2}\cos 9^\circ.\end{aligned}$$

53. **Prove that $\tan 11^\circ 15' + 2\tan 22^\circ 30' + 4\tan 45^\circ = \cot 11^\circ 15'$.**

By Prob. 11, Ex. XXIV,

$$\begin{aligned} \cot 11^\circ 15' - \tan 11^\circ 15' &= 2\cot 22^\circ 30'. \\ 2\cot 22^\circ 30' - 2\tan 22^\circ 30' &= 4\cot 45^\circ. \end{aligned}$$

$$\therefore \cot 11^\circ 15' - \tan 11^\circ 15' - 2\tan 22^\circ 30' = 4\cot 45^\circ = 4\tan 45^\circ.$$

$$\therefore \tan 11^\circ 15' + 2\tan 22^\circ 30' + 4\tan 45^\circ = \cot 11^\circ 15'.$$

54. **If A, B, C are the angles of a plane triangle, prove that**

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C.$$

$$A + B + C = 180^\circ.$$

$$\begin{aligned} &\text{By [20],} & \sin 2A + \sin 2B &= 2\sin(A+B)\cos(A-B) \\ & & &= 2\sin C\cos(A-B). \\ &\text{By [12],} & \sin 2C &= 2\sin C\cos C. \\ & & \therefore \sin 2A + \sin 2B + \sin 2C &= 2\sin C\cos(A-B) + 2\sin C\cos C \\ & & &= 2\sin C\,[\cos(A-B) - \cos(A+B)] \\ &\text{By [23],} & &= 2\sin C\,[-2\sin A\sin(-B)] \\ & & &= 4\sin C\sin A\sin B. \end{aligned}$$

55. **If A, B, C are the angles of a plane triangle, prove that**

$$\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C.$$

$$\begin{aligned} &\text{By [22],} & \cos 2A + \cos 2B &= 2\cos(A+B)\cos(A-B) \\ & & &= -2\cos C\cos(A-B). \\ & & \therefore \cos 2A + \cos 2B + \cos 2C &= -2\cos C\cos(A-B) + \cos 2C \\ &\text{By [13],} & &= -2\cos C\cos(A-B) + 2\cos^2 C - 1 \\ & & &= 2\cos C\,[\cos C - \cos(A-B)] - 1 \\ & & &= 2\cos C\,[-\cos(A+B) - \cos(A-B)] - 1 \\ &\text{By [22],} & &= 2\cos C\,(-2\cos A\cos B) - 1 \\ & & &= -4\cos A\cos B\cos C - 1. \end{aligned}$$

56. **If A, B, C are the angles of a plane triangle, prove that**

$$\sin 3A + \sin 3B + \sin 3C = -4\cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2}.$$

$$\begin{aligned} &\text{By [20],} & \sin 3A + \sin 3B &= 2\sin\tfrac{3}{2}(A+B)\cos\tfrac{3}{2}(A-B) \\ & & &= 2\sin\tfrac{3}{2}(180^\circ - C)\cos\tfrac{3}{2}(A-B) \\ & & &= -2\cos\tfrac{3}{2}C\cos\tfrac{3}{2}(A-B). \\ &\text{By [12],} & \sin 3C &= 2\sin\tfrac{3}{2}C\cos\tfrac{3}{2}C \\ & & &= 2\sin\tfrac{3}{2}(180^\circ - A - B)\cos\tfrac{3}{2}C \\ & & &= -2\cos\tfrac{3}{2}(A+B)\cos\tfrac{3}{2}C. \\ & & \therefore \sin 3A + \sin 3B + \sin 3C &= -2\cos\tfrac{3}{2}C\,[\cos\tfrac{3}{2}(A-B) + \cos\tfrac{3}{2}(A+B)] \\ & & &= -2\cos\tfrac{3}{2}C\,(2\cos\tfrac{3}{2}A\cos\tfrac{3}{2}B) \\ & & &= -4\cos\tfrac{3}{2}A\cos\tfrac{3}{2}B\cos\tfrac{3}{2}C. \end{aligned}$$

57. If A, B, C are the angles of a plane triangle, prove that

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C.$$

By [16],

$$\cos^2 A = \frac{1+\cos 2A}{2}.$$

$$\cos^2 B = \frac{1+\cos 2B}{2}.$$

$$\cos^2 C = \frac{1+\cos 2C}{2}.$$

$$\cos^2 A + \cos^2 B + \cos^2 C = \frac{3+\cos 2A + \cos 2B + \cos 2C}{2}.$$

By Prob. 55, Ex. XXIV,

$$\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C.$$

$$\begin{aligned}\therefore \cos^2 A + \cos^2 B + \cos^2 C &= \frac{3-1-4\cos A\cos B\cos C}{2}\\ &= 1 - 2\cos A\cos B\cos C.\end{aligned}$$

58. If $A + B + C = 90°$, prove that

$$\tan A\tan B + \tan B\tan C + \tan C\tan A = 1.$$

$$\begin{aligned}\tan A\tan B &+ \tan B\tan C + \tan C\tan A\\ &= \tan A\tan B + (\tan A + \tan B)\tan C\\ &= \tan A\tan B + \frac{\tan A + \tan B}{\tan(A+B)}\end{aligned}$$

By [6],

$$\begin{aligned}&= \tan A\tan B + \frac{\tan A + \tan B}{\dfrac{\tan A + \tan B}{1-\tan A\tan B}}\\ &= \tan A\tan B + 1 - \tan A\tan B\\ &= 1.\end{aligned}$$

59. If $A + B + C = 90°$, prove that

$$\sin^2 A + \sin^2 B + \sin^2 C = 1 - 2\sin A\sin B\sin C.$$

$$\sin C = \cos(A+B)$$

By [5],

$$\begin{aligned}&= \cos A\cos B - \sin A\sin B.\\ \sin C + \sin A\sin B &= \cos A\cos B.\\ \sin^2 C + 2\sin A\sin B\sin C + \sin^2 A\sin^2 B &= \cos^2 A\cos^2 B.\\ \sin^2 C + 2\sin A\sin B\sin C &= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B\\ &= (1-\sin^2 A)(1-\sin^2 B) - \sin^2 A\sin^2 B\\ &= 1 - \sin^2 A - \sin^2 B.\\ \therefore \sin^2 A + \sin^2 B + \sin^2 C &= 1 - 2\sin A\sin B\sin C.\end{aligned}$$

60. If $A + B + C = 90^\circ$, prove that

$$\sin 2A + \sin 2B + \sin 2C = 4\cos A \cos B \cos C.$$

By [20], $\sin 2A + \sin 2B = 2\sin(A + B)\cos(A - B)$

$$= 2\cos C \cos(A - B).$$

By [12], $\sin 2C = 2\sin C \cos C.$

$$\begin{aligned}\therefore \sin 2A + \sin 2B + \sin 2C &= 2\cos C\cos(A - B) + 2\sin C\cos C\\ &= 2\cos C[\cos(A - B) + \sin C]\\ &= 2\cos C[\cos(A - B) + \cos(A + B)]\end{aligned}$$

By [22], $= 4\cos A \cos B \cos C.$

61. Prove that $\sin(\sin^{-1}x + \sin^{-1}y) = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}.$

By [4], $\sin(\sin^{-1}x + \sin^{1-}y) = \sin(\sin^{-1}x)\cos(\sin^{-1}y)$

$$\begin{aligned}&+ \cos(\sin^{-1}x)\sin(\sin^{-1}y)\\ &= x\sqrt{1 - y^2} + y\sqrt{1 - x^2}.\end{aligned}$$

62. Prove that $\tan(\tan^{-1}x + \tan^{-1}y) = \dfrac{x + y}{1 - xy}.$

By [6], $\tan(\tan^{-1}x + \tan^{-1}y) = \dfrac{\tan(\tan^{-1}x) + \tan(\tan^{-1}y)}{1 - \tan(\tan^{-1}x)\tan(\tan^{-1}y)} = \dfrac{x + y}{1 - xy}.$

63. Prove that $2\tan^{-1}x = \tan^{-1}\dfrac{2x}{1 - x^2}.$

By [14], $\tan(2\tan^{-1}x) = \dfrac{2\tan(\tan^{-1}x)}{1 - \tan^2(\tan^{-1}x)} = \dfrac{2x}{1 - x^2}.$

$$\therefore 2\tan^{-1}x = \tan^{-1}\frac{2x}{1 - x^2}.$$

64. Prove that $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1 - x^2}).$

By [12], $\sin(2\sin^{-1}x) = 2\sin(\sin^{-1}x)\cos(\sin^{-1}x) = 2x\sqrt{1 - x^2}.$

$$\therefore 2\sin^{-1}x = \sin^{-1}(2x\sqrt{1 - x^2}).$$

65. Prove that $2\cos^{-1}x = \cos^{-1}(2x^2 - 1).$

By [13], $\cos(2\cos^{-1}x) = 2\cos^2(\cos^{-1}x) - 1 = 2x^2 - 1.$

$$\therefore 2\cos^{-1}x = \cos^{-1}(2x^2 - 1).$$

66. Prove that $3\tan^{-1}x = \tan^{-1}\dfrac{3x - x^3}{1 - 3x^2}.$

$$\tan(3\tan^{-1}x) = \tan(\tan^{-1}x + 2\tan^{-1}x)$$

By [6],

$$\begin{aligned}&= \frac{\tan(\tan^{-1}x) + \tan(2\tan^{-1}x)}{1 - \tan(\tan^{-1}x)\tan(2\tan^{-1}x)}\\ &= \frac{x + \tan(2\tan^{-1}x)}{1 - x\tan(2\tan^{-1}x)}\end{aligned}$$

By Prob. 63, Ex. XXIV, $$= \frac{x + \dfrac{2x}{1 - x^2}}{1 - x\dfrac{2x}{1 - x^2}} = \frac{3x - x^3}{1 - 3x^2}.$$

$$\therefore 3\tan^{-1}x = \tan^{-1}\frac{3x - x^3}{1 - 3x^2}.$$

67. Prove that
$$\sin^{-1}\sqrt{\frac{x}{y}} = \tan^{-1}\sqrt{\frac{x}{y - x}}.$$

Let $\sin^{-1}\sqrt{\dfrac{x}{y}} = n.$

Then $$\sqrt{\frac{x}{y}} = \sin n.$$
$$\sqrt{\frac{y - x}{y}} = \cos n.$$
$$\sqrt{\frac{x}{y - x}} = \tan n.$$
$$\therefore n = \tan^{-1}\sqrt{\frac{x}{y - x}}.$$
$$\therefore \sin^{-1}\sqrt{\frac{x}{y}} = \tan^{-1}\sqrt{\frac{y}{y - x}}.$$

68. Prove that
$$\sin^{-1}\sqrt{\frac{x - y}{x - z}} = \tan^{-1}\sqrt{\frac{x - y}{y - z}}.$$

Let
$$\sin^{-1}\sqrt{\frac{x - y}{x - z}} = n.$$

Then $$\sqrt{\frac{x - y}{x - z}} = \sin n.$$
$$\sqrt{\frac{y - z}{x - z}} = \cos n.$$
$$\sqrt{\frac{x - y}{y - z}} = \tan n.$$
$$n = \tan^{-1}\sqrt{\frac{x - y}{y - z}}.$$
$$\therefore \sin^{-1}\sqrt{\frac{x - y}{x - z}} = \tan^{-1}\sqrt{\frac{x - y}{y - z}}.$$

69. Prove that
$$\sin^{-1}x = \sec^{-1}\frac{1}{\sqrt{1 - x^2}}.$$

Let $\sin^{-1}x = n.$

Then $$x = \sin n.$$
$$\sqrt{1 - x^2} = \cos n.$$
$$\frac{1}{\sqrt{1 - x^2}} = \sec n.$$
$$n = \sec^{-1}\frac{1}{\sqrt{1 - x^2}}.$$
$$\therefore \sin^{-1}x = \sec^{-1}\frac{1}{\sqrt{1 - x^2}}.$$

70. Prove that
$$2\sec^{-1}x = \tan^{-1}\frac{2\sqrt{x^2 - 1}}{2 - x^2}.$$

Let $2\sec^{-1}x = n.$

Then $$x = \sec \tfrac{1}{2}n.$$
$$\frac{1}{x} = \cos \tfrac{1}{2}n.$$

By [18],
$$2\left(\frac{1}{x}\right)^2 - 1 = \cos n.$$
$$\frac{2 - x^2}{x^2} = \cos n.$$
$$\frac{x^2}{2 - x^2} = \sec n.$$

By Prob. 2, Ex. V,
$$\left(\frac{x^2}{2 - x^2}\right)^2 - 1 = \tan^2 n.$$
$$\frac{4x^2 - 4}{(2 - x^2)^2} = \tan^2 n.$$
$$\tan n = \frac{2\sqrt{x^2 - 1}}{2 - x^2}.$$
$$n = \tan^{-1}\frac{2\sqrt{x^2 - 1}}{2 - x^2}.$$
$$\therefore 2\sec^{-1}x = \tan^{-1}\frac{2\sqrt{x^2 - 1}}{2 - x^2}.$$

71. Prove that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = 45°$.

By [6],
$$\tan(\tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{3}) = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}} = 1.$$
$$\therefore \tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{3} = \tan^{-1}1 = 45°.$$

72. Prove that $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{4}{7}$.

By [6],
$$\tan(\tan^{-1}\tfrac{1}{3} + \tan^{-1}\tfrac{1}{5}) = \frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{3}\times\frac{1}{5}} = \tfrac{4}{7}.$$
$$\therefore \tan^{-1}\tfrac{1}{3} + \tan^{-1}\tfrac{1}{5} = \tan^{-1}\tfrac{4}{7}.$$

73. Prove that $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{12}{13} = \sin^{-1}\frac{63}{65}$.

By [4],
$$\begin{aligned}&\sin(\sin^{-1}\tfrac{3}{5} + \sin^{-1}\tfrac{12}{13})\\ &= \sin(\sin^{-1}\tfrac{3}{5})\cos(\sin^{-1}\tfrac{12}{13}) + \cos(\sin^{-1}\tfrac{3}{5})\sin(\sin^{-1}\tfrac{12}{13})\\ &= \tfrac{3}{5}\times\tfrac{5}{13} + \tfrac{4}{5}\times\tfrac{12}{13} = \tfrac{63}{65}.\end{aligned}$$
$$\therefore \sin^{-1}\tfrac{3}{5} + \sin^{-1}\tfrac{12}{13} = \sin^{-1}\tfrac{63}{65}.$$

74. Prove that $\sin^{-1}\dfrac{1}{\sqrt{82}} + \sin^{-1}\dfrac{4}{\sqrt{41}} = 45°$.

By [4],
$$\sin\left(\sin^{-1}\frac{1}{\sqrt{82}} + \sin^{-1}\frac{4}{\sqrt{41}}\right) = \frac{1}{\sqrt{82}}\times\frac{5}{\sqrt{41}} + \frac{9}{\sqrt{82}}\times\frac{4}{\sqrt{41}}$$
$$= \frac{5}{41\sqrt{2}} + \frac{36}{41\sqrt{2}} = \frac{1}{\sqrt{2}} = \tfrac{1}{2}\sqrt{2}.$$
$$\therefore \sin^{-1}\frac{1}{\sqrt{82}} + \sin^{-1}\frac{4}{\sqrt{41}} = \sin^{-1}\tfrac{1}{2}\sqrt{2} = 45°.$$

75. Prove that
$$\sec^{-1}\tfrac{5}{3} + \sec^{-1}\tfrac{13}{12} = 75°\ 45'.$$
$$\sec^{-1}\tfrac{5}{3} + \sec^{-1}\tfrac{13}{12} = \cos^{-1}\tfrac{3}{5} + \cos^{-1}\tfrac{12}{13}.$$

By [5],
$$\cos(\cos^{-1}\tfrac{3}{5} + \cos^{-1}\tfrac{12}{13}) = \tfrac{3}{5}\times\tfrac{12}{13} - \tfrac{4}{5}\times\tfrac{5}{13} = \tfrac{16}{65}.$$
$$\therefore \cos^{-1}\tfrac{3}{5} + \cos^{-1}\tfrac{12}{13} = \cos^{-1}\tfrac{16}{65}.$$
$$\sec^{-1}\tfrac{5}{3} + \sec^{-1}\tfrac{13}{12} = \sec^{-1}\tfrac{65}{16} = 75°\ 45'.$$

76. Prove that $\tan^{-1}(2+\sqrt{3}) - \tan^{-1}(2-\sqrt{3}) = \sec^{-1}2$.

Let $\tan^{-1}(2+\sqrt{3}) - \tan^{-1}(2-\sqrt{3}) = n$.

By [10],
$$\tan n = \frac{(2+\sqrt{3})-(2-\sqrt{3})}{1+(2+\sqrt{3})(2-\sqrt{3})}$$
$$= \frac{2\sqrt{3}}{2} = \sqrt{3}.$$
$$\therefore n = 60°.$$
$$\sec n = 2.$$
$$\therefore \tan^{-1}(2+\sqrt{3}) - \tan^{-1}(2-\sqrt{3}) = \sec^{-1}2.$$

77. Prove that $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = 45°$.

Let $$\tan^{-1}\tfrac{1}{3} + \tan^{-1}\tfrac{1}{5} = n,$$
and $$\tan^{-1}\tfrac{1}{7} + \tan^{-1}\tfrac{1}{8} = v.$$

By [6], $$\tan n = \frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{3}\times\frac{1}{5}} = \tfrac{4}{7}.$$
$$\tan v = \frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\frac{1}{8}} = \tfrac{3}{11}.$$
$$\tan(n+v) = \frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{4}{7}\times\frac{3}{11}} = 1.$$
$$n + v = \tan^{-1}1 = 45°.$$
$$\tan^{-1}\tfrac{1}{3} + \tan^{-1}\tfrac{1}{5} + \tan^{-1}\tfrac{1}{7} + \tan^{-1}\tfrac{1}{8} = 45°.$$

78. Prove that $\tan^{-1}\frac{1}{1-2x+4x^2} + \tan^{-1}\frac{1}{1+2x+4x^2} = \tan^{-1}\frac{1}{2x^2}$.

By [6], $$\tan\left(\tan^{-1}\frac{1}{1-2x+4x^2} + \tan^{-1}\frac{1}{1+2x+4x^2}\right)$$
$$= \frac{\frac{1}{1-2x+4x^2}+\frac{1}{1+2x+4x^2}}{1-\frac{1}{(1-2x+4x^2)(1+2x+4x^2)}}$$
$$= \frac{1+2x+4x^2+1-2x+4x^2}{(1-2x+4x^2)(1+2x+4x^2)-1}$$
$$= \frac{2+8x^2}{4x^2+16x^4}$$
$$= \frac{1}{2x^2}.$$
$$\therefore \tan^{-1}\frac{1}{1-2x+4x^2} + \tan^{-1}\frac{1}{1+2x+4x^2} = \tan^{-1}\frac{1}{2x^2}.$$

79. Given $\cos x = \frac{3}{5}$; find $\sin\frac{1}{2}x$ and $\cos\frac{1}{2}x$.

By [16], $$\sin\tfrac{1}{2}x = \pm\sqrt{\frac{1-\frac{3}{5}}{2}} = \pm\sqrt{\tfrac{1}{5}} = \pm\tfrac{1}{5}\sqrt{5}.$$

By [17], $$\cos\tfrac{1}{2}x = \pm\sqrt{\frac{1+\frac{3}{5}}{2}} = \pm\sqrt{\tfrac{4}{5}} = \pm\tfrac{2}{5}\sqrt{5}.$$

80. Given $\tan x = \frac{1}{2}$; find $\tan\frac{1}{2}x$.

By [14], $$\tan x = \frac{2\tan\frac{1}{2}x}{1-\tan^2\frac{1}{2}x}.$$
$$\tfrac{1}{2} = \frac{2\tan\frac{1}{2}x}{1-\tan^2\frac{1}{2}x}.$$

$$
\begin{aligned}
1 - \tan^2 \tfrac{1}{2}x &= 4 \tan \tfrac{1}{2}x. \\
\tan^2 \tfrac{1}{2}x + 4 \tan \tfrac{1}{2}x &= 1. \\
\tan^2 \tfrac{1}{2}x + 4 \tan \tfrac{1}{2}x + 4 &= 5. \\
\tan \tfrac{1}{2}x + 2 &= \pm \sqrt{5}. \\
\therefore \tan \tfrac{1}{2}x &= \pm \sqrt{5} - 2.
\end{aligned}
$$

81. Given $\sin x + \cos x = \sqrt{\tfrac{1}{2}}$; find $\cos 2x$.

$$
\begin{aligned}
\sin x + \cos x &= \sqrt{\tfrac{1}{2}}. \\
\sin^2 x + 2 \sin x \cos x + \cos^2 x &= \tfrac{1}{2}. \\
1 + 2 \sin x \cos x &= \tfrac{1}{2}. \\
2 \sin x \cos x &= -\tfrac{1}{2}. \\
\text{By [12],} \qquad \sin 2x &= -\tfrac{1}{2}. \\
\cos 2x &= \pm \sqrt{1 - (-\tfrac{1}{2})^2} \\
&= \pm \sqrt{\tfrac{3}{4}} = \pm \tfrac{1}{2}\sqrt{3}.
\end{aligned}
$$

82. Given $\tan 2x = \tfrac{24}{7}$; find $\sin x$.

$$
\begin{aligned}
\tan 2x &= \tfrac{24}{7}. \\
\text{By Prob. 19, Ex. XIV,} \qquad \sec^2 2x &= 1 + \tan^2 2x \\
&= \tfrac{625}{49}. \\
\cos^2 2x &= \tfrac{49}{625}. \\
\cos 2x &= \pm \tfrac{7}{25}. \\
\text{By [13],} \qquad 1 - 2\sin^2 x &= \pm \tfrac{7}{25}. \\
\sin^2 x &= \tfrac{9}{25} \text{ or } \tfrac{16}{25}. \\
\therefore \sin x &= \pm \tfrac{3}{5} \text{ or } \pm \tfrac{4}{5}.
\end{aligned}
$$

83. Given $\cos 3x = \tfrac{23}{27}$; find $\tan x$.

$$
\begin{aligned}
\text{By Prob. 19, Ex. XIV,} \qquad \cos 3x &= 4\cos^3 x - 3\cos x. \\
4\cos^3 x - 3\cos x &= \tfrac{23}{27}. \\
108\cos^3 x - 81\cos x &= 23. \\
108\cos^3 x - 81\cos x - 23 &= 0. \\
(3\cos x + 1)(36\cos^2 x - 12\cos x - 23) &= 0.
\end{aligned}
$$

(i)
$$
\begin{aligned}
3\cos x + 1 &= 0. \\
\cos x &= -\tfrac{1}{3}.
\end{aligned}
$$

(ii)
$$
\begin{aligned}
36\cos^2 x - 12\cos x - 23 &= 0. \\
36\cos^2 x - 12\cos x + 1 &= 24. \\
6\cos x - 1 &= \pm 2\sqrt{6}. \\
\cos x &= \tfrac{1}{6}(1 \pm 2\sqrt{6}).
\end{aligned}
$$

$$
\begin{aligned}
\sin x &= \pm\sqrt{1 - \cos^2 x} \\
&= \pm\sqrt{1 - (-\tfrac{1}{3})^2} \text{ or } \pm\sqrt{1 - [\tfrac{1}{6}(1 \pm 2\sqrt{6})]^2} \\
&= \pm\sqrt{\tfrac{8}{9}} \text{ or } \pm\sqrt{\tfrac{1}{36}(11 \mp 4\sqrt{6})} \\
&= \pm\tfrac{2}{3}\sqrt{2} \text{ or } \pm\tfrac{1}{6}(\sqrt{3} \mp 2\sqrt{2}).
\end{aligned}
$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$= \frac{\pm \frac{2}{3}\sqrt{2}}{-\frac{1}{3}} \text{ or } \frac{\pm \frac{1}{6}(\sqrt{3} \mp 2\sqrt{2})}{\frac{1}{6}(1 \pm 2\sqrt{6})}$$

$$= \mp 2\sqrt{2} \text{ or } \frac{\pm(\sqrt{3} \mp 2\sqrt{2})}{1 \pm 2\sqrt{6}}$$

$$= \mp 2\sqrt{2}, \text{ or } \frac{\pm(3 - 2\sqrt{2})(1 - 2\sqrt{6})}{(1 + 2\sqrt{6})(1 - 2\sqrt{6})}, \text{ or } \frac{\pm(\sqrt{3} + 2\sqrt{2})(1 + 2\sqrt{6})}{(1 - 2\sqrt{6})(1 + 2\sqrt{6})}$$

$$= \mp 2\sqrt{2}, \text{ or } \frac{\pm(9\sqrt{3} - 8\sqrt{2})}{-23}, \text{ or } \frac{\pm(9\sqrt{3} + 8\sqrt{2})}{-23}.$$

$$\therefore \tan x = \pm 2\sqrt{2},\ \pm \tfrac{1}{23}(9\sqrt{3} + 8\sqrt{2}), \text{ or } \pm \tfrac{1}{23}(9\sqrt{3} - 8\sqrt{2}).$$

84. Given $2 \csc x - \cot x = \sqrt{3}$; find $\sin \frac{1}{2} x$.

$$2 \csc x - \cot x = \sqrt{3}.$$

$$\frac{2}{\sin x} - \frac{\cos x}{\sin x} = \sqrt{3}.$$

$$2 - \cos x = \sqrt{3} \sin x.$$

$$4 - 4 \cos x + \cos^2 x = 3 \sin^2 x$$

$$= 3 - 3 \cos^2 x.$$

$$4 \cos^2 x - 4 \cos x + 1 = 0.$$

$$2 \cos x - 1 = 0.$$

$$\cos x = \tfrac{1}{2}.$$

By [16], $\sin \frac{1}{2} x = \pm \sqrt{\dfrac{1 - \frac{1}{2}}{2}}.$

$$\therefore \sin \tfrac{1}{2} x = \pm \tfrac{1}{2}.$$

85. Find sin 18° and cos 36°.

(i) $54° = 90° - 36°.$

$$3 \times 18° = 90° - 2 \times 18°.$$

$$\cos(3 \times 18°) = \sin(2 \times 18°).$$

By Prob. 19, Ex. XIV, and [12],

$$4 \cos^3 18° - 3 \cos 18°$$

$$= 2 \sin 18° \cos 18°.$$

$$4 \cos^2 18° - 3 = 2 \sin 18°.$$

$$4 - 4 \sin^2 18° - 3 = 2 \sin 18°.$$

$$4 \sin^2 18° + 2 \sin 18° = 1.$$

$$16 \sin^2 18° + (\ ) + 1 = 5.$$

$$4 \sin 18° + 1 = \pm \sqrt{5}.$$

$$4 \sin 18° = \pm \sqrt{5} - 1.$$

$$\therefore \sin 18° = \frac{\sqrt{5} - 1}{4}.$$

(ii) $\cos 36° = 1 - 2 \sin^2 18°$

$$= 1 - 2\left(\frac{\sqrt{5} - 1}{4}\right)^2$$

$$= \frac{\sqrt{5} + 1}{4}.$$

86. Find the value of $a \sec x + b \csc x$, when $\tan x = \sqrt[3]{\dfrac{b}{a}}.$

$$\tan x = \frac{b^{\frac{1}{3}}}{a^{\frac{1}{3}}}.$$

By Prob. 2, Ex. V,

$$\sec^2 x = 1 + \frac{b^{\frac{2}{3}}}{a^{\frac{2}{3}}}$$

$$= \frac{a^{\frac{2}{3}} + b^{\frac{2}{3}}}{a^{\frac{2}{3}}}.$$

$$\cot x = \frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}.$$

By Prob. 3, Ex. V,

$$\csc^2 x = \frac{a^{\frac{2}{3}} + b^{\frac{2}{3}}}{b^{\frac{2}{3}}}.$$

$$\therefore \sec x = \frac{(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}}}{a^{\frac{1}{3}}}.$$

$$\csc x = \frac{(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}}}{b^{\frac{1}{3}}}.$$

$$\begin{aligned} a\sec x + b\csc x &= a^{\frac{2}{3}}(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{1}{2}} + b^{\frac{2}{3}}(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{1}{2}} \\ &= (a^{\frac{2}{3}}+b^{\frac{2}{3}})(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{1}{2}} \\ &= (a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}}. \end{aligned}$$

87. Find the value of $\sin 3x$, when $\sin 2x = \sqrt{1-m^2}$.

$$\begin{aligned} \sin 2x &= \sqrt{1-m^2}. \\ \cos^2 2x &= m^2. \\ \cos 2x &= \pm m. \\ 1 - 2\sin^2 x &= \pm m. \\ 2\sin^2 x &= 1 \pm m. \\ \sin x &= \sqrt{\frac{1 \pm m}{2}}. \end{aligned}$$

By Prob. 18, Ex. XIV,

$$\begin{aligned} \sin 3x &= 3\sin x - 4\sin^3 x \\ &= 3\left(\frac{1 \pm m}{2}\right)^{\frac{1}{2}} - 4\left(\frac{1 \pm m}{2}\right)^{\frac{3}{2}} \\ &= \left(\frac{1 \pm m}{2}\right)^{\frac{1}{2}}\left(3 - 4\frac{\{1 \pm m\}}{2}\right) \\ &= \left(\frac{1 \pm m}{2}\right)^{\frac{1}{2}}(1 \mp 2m). \end{aligned}$$

88. Find the value of $\sin x$, when $\tan^2 x + 3\cot^2 x = 4$.

$$\begin{aligned} \tan^2 x + 3\cot^2 x &= 4. \\ \tan^2 x + \frac{3}{\tan^2 x} &= 4. \\ \tan^4 x - 4\tan^2 x + 3 &= 0. \\ (\tan^2 x - 1)(\tan^2 x - 3) &= 0. \\ \therefore \tan^2 x &= 1 \text{ or } 3. \\ \cot^2 x &= 1 \text{ or } \tfrac{1}{3}. \\ \csc^2 x &= 2 \text{ or } \tfrac{4}{3}. \\ \sin^2 x &= \tfrac{1}{2} \text{ or } \tfrac{3}{4}. \\ \therefore \sin x &= \pm \tfrac{1}{2}\sqrt{2} \\ &\text{ or } \pm \tfrac{1}{2}\sqrt{3}. \end{aligned}$$

89. Find the value of $\dfrac{\csc^2 x - \sec^2 x}{\csc^2 x + \sec^2 x}$, when $\tan x = \sqrt{\tfrac{1}{7}}$.

$$\begin{aligned} \tan x &= \sqrt{\tfrac{1}{7}}. \\ \sec^2 x &= 1 + \tfrac{1}{7} = \tfrac{8}{7}. \\ \cot x &= \sqrt{7}. \\ \csc^2 x &= 1 + 7 = 8. \\ \therefore \frac{\csc^2 x - \sec^2 x}{\csc^2 x + \sec^2 x} &= \frac{8 - \frac{8}{7}}{8 + \frac{8}{7}} = \tfrac{3}{4}. \end{aligned}$$

90. Find the value of $\cos x$, when $5\tan x + \sec x = 5$.

$$\begin{aligned} 5\tan x + \sec x &= 5. \\ \frac{5\sin x}{\cos x} + \frac{1}{\cos x} &= 5. \\ 5\sin x + 1 &= 5\cos x. \\ 5\sin x &= 5\cos x - 1. \\ 25(1 - \cos^2 x) &= 25\cos^2 x - 10\cos x + 1. \\ 50\cos^2 x - 10\cos x - 24 &= 0. \\ 25\cos^2 x - 5\cos x - 12 &= 0. \\ (5\cos x - 4)(5\cos x + 3) &= 0. \\ 5\cos x &= 4 \text{ or } -3. \\ \therefore \cos x &= \tfrac{4}{5} \text{ or } -\tfrac{3}{5}. \end{aligned}$$

91. Find the value of $\sec x$, when $\tan x = \dfrac{a}{\sqrt{2a+1}}$.

$$\begin{aligned} \tan x &= \frac{a}{\sqrt{2a+1}}. \\ \sec^2 x &= 1 + \frac{a^2}{2a+1} \\ &= \frac{a^2 + 2a + 1}{2a+1}. \\ \sec x &= \pm\frac{a+1}{\sqrt{2a+1}}. \end{aligned}$$

92. Simplify $\frac{(\cos x + \cos y)^2 + (\sin x + \sin y)^2}{\cos^2 \frac{1}{2}(x - y)}$.

By [22] and [20],

$$\begin{aligned}
&\frac{(\cos x + \cos y)^2 + (\sin x + \sin y)^2}{\cos^2 \frac{1}{2}(x - y)}\\
&\quad = \frac{[2 \cos \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)]^2 + [2 \sin \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)]^2}{\cos^2 \frac{1}{2}(x - y)}\\
&\quad = 4 \cos^2 \tfrac{1}{2}(x + y) + 4 \sin^2 \tfrac{1}{2}(x + y)\\
&\quad = 4.
\end{aligned}$$

93. Simplify $\frac{\sin (x + 2y) - 2 \sin (x + y) + \sin x}{\cos (x + 2y) - 2 \cos (x + y) + \cos x}$.

$$\begin{aligned}
&\frac{\sin (x + 2y) - 2 \sin (x + y) + \sin x}{\cos (x + 2y) - 2 \cos (x + y) + \cos x}\\
&\qquad = \frac{[\sin (x + 2y) + \sin x] - 2 \sin (x + y)}{[\cos (x + 2y) + \cos x] - 2 \cos (x + y)}
\end{aligned}$$

By [20] and [22],
$$\begin{aligned}
&= \frac{2 \sin (x + y) \cos y - 2 \sin (x + y)}{2 \cos (x + y) \cos y - 2 \cos (x + y)}\\
&= \frac{\sin (x + y)(\cos y - 1)}{\cos (x + y)(\cos y - 1)}\\
&= \frac{\sin (x + y)}{\cos (x + y)}\\
&= \tan (x + y).
\end{aligned}$$

94. Simplify $\frac{\sin (x - z) + 2 \sin x + \sin (x + z)}{\sin (y - z) + 2 \sin y + \sin (y + z)}$.

$$\begin{aligned}
&\frac{\sin (x - z) + 2 \sin x + \sin (x + z)}{\sin (y - z) + 2 \sin y + \sin (y + z)}\\
&\qquad = \frac{[\sin (x + z) + \sin (x - z)] + 2 \sin x}{[\sin (y + z) + \sin (y - z)] + 2 \sin y}
\end{aligned}$$

By [20],
$$\begin{aligned}
&= \frac{2 \sin x \cos z + 2 \sin x}{2 \sin y \cos z + 2 \sin y}\\
&= \frac{\sin x (\cos z + 1)}{\sin y (\cos z + 1)}\\
&= \frac{\sin x}{\sin y}.
\end{aligned}$$

95. Simplify $\frac{\cos 6x - \cos 4x}{\sin 6x + \sin 4x}$.

By [23] and [20],

$$\frac{\cos 6x - \cos 4x}{\sin 6x + \sin 4x} = \frac{-2 \sin 5x \sin x}{2 \sin 5x \cos x} = \frac{-\sin x}{\cos x} = -\tan x.$$

96. Simplify $\tan^{-1}(2x+1)+\tan^{-1}(2x-1)$.

$$\tan[\tan^{-1}(2x+1)+\tan^{-1}(2x-1)]$$

By [6],
$$= \frac{2x+1+2x-1}{1-(2x+1)(2x-1)}$$
$$= \frac{4x}{2-4x^2}$$
$$= \frac{2x}{1-2x^2}.$$

$$\therefore \tan^{-1}(2x+1)+\tan^{-1}(2x-1)=\tan^{-1}\frac{2x}{1-2x^2}.$$

97. Simplify $\dfrac{1}{1+\sin^2 x}+\dfrac{1}{1+\cos^2 x}+\dfrac{1}{1+\sec^2 x}+\dfrac{1}{1+\csc^2 x}$.

$$\frac{1}{1+\sin^2 x}+\frac{1}{1+\cos^2 x}+\frac{1}{1+\sec^2 x}+\frac{1}{1+\csc^2 x}$$
$$= \left(\frac{1}{1+\sin^2 x}+\frac{1}{1+\csc^2 x}\right)+\left(\frac{1}{1+\cos^2 x}+\frac{1}{1+\sec^2 x}\right)$$
$$= \left(\frac{1}{1+\sin^2 x}+\frac{\sin^2 x}{1+\sin^2 x}\right)+\left(\frac{1}{1+\cos^2 x}+\frac{\cos^2 x}{1+\cos^2 x}\right)$$
$$= \frac{1+\sin^2 x}{1+\sin^2 x}+\frac{1+\cos^2 x}{1+\cos^2 x}$$
$$= 1+1$$
$$= 2.$$

98. Simplify $2\sec^2 x-\sec^4 x-2\csc^2 x+\csc^4 x$.

$$2\sec^2 x-\sec^4 x-2\csc^2 x+\csc^4 x$$
$$= 1-2\csc^2 x+\csc^4 x-1+2\sec^2 x-\sec^4 x$$
$$= (\csc^2 x-1)^2-(\sec^2 x-1)^2$$
$$= \cot^2 x-\tan^2 x.$$

99. Solve $\sin x = 2\sin(\frac{1}{3}\pi + x)$.

$$\sin x = 2\sin(\tfrac{1}{3}\pi + x)$$

By [4],
$$= 2\sin\tfrac{1}{3}\pi\cos x + 2\cos\tfrac{1}{3}\pi\sin x$$
$$= \sqrt{3}\cos x + \sin x.$$
$$\sqrt{3}\cos x = 0.$$
$$\cos x = 0.$$
$$\therefore x = \tfrac{1}{2}\pi \text{ or } \tfrac{3}{2}\pi.$$

100. Solve $\sin 2x = 2\cos x$.

$$\sin 2x = 2\cos x.$$
$$2\sin x\cos x = 2\cos x.$$
$$2\cos x(\sin x - 1) = 0.$$

(i)
$$2\cos x = 0.$$
$$\cos x = 0.$$
$$\therefore x = 90^\circ \text{ or } 270^\circ.$$

(ii)
$$\sin x - 1 = 0.$$
$$\sin x = 1.$$
$$\therefore x = 90^\circ.$$
$$\therefore x = 90^\circ \text{ or } 270^\circ.$$

101. Solve $\cos 2x = 2\sin x$.

$$\cos 2x = 2\sin x.$$
$$1 - 2\sin^2 x = 2\sin x.$$
$$2\sin^2 x + 2\sin x = 1.$$
$$4\sin^2 x + 4\sin x + 1 = 3.$$
$$2\sin x + 1 = \pm\sqrt{3}.$$
$$2\sin x = -1 \pm \sqrt{3}.$$
$$\sin x = \frac{-1 \pm \sqrt{3}}{2}$$
$$= \frac{-1 \pm 1.7320}{2}$$
$$= 0.3660 \text{ or } -1.3660.$$
$$\therefore x = 21^\circ\, 28' \text{ or } 158^\circ\, 32'.$$

102. Solve $\sin x + \cos x = 1$.

$$\sin x + \cos x = 1.$$
$$\sin^2 x + 2\sin x \cos x + \cos^2 x = 1.$$
$$2\sin x \cos x = 0.$$

(i)
$$\sin x = 0.$$
$$\therefore x = 0^\circ \text{ or } 180^\circ.$$

(ii)
$$\cos x = 0.$$
$$\therefore x = 90^\circ \text{ or } 270^\circ.$$
$$\therefore x = 0^\circ, 90^\circ, 180^\circ, \text{ or } 270^\circ.$$

But the values $x = 180^\circ$ and $x = 270^\circ$ do not satisfy the given equation.
$$\therefore x = 0^\circ \text{ or } 90^\circ.$$

103. Solve $\sin x + \cos 2x = 4\sin^2 x$.

$$\sin x + \cos 2x = 4\sin^2 x.$$
$$\sin x + 1 - 2\sin^2 x = 4\sin^2 x.$$
$$6\sin^2 x - \sin x - 1 = 0.$$
$$(2\sin x - 1)(3\sin x + 1) = 0.$$

(i)
$$\begin{aligned} 2\sin x - 1 &= 0. \\ 2\sin x &= 1. \\ \sin x &= \tfrac{1}{2}. \\ \therefore x &= 30^\circ \text{ or } 150^\circ. \end{aligned}$$

(ii)
$$\begin{aligned} 3\sin x + 1 &= 0. \\ 3\sin x &= -1. \\ \sin x &= -\tfrac{1}{3}. \\ \therefore x &= 199^\circ\, 28' \text{ or } 340^\circ\, 32'. \\ \therefore x &= 30^\circ,\ 150^\circ,\ 199^\circ\, 28',\ \text{or } 340^\circ\, 32'. \end{aligned}$$

104. Solve $4\cos 2x + 3\cos x = 1$.

$$\begin{aligned} 4\cos 2x + 3\cos x &= 1. \\ 8\cos^2 x - 4 + 3\cos x &= 1. \\ 8\cos^2 x + 3\cos x - 5 &= 0. \\ (\cos x + 1)(8\cos x - 5) &= 0. \end{aligned}$$

(i)
$$\begin{aligned} \cos x + 1 &= 0. \\ \cos x &= -1. \\ \therefore x &= 180^\circ. \end{aligned}$$

(ii)
$$\begin{aligned} 8\cos x - 5 &= 0. \\ 8\cos x &= 5. \\ \cos x &= \tfrac{5}{8} = 0.6250. \\ \therefore x &= 51^\circ\, 19' \text{ or } 308^\circ\, 41'. \\ \therefore x &= 51^\circ\, 19',\ 180^\circ,\ \text{or } 308^\circ\, 41'. \end{aligned}$$

105. Solve $\sin x + \sin 2x = \sin 3x$.

$$\sin x + \sin 2x = \sin 3x.$$

By [12] and Prob. 18, Ex. XIV,

$$\begin{aligned} \sin x + 2\sin x\cos x &= 3\sin x - 4\sin^3 x. \\ 4\sin^3 x - 2\sin x + 2\sin x\cos x &= 0. \\ \sin x(2\sin^2 x - 1 + \cos x) &= 0. \\ \sin x(1 - 2\cos^2 x + \cos x) &= 0. \\ \sin x(1 - \cos x)(1 + 2\cos x) &= 0. \end{aligned}$$

(i)
$$\begin{aligned} \sin x &= 0. \\ \therefore x &= 0^\circ \text{ or } 180^\circ. \end{aligned}$$

(ii)
$$\begin{aligned} 1 - \cos x &= 0. \\ \cos x &= 1. \\ \therefore x &= 0^\circ. \end{aligned}$$

(iii)
$$\begin{aligned} 1 + 2\cos x &= 0. \\ 2\cos x &= -1. \\ \cos x &= -\tfrac{1}{2}. \\ \therefore x &= 120^\circ \text{ or } 240^\circ. \\ \therefore x &= 0^\circ,\ 120^\circ,\ 180^\circ,\ \text{or } 240^\circ. \end{aligned}$$

106. Solve $\sin 2x = 3\sin^2 x - \cos^2 x$.

$$\sin 2x = 3\sin^2 x - \cos^2 x.$$

By [12], $2\sin x\cos x = 3\sin^2 x - \cos^2 x.$

$$3\sin^2 x - 2\sin x\cos x - \cos^2 x = 0.$$
$$(3\sin x + \cos x)(\sin x - \cos x) = 0.$$

(i) $3\sin x + \cos x = 0.$

$$3\tan x + 1 = 0.$$
$$\tan x = -\tfrac{1}{3}.$$
$$\therefore x = \tan^{-1}(-\tfrac{1}{3}) = 161^\circ\ 34' \text{ or } 341^\circ\ 34'.$$

(ii) $\sin x - \cos x = 0.$

$$\sin x = \cos x.$$
$$\therefore x = 45^\circ \text{ or } 225^\circ.$$
$$\therefore x = 45^\circ,\ 161^\circ\ 34',\ 225^\circ, \text{ or } 341^\circ\ 34'.$$

107. Solve $\cot\theta = \frac{1}{3}\tan\theta$.

$$\cot\theta = \tfrac{1}{3}\tan\theta.$$
$$\frac{1}{\tan\theta} = \frac{\tan\theta}{3}.$$
$$\tan^2\theta = 3.$$
$$\tan\theta = \pm\sqrt{3}.$$
$$\therefore \theta = 60^\circ,\ 120^\circ,\ 240^\circ, \text{ or } 300^\circ.$$

108. Solve $2\sin\theta = \cos\theta$.

$$2\sin\theta = \cos\theta.$$
$$4\sin^2\theta = \cos^2\theta.$$
$$4\sin^2\theta = 1 - \sin^2\theta.$$
$$5\sin^2\theta = 1.$$
$$\sin^2\theta = \tfrac{1}{5}.$$
$$\sin\theta = \pm\tfrac{1}{5}\sqrt{5} = \pm 0.4472.$$
$$\therefore \theta = 26^\circ\ 34',\ 153^\circ\ 26',\ 206^\circ\ 34', \text{ or } 333^\circ\ 26'.$$

But the values $\theta = 153^\circ\ 26'$ and $\theta = 333^\circ\ 26'$ do not satisfy the given equation.

$$\therefore \theta = 26^\circ\ 34' \text{ or } 206^\circ\ 34'.$$

109. Solve $2\sin^2 x + 5\sin x = 3$.

$$2\sin^2 x + 5\sin x = 3.$$
$$2\sin^2 x + 5\sin x - 3 = 0.$$
$$(\sin x + 3)(2\sin x - 1) = 0.$$

(i) $\sin x + 3 = 0.$

$$\sin x = -3.$$
$$\therefore x \text{ is impossible.}$$

(ii)
$$2\sin x - 1 = 0.$$
$$2\sin x = 1.$$
$$\sin x = \tfrac{1}{2}.$$
$$\therefore x = 30^\circ \text{ or } 150^\circ.$$

110. Solve $\tan x \sec x = \sqrt{2}$.

$$\tan x \sec x = \sqrt{2}.$$
$$\tan^2 x \sec^2 x = 2.$$

By Prob. 2, Ex. V,

$$\tan^2 x\,(1 + \tan^2 x) = 2.$$
$$\tan^2 x + \tan^4 x = 2.$$
$$\tan^4 x + \tan^2 x - 2 = 0.$$
$$(\tan^2 x - 1)(\tan^2 x + 2) = 0.$$

(i)
$$\tan^2 x - 1 = 0.$$
$$\tan^2 x = 1.$$
$$\tan x = \pm 1.$$
$$\therefore x = 45^\circ,\ 135^\circ,\ 225^\circ, \text{ or } 315^\circ.$$

(ii)
$$\tan^2 x + 2 = 0.$$
$$\tan^2 x = -2.$$
$$\tan x = \pm\sqrt{-2}.$$
$$\therefore x \text{ is impossible.}$$
$$\therefore x = 45^\circ,\ 135^\circ,\ 225^\circ, \text{ or } 315^\circ.$$

But the values $x = 225^\circ$ and $x = 315^\circ$ do not satisfy the given equation.

$$\therefore x = 45^\circ \text{ or } 135^\circ.$$

111. Solve $\sin x = \cos 2x$.

$$\sin x = \cos 2x.$$

By [13],
$$\sin x = \cos^2 x - \sin^2 x.$$
$$\sin x = 1 - \sin^2 x - \sin^2 x.$$
$$2\sin^2 x + \sin x - 1 = 0.$$
$$(\sin x + 1)(2\sin x - 1) = 0.$$

(i)
$$\sin x + 1 = 0.$$
$$\sin x = -1.$$
$$\therefore x = 270^\circ.$$

(ii)
$$2\sin x - 1 = 0.$$
$$2\sin x = 1.$$
$$\sin x = \tfrac{1}{2}.$$
$$\therefore x = 30^\circ \text{ or } 150^\circ.$$
$$\therefore x = 30^\circ,\ 150^\circ, \text{ or } 270^\circ.$$

112. Solve $\tan x \tan 2x = 2$.

$$\tan x \tan 2x = 2.$$

By [14],
$$\tan x \frac{2 \tan x}{1 - \tan^2 x} = 2.$$
$$2 \tan^2 x = 2 - 2 \tan^2 x.$$
$$4 \tan^2 x = 2.$$
$$\tan^2 x = \tfrac{1}{2}.$$
$$\tan x = \pm \sqrt{\tfrac{1}{2}} = \pm \tfrac{1}{2} \sqrt{2} = \pm 0.7071.$$
$$\therefore x = 35° \, 16', \; 144° \, 44', \; 215° \, 16', \text{ or } 324° \, 44'.$$

113. Solve $\sec x = 4 \csc x$.

$$\sec x = 4 \csc x.$$
$$\frac{1}{\cos x} = \frac{4}{\sin x}.$$
$$\sin x = 4 \cos x.$$
$$\sin^2 x = 16 \cos^2 x.$$
$$1 - \cos^2 x = 16 \cos^2 x.$$
$$17 \cos^2 x = 1.$$
$$\cos x = \pm \tfrac{1}{17} \sqrt{17} = \pm 0.2425.$$
$$\therefore x = 75° \, 58', \; 104° \, 2', \; 255° \, 58', \text{ or } 284° \, 2'.$$

But the values $x = 104° \, 2'$ and $x = 284° \, 2'$ do not satisfy the given equation.

$$\therefore x = 75° \, 58' \text{ or } 255° \, 58'.$$

114. Solve $\cos \theta + \cos 2\theta = 0$.

$$\cos \theta + \cos 2\theta = 0.$$

By [13],
$$\cos \theta + \cos^2 \theta - \sin^2 \theta = 0.$$
$$\cos \theta + \cos^2 \theta - 1 + \cos^2 \theta = 0.$$
$$2 \cos^2 \theta + \cos \theta - 1 = 0.$$
$$(\cos \theta + 1)(2 \cos \theta - 1) = 0.$$

(i)
$$\cos \theta + 1 = 0.$$
$$\cos \theta = -1.$$
$$\therefore \theta = 180°.$$

(ii)
$$2 \cos \theta - 1 = 0.$$
$$2 \cos \theta = 1.$$
$$\cos \theta = \tfrac{1}{2}.$$
$$\therefore \theta = 60° \text{ or } 300°.$$
$$\therefore \theta = 60°, \; 180°, \text{ or } 300°.$$

115. Solve $\cot \frac{1}{2}\theta + \csc \theta = 2$.

$$\cot \tfrac{1}{2}\theta + \csc \theta = 2.$$

By [19], $$\pm\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} + \frac{1}{\sin\theta} = 2.$$

$$\pm\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = 2 - \frac{1}{\sin\theta}.$$

$$\frac{1+\cos\theta}{1-\cos\theta} = 4 - \frac{4}{\sin\theta} + \frac{1}{\sin^2\theta}.$$

$$\frac{1+2\cos\theta+\cos^2\theta}{1-\cos^2\theta} = 4 - \frac{4}{\sin\theta} - \frac{1}{\sin^2\theta}.$$

$$\frac{1+2\cos\theta+\cos^2\theta}{\sin^2\theta} = 4 - \frac{4}{\sin\theta} - \frac{1}{\sin^2\theta}.$$

Extract square root, $$\frac{1+\cos\theta}{\sin\theta} = 2 - \frac{1}{\sin\theta}.$$

$$\begin{aligned}
1+\cos\theta &= 2\sin\theta - 1.\\
2+\cos\theta &= 2\sin\theta.\\
2+\cos\theta &= 2\sqrt{1-\cos^2\theta}.\\
4+4\cos\theta+\cos^2\theta &= 4 - 4\cos^2\theta.\\
5\cos^2\theta + 4\cos\theta &= 0.\\
\cos\theta(5\cos\theta+4) &= 0.\\
\therefore \cos\theta &= 0 \text{ or } -0.8.
\end{aligned}$$

Also $$\begin{aligned}
2+\sqrt{1-\sin^2\theta} &= 2\sin\theta.\\
1-\sin^2\theta &= 4\sin^2\theta - 8\sin\theta + 4.\\
5\sin^2\theta - 8\sin\theta + 3 &= 0.\\
(\sin\theta-1)(5\sin\theta-3) &= 0.\\
\therefore \sin\theta &= 1 \text{ or } 0.6.
\end{aligned}$$

But $$\begin{aligned}
\cos\theta &= 0 \text{ or } -0.8.\\
\therefore \theta &= 90^\circ \text{ or } 143^\circ\, 8'.
\end{aligned}$$

116. Solve $\cot x \tan 2x = 3$.

$$\cot x \tan 2x = 3.$$

By [14], $$\frac{1}{\tan x} \times \frac{2\tan x}{1-\tan^2 x} = 3.$$

$$\begin{aligned}
\frac{2}{1-\tan^2 x} &= 3.\\
2 &= 3 - 3\tan^2 x.\\
3\tan^2 x &= 1.\\
\tan^2 x &= \tfrac{1}{3}.\\
\tan x &= \pm\sqrt{\tfrac{1}{3}} = \pm\tfrac{1}{3}\sqrt{3}.\\
\therefore x &= 30^\circ, 150^\circ, 210^\circ, \text{ or } 330^\circ.
\end{aligned}$$

117. Solve $\sin x \sec 2x = 1$.

$$\sin x \sec 2x = 1.$$

$$\frac{\sin x}{\cos 2x} = 1.$$

$$\sin x = \cos 2x.$$

By [13],

$$\sin x = \cos^2 x - \sin^2 x.$$

$$\sin x = 1 - 2\sin^2 x.$$

$$2\sin^2 x + \sin x - 1 = 0.$$

$$(\sin x + 1)(2\sin x - 1) = 0.$$

(i)

$$\sin x + 1 = 0.$$

$$\sin x = -1.$$

$$\therefore x = 270^\circ.$$

(ii)

$$2\sin x - 1 = 0.$$

$$2\sin x = 1.$$

$$\sin x = \tfrac{1}{2}.$$

$$\therefore x = 30^\circ \text{ or } 150^\circ.$$

$$\therefore x = 30^\circ,\ 150^\circ, \text{ or } 270^\circ.$$

118. Solve $\sin^2 x + \sin 2x = 1$.

$$\sin^2 x + \sin 2x = 1.$$

By [12],

$$\sin^2 x + 2\sin x \cos x = 1.$$

$$\sin^2 x + 2\sin x\sqrt{1 - \sin^2 x} = 1.$$

$$2\sin x\sqrt{1 - \sin^2 x} = 1 - \sin^2 x.$$

$$4\sin^2 x - 4\sin^4 x = 1 - 2\sin^2 x + \sin^4 x.$$

$$5\sin^4 x - 6\sin^2 x + 1 = 0.$$

$$(\sin^2 x - 1)(5\sin^2 x - 1) = 0.$$

(i)

$$\sin^2 x - 1 = 0.$$

$$\sin^2 x = 1.$$

$$\sin x = \pm 1.$$

$$\therefore x = 90^\circ \text{ or } 270^\circ.$$

(ii)

$$5\sin^2 x - 1 = 0.$$

$$5\sin^2 x = 1.$$

$$\sin^2 x = \tfrac{1}{5}.$$

$$\sin x = \pm\sqrt{\tfrac{1}{5}} = \pm\tfrac{1}{5}\sqrt{5} = \pm 0.4472.$$

$$\therefore x = 26^\circ\,34',\ 153^\circ\,26',\ 206^\circ\,34', \text{ or } 333^\circ\,26'.$$

$$\therefore x = 26^\circ\,34',\ 90^\circ,\ 153^\circ\,26',\ 206^\circ\,34',\ 270^\circ, \text{ or } 333^\circ\,26'.$$

But the values $x = 153^\circ\,26'$ and $x = 333^\circ\,26'$ do not satisfy the given equation.

$$\therefore x = 26^\circ\,34',\ 90^\circ,\ 206^\circ\,34', \text{ or } 270^\circ.$$

119. Solve $\cos x \sin 2x \csc x = 1$.

$$\cos x \sin 2x \csc x = 1.$$

By [12], $$\cos x(2 \sin x \cos x) \times \frac{1}{\sin x} = 1.$$

$$2 \cos^2 x = 1.$$
$$\cos^2 x = \tfrac{1}{2}.$$
$$\cos x = \pm \sqrt{\tfrac{1}{2}} = \pm \tfrac{1}{2} \sqrt{2}.$$
$$\therefore x = 45^\circ,\ 135^\circ,\ 225^\circ, \text{ or } 315^\circ.$$

120. Solve $\cot x \tan 2x = \sec 2x$.

$$\cot x \tan 2x = \sec 2x.$$

By [14] and [13], $$\frac{1}{\tan x} \times \frac{2 \tan x}{1 - \tan^2 x} = \frac{1}{\cos^2 x - \sin^2 x}.$$

$$\frac{2}{1 - \tan^2 x} = \frac{1}{\cos^2 x - \sin^2 x}.$$
$$2 \cos^2 x - 2 \sin^2 x = 1 - \frac{\sin^2 x}{\cos^2 x}.$$
$$2 - 2 \sin^2 x - 2 \sin^2 x = 1 - \frac{\sin^2 x}{1 - \sin^2 x}.$$
$$1 - 4 \sin^2 x = - \frac{\sin^2 x}{1 - \sin^2 x}.$$
$$1 - 5 \sin^2 x + 4 \sin^4 x = - \sin^2 x.$$
$$1 - 4 \sin^2 x + 4 \sin^4 x = 0.$$

Extract root, $$1 - 2 \sin^2 x = 0.$$

$$2 \sin^2 x = 1.$$
$$\sin^2 x = \tfrac{1}{2}.$$
$$\sin x = \pm \sqrt{\tfrac{1}{2}} = \pm \tfrac{1}{2} \sqrt{2}.$$
$$\therefore x = 45^\circ,\ 135^\circ,\ 225^\circ, \text{ or } 315^\circ.$$

121. Solve $\sin 2x = \cos 4x$.

$$\sin 2x = \cos 4x.$$

By [13], $$\sin 2x = \cos^2 2x - \sin^2 2x.$$

$$\sin 2x = 1 - \sin^2 2x - \sin^2 2x.$$
$$2 \sin^2 2x + \sin 2x - 1 = 0.$$
$$(\sin 2x + 1)(2 \sin 2x - 1) = 0.$$

(i) $$\sin 2x + 1 = 0.$$

$$\sin 2x = -1.$$
$$\therefore 2x = 270^\circ \text{ or } 630^\circ.$$
$$\therefore x = 135^\circ \text{ or } 315^\circ.$$

(ii)
$$2 \sin 2x - 1 = 0.$$
$$2 \sin 2x = 1.$$
$$\sin 2x = \tfrac{1}{2}.$$
$$\therefore 2x = 30^\circ,\ 150^\circ,\ 390^\circ, \text{ or } 510^\circ.$$
$$\therefore x = 15^\circ,\ 75^\circ,\ 195^\circ, \text{ or } 255^\circ.$$
$$\therefore x = 15^\circ,\ 75^\circ,\ 135^\circ,\ 195^\circ, \text{ or } 255^\circ.$$

122. Solve $\sin 2z \cot z - \sin^2 z = \tfrac{1}{2}$.

$$\sin 2z \cot z - \sin^2 z = \tfrac{1}{2}.$$

By [12],

$$2 \sin z \cos z \times \frac{\cos z}{\sin z} - \sin^2 z = \tfrac{1}{2}.$$
$$2 \cos^2 z - \sin^2 z = \tfrac{1}{2}.$$
$$2 - 2 \sin^2 z - \sin^2 z = \tfrac{1}{2}.$$
$$3 \sin^2 z = \tfrac{3}{2}.$$
$$\sin^2 z = \tfrac{1}{2}.$$
$$\sin z = \pm \sqrt{\tfrac{1}{2}} = \pm \tfrac{1}{2}\sqrt{2}.$$
$$\therefore z = 45^\circ,\ 135^\circ,\ 225^\circ, \text{ or } 315^\circ.$$

123. Solve $\tan x + \tan 2x = \tan 3x$.

$$\tan x + \tan 2x = \tan 3x.$$

By [14], and Prob. 19, Ex. XXIV,

$$\tan x + \frac{2 \tan x}{1 - \tan^2 x} = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}.$$
$$\tan x\left(1 + \frac{2}{1 - \tan^2 x} - \frac{3 - \tan^2 x}{1 - 3 \tan^2 x}\right) = 0.$$

(i)
$$\tan x = 0.$$
$$\therefore x = 0^\circ \text{ or } 180^\circ.$$

(ii) $$1 + \frac{2}{1 - \tan^2 x} - \frac{3 - \tan^2 x}{1 - 3 \tan^2 x} = 0.$$
$$1 - 4 \tan^2 x + 3 \tan^4 x + 2 - 6 \tan^2 x - 3 + 4 \tan^2 x - \tan^4 x + = 0.$$
$$2 \tan^4 x - 6 \tan^2 x = 0.$$
$$\tan^2 x (\tan^2 x - 3) = 0.$$
$$\tan^2 x = 0 \text{ or } 3.$$
$$\tan x = 0.$$
$$\therefore x = 0^\circ \text{ or } 180^\circ.$$
$$\tan x = \pm \sqrt{3}.$$
$$\therefore x = 60^\circ,\ 120^\circ,\ 240^\circ, \text{ or } 300^\circ.$$
$$\therefore x = 0^\circ,\ 60^\circ,\ 120^\circ,\ 180^\circ,\ 240^\circ, \text{ or } 300^\circ.$$

124. Solve $\cot x - \tan x = \sin x + \cos x$.

$$\cot x - \tan x = \sin x + \cos x.$$

$$\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} = \sin x + \cos x.$$

$$\frac{\cos^2 x - \sin^2 x}{\sin x \cos x} = \sin x + \cos x.$$

$$\cos^2 x - \sin^2 x = \sin x \cos x (\sin x + \cos x).$$

$$(\sin x + \cos x)(\cos x - \sin x - \sin x \cos x) = 0.$$

(i) $$\sin x + \cos x = 0.$$

$$\sin x = -\cos x.$$

$$\therefore x = 135° \text{ or } 315°.$$

(ii) $$\cos x - \sin x - \sin x \cos x = 0.$$

$$\cos x - \sin x = \sin x \cos x.$$

Square, $$\cos^2 x + \sin^2 x - 2 \sin x \cos x = \sin^2 x \cos^2 x.$$

$$1 - 2 \sin x \cos x = \sin^2 x \cos^2 x.$$

$$\sin^2 x \cos^2 x + 2 \sin x \cos x = 1.$$

$$\sin^2 x \cos^2 x + 2 \sin x \cos x + 1 = 2.$$

Extract the root, $$\sin x \cos x + 1 = \pm \sqrt{2}.$$

$$\sin x \cos x = -1 \pm \sqrt{2}.$$

$$2 \sin x \cos x = -2 \pm 2\sqrt{2}.$$

By [12], $$\sin 2x = -2 \pm 2\sqrt{2}.$$

$$\sin 2x = 0.8284 \text{ or } -4.8284.$$

$$\therefore 2x = 55° 56', 124° 4', 415° 56', \text{ or } 484° 4'.$$

$$\therefore x = 27° 58', 62° 2', 207° 58', \text{ or } 242° 2'.$$

$$\therefore x = 27° 58', 62° 2', 135°, 207° 58', 242° 2', \text{ or } 315°.$$

But the values $x = 62° 2'$ and $x = 207° 58'$ do not satisfy the given equation.

$$\therefore x = 27° 58', 135°, 242° 2', \text{ or } 315°.$$

125. Solve $\tan^2 x = \sin 2x$.

$$\tan^2 x = \sin 2x.$$

By [12], $$\tan^2 x = 2 \sin x \cos x.$$

$$\tan^2 x = 2 \tan x \cos^2 x.$$

$$\tan^2 x = \frac{2 \tan x}{\sec^2 x}.$$

By Prob. 2, Ex. V, $$\tan^2 x = \frac{2 \tan x}{1 + \tan^2 x}.$$

$$\tan^2 x + \tan^4 x = 2 \tan x.$$

$$\tan x (\tan^3 x + \tan x - 2) = 0.$$

$$\tan x (\tan x - 1)(\tan^2 x + \tan x + 2) = 0.$$

(i)
$$\tan x = 0.$$
$$\therefore x = 0^\circ \text{ or } 180^\circ.$$

(ii)
$$\tan x - 1 = 0.$$
$$\tan x = 1.$$
$$\therefore x = 45^\circ \text{ or } 225^\circ.$$

(iii)
$$\tan^2 x + \tan x + 2 = 0.$$
$$4\tan^2 x + (\,) + 1 = -7.$$
$$2\tan x + 1 = \pm\sqrt{-7}.$$
$$\tan x = \tfrac{1}{2}(-1 \pm \sqrt{-7}).$$
$$\therefore x \text{ is impossible.}$$
$$\therefore x = 0^\circ,\ 45^\circ,\ 180^\circ, \text{ or } 225^\circ.$$

126. Solve $\tan x + \cot x = \tan 2x$.

$$\tan x + \cot x = \tan 2x.$$

By [14],
$$\tan x + \frac{1}{\tan x} = \frac{2\tan x}{1 - \tan^2 x}.$$
$$\frac{\tan^2 x + 1}{\tan x} = \frac{2\tan x}{1 - \tan^2 x}.$$
$$1 - \tan^4 x = 2\tan^2 x.$$
$$\tan^4 x + 2\tan^2 x = 1.$$
$$\tan^4 x + 2\tan^2 x + 1 = 2.$$
$$\tan^2 x + 1 = \pm\sqrt{2}.$$
$$\tan^2 x = -1 \pm \sqrt{2}.$$
$$\tan^2 x = 0.4142 \text{ or } -2.4142.$$
$$\tan x = \pm 0.6436.$$
$$\therefore x = 32^\circ 46',\ 147^\circ 14',\ 212^\circ 46', \text{ or } 327^\circ 14'.$$

127. Solve $\dfrac{1 - \tan x}{1 + \tan x} = \cos 2x$.

$$\frac{1 - \tan x}{1 + \tan x} = \cos 2x.$$

By [18],
$$\frac{\cos x - \sin x}{\cos x + \sin x} = \cos^2 x - \sin^2 x.$$
$$\frac{\cos x - \sin x}{\cos x + \sin x} = (\cos x - \sin x)(\cos x + \sin x).$$
$$\cos x - \sin x = (\cos x - \sin x)(\cos x + \sin x)^2.$$
$$(\cos x - \sin x)\,[1 - (\cos x + \sin x)^2] = 0.$$

(i)
$$\cos x - \sin x = 0.$$
$$\cos x = \sin x.$$
$$\therefore x = 45^\circ \text{ or } 225^\circ.$$

(ii) $1 - (\cos x + \sin x)^2 = 0.$

$$1 - (\cos^2 x + \sin^2 x + 2 \sin x \cos x) = 0.$$

$$1 - (1 + 2 \sin x \cos x) = 0.$$

$$2 \sin x \cos x = 0.$$

$$\sin x \cos x = 0.$$

$$\therefore x = 0^\circ, 90^\circ, 180^\circ, \text{ or } 270^\circ.$$

$$\therefore x = 0^\circ, 45^\circ, 90^\circ, 180^\circ, 225^\circ, \text{ or } 270^\circ.$$

128. Solve $\sin x + \sin 2x = 1 - \cos 2x$.

$$\sin x + \sin 2x = 1 - \cos 2x.$$

By [12], $\sin x + 2 \sin x \cos x = 1 - \cos 2x.$

By [16], $\sin x + 2 \sin x \cos x = 2 \sin^2 x.$

$$\sin x (1 + 2 \cos x - 2 \sin x) = 0.$$

(i) $\sin x = 0.$

$$\therefore x = 0^\circ \text{ or } 180^\circ.$$

(ii) $1 + 2 \cos x - 2 \sin x = 0.$

$$\sin x - \cos x = \tfrac{1}{2}.$$

$$\sin^2 x + \cos^2 x - 2 \sin x \cos x = \tfrac{1}{4}.$$

$$1 - 2 \sin x \cos x = \tfrac{1}{4}.$$

$$2 \sin x \cos x = \tfrac{3}{4}.$$

By [12], $\sin 2x = \tfrac{3}{4} = 0.75.$

$$\therefore 2x = 48^\circ\, 36', 131^\circ\, 24', 408^\circ\, 36', \text{ or } 491^\circ\, 24'.$$

$$\therefore x = 24^\circ\, 18', 65^\circ\, 42', 204^\circ\, 18', \text{ or } 245^\circ\, 42'.$$

$$\therefore x = 0^\circ, 24^\circ\, 18', 65^\circ\, 42', 180^\circ, 204^\circ\, 18', \text{ or } 245^\circ\, 42'.$$

But the values $x = 24^\circ\, 18'$ and $x = 245^\circ\, 42'$ do not satisfy the given equation.

$$\therefore x = 0^\circ, 65^\circ\, 42', 180^\circ, \text{ or } 204^\circ\, 18'.$$

129. Solve $\sec 2x + 1 = 2 \cos x$.

$$\sec 2x + 1 = 2 \cos x.$$

$$\frac{1}{\cos 2x} + 1 = 2 \cos x.$$

$$1 + \cos 2x = 2 \cos x \cos 2x.$$

By [13], $1 + \cos^2 x - \sin^2 x = 2 \cos x (\cos^2 x - \sin^2 x).$

$$1 + \cos^2 x - 1 + \cos^2 x = 2 \cos x (\cos^2 x - 1 + \cos^2 x).$$

$$2 \cos^2 x = 2 \cos x (2 \cos^2 x - 1).$$

$$\cos x (2 \cos^2 x - \cos x - 1) = 0.$$

$$\cos x (\cos x - 1)(2 \cos x + 1) = 0.$$

(i) $\cos x = 0.$

$$\therefore x = 90^\circ \text{ or } 270^\circ.$$

(ii)
$$\cos x - 1 = 0.$$
$$\cos x = 1.$$
$$\therefore x = 0°.$$

(iii)
$$2\cos x + 1 = 0.$$
$$\cos x = -\tfrac{1}{2}.$$
$$\therefore x = 120° \text{ or } 240°.$$
$$\therefore x = 0°, 90°, 120°, 240° \text{ or } 270°.$$

130. Solve $\tan 2x + \tan 3x = 0$.

$$\tan 2x + \tan 3x = 0.$$
$$\tan 2x = -\tan 3x = \tan(-3x).$$
$$\therefore 2x = -3x \text{ or } 180° - 3x.$$

(i)
$$5x = 0° + n\,360°.$$
$$\therefore x = 0°, 72°, 144°, 216°, \text{ or } 288°.$$

(ii)
$$5x = 180° + n\,360°.$$
$$\therefore x = 36°, 108°, 180°, 252°, \text{ or } 324°.$$
$$\therefore x = 0°, 36°, 72°, 108°, 144°, 180°, 216°, 252°, 288°, \text{ or } 324°.$$

131. Solve $\tan(\tfrac{1}{4}\pi + x) + \tan(\tfrac{1}{4}\pi - x) = 4$.

$$\tan(\tfrac{1}{4}\pi + x) + \tan(\tfrac{1}{4}\pi - x) = 4.$$

By [6] and [10],
$$\frac{1+\tan x}{1-\tan x} + \frac{1-\tan x}{1+\tan x} = 4.$$
$$1 + 2\tan x + \tan^2 x + 1 - 2\tan x + \tan^2 x = 4 - 4\tan^2 x.$$
$$6\tan^2 x = 2.$$
$$\tan^2 x = \tfrac{1}{3}.$$
$$\tan x = \pm\sqrt{\tfrac{1}{3}} = \pm\tfrac{1}{3}\sqrt{3}.$$
$$\therefore x = 30°, 150°, 210°, \text{ or } 330°.$$

132. Solve $\sqrt{1+\sin x} - \sqrt{1-\sin x} = 2\cos x$.

$$\sqrt{1+\sin x} - \sqrt{1-\sin x} = 2\cos x.$$

Square,
$$1 + \sin x - 2\sqrt{1-\sin^2 x} + 1 - \sin x = 4\cos^2 x.$$
$$2 - 2\sqrt{1-\sin^2 x} = 4\cos^2 x.$$
$$1 - \sqrt{\cos^2 x} = 2\cos^2 x.$$
$$1 - \cos x = 2\cos^2 x.$$
$$2\cos^2 x + \cos x - 1 = 0.$$
$$(\cos x + 1)(2\cos x - 1) = 0.$$

(i)
$$\cos x + 1 = 0.$$
$$\cos x = -1.$$
$$\therefore x = 180°.$$

(ii)
$$2\cos x - 1 = 0.$$
$$\cos x = \tfrac{1}{2}.$$
$$\therefore x = 60^\circ \text{ or } 300^\circ.$$
$$\therefore x = 60^\circ,\ 180^\circ, \text{ or } 300^\circ.$$

But the values $x = 180^\circ$ and $x = 300^\circ$ do not satisfy the given equation.
$$\therefore x = 60^\circ.$$

133. Solve $\tan x \tan 3x = -\tfrac{2}{5}$.
$$\tan x \tan 3x = -\tfrac{2}{5}.$$

By Prob. 19, Ex. XXIV,
$$\tan x \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} = -\tfrac{2}{5}.$$
$$\frac{3\tan^2 x - \tan^4 x}{1 - 3\tan^2 x} = -\tfrac{2}{5}.$$
$$15\tan^2 x - 5\tan^4 x = -2 + 6\tan^2 x.$$
$$5\tan^4 x - 9\tan^2 x - 2 = 0.$$
$$(\tan^2 x - 2)(5\tan^2 x + 1) = 0.$$

(i)
$$\tan^2 x - 2 = 0.$$
$$\tan^2 x = 2.$$
$$\tan x = \pm\sqrt{2} = \pm 1.4142.$$
$$\therefore x = 54^\circ 44',\ 125^\circ 16',\ 234^\circ 44', \text{ or } 305^\circ 16'.$$

(ii)
$$5\tan^2 x + 1 = 0.$$
$$5\tan^2 x = -1.$$
$$\tan^2 x = -\tfrac{1}{5}.$$
$$\tan x = \pm\sqrt{-\tfrac{1}{5}}.$$
$$\therefore x \text{ is impossible.}$$
$$\therefore x = 54^\circ 44',\ 125^\circ 16',\ 234^\circ 44', \text{ or } 305^\circ 16'.$$

134. Solve $\sin(45^\circ + x) + \cos(45^\circ - x) = 1$.
$$\sin(45^\circ + x) + \cos(45^\circ - x) = 1.$$

Now
$$45^\circ + x = 90^\circ - (45^\circ - x).$$
$$\therefore \cos(45^\circ - x) + \cos(45^\circ - x) = 1.$$
$$2\cos(45^\circ - x) = 1.$$
$$\cos(45^\circ - x) = \tfrac{1}{2}.$$
$$45^\circ - x = 60^\circ \text{ or } 300^\circ.$$
$$\therefore x = -15^\circ \text{ or } -255^\circ.$$
$$\therefore x = 105^\circ \text{ or } 345^\circ.$$

135. Solve $\tan x + \sec x = a$.

$$\tan x + \sec x = a.$$

$$\sec x = a - \tan x.$$

$$\sec^2 x = a^2 - 2a\tan x + \tan^2 x.$$

By Prob. 2, Ex. V, $$1 + \tan^2 x = a^2 - 2a\tan x + \tan^2 x.$$

$$2a\tan x = a^2 - 1.$$

$$\tan x = \frac{a^2-1}{2a}.$$

$$\therefore x = \tan^{-1}\frac{a^2-1}{2a}.$$

136. Solve $\cos 2x = a(1 - \cos x)$.

$$\cos 2x = a(1 - \cos x).$$

By [13], $$\cos^2 x - \sin^2 x = a(1 - \cos x).$$

$$\cos^2 x - 1 + \cos^2 x = a - a\cos x.$$

$$2\cos^2 x + a\cos x = a + 1.$$

$$16\cos^2 x + (\ ) + a^2 = a^2 + 8a + 8.$$

$$4\cos x + a = \pm\sqrt{a^2 + 8a + 8}.$$

$$\cos x = \frac{-a \pm \sqrt{a^2 + 8a + 8}}{4}.$$

$$x = \cos^{-1}\left(\frac{-a \pm \sqrt{a^2 + 8a + 8}}{4}\right).$$

137. Solve $(1 - \tan x)\cos 2x = a(1 + \tan x)$.

$$(1 - \tan x)\cos 2x = a(1 + \tan x).$$

$$\cos 2x = a\frac{1 + \tan x}{1 - \tan x}.$$

$$\cos 2x = \frac{a(\cos x + \sin x)}{\cos x - \sin x}.$$

By [13], $$\cos^2 x - \sin^2 x = \frac{a(\cos x + \sin x)}{\cos x - \sin x}.$$

$$(\cos x - \sin x)^2(\cos x + \sin x) - a(\cos x + \sin x) = 0.$$

$$(\cos x + \sin x)[(\cos x - \sin x)^2 - a] = 0.$$

(i) $$\cos x + \sin x = 0.$$

$$\sin x = -\cos x.$$

$$\therefore x = 135° \text{ or } 315°.$$

(ii) $$(\cos x - \sin x)^2 - a = 0.$$

$$\cos^2 x + \sin^2 x - 2\sin x\cos x = a.$$

$$1 - 2\sin x\cos x = a.$$

$$2 \sin x \cos x = 1 - a.$$

$$\sin 2x = 1 - a.$$

$$\therefore 2x = \sin^{-1}(1 - a).$$

$$\therefore x = \tfrac{1}{2} \sin^{-1}(1 - a).$$

$$\therefore x = 135°, 315°, \text{ or } \tfrac{1}{2} \sin^{-1}(1 - a).$$

138. Solve $\sin^6 x + \cos^6 x = \frac{7}{12} \sin^2 2x$.

$$\sin^6 x + \cos^6 x = \tfrac{7}{12} \sin^2 2x.$$

$$(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x) = \tfrac{7}{12} \sin^2 2x.$$

$$\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x = \tfrac{7}{12} \sin^2 2x.$$

$$(\sin^4 x + 2 \sin^2 x \cos^2 x + \cos^4 x) - 3 \sin^2 x \cos^2 x = \tfrac{7}{3} \sin^2 x \cos^2 x.$$

$$(\sin^2 x + \cos^2 x)^2 = \tfrac{16}{3} \sin^2 x \cos^2 x.$$

$$1 = \tfrac{16}{3} \sin^2 x \cos^2.$$

$$16 \sin^2 x \cos^2 x = 3.$$

$$4 \sin^2 x \cos^2 x = \tfrac{3}{4}.$$

$$\sin^2 2x = \pm \tfrac{1}{2} \sqrt{3}.$$

$$\therefore 2x = 60°, 120°, 240°, 300°, 420°, 480°, 600°, \text{ or } 660°.$$

$$\therefore x = 30°, 60°, 120°, 150°, 210°, 240°, 300°, \text{ or } 330°.$$

139. Solve $\cos 3x + 8 \cos^3 x = 0$.

$$\cos 3x + 8 \cos^3 x = 0.$$

By Prob. 19, Ex. XIV,

$$4 \cos^3 x - 3 \cos x + 8 \cos^3 x = 0.$$

$$12 \cos^3 x - 3 \cos x = 0.$$

$$\cos x (4 \cos^2 x - 1) = 0.$$

(i)

$$\cos x = 0.$$

$$\therefore x = 90° \text{ or } 270°.$$

(ii)

$$4 \cos^2 x - 1 = 0.$$

$$4 \cos^2 x = 1.$$

$$\cos^2 x = \tfrac{1}{4}.$$

$$\cos x = \pm \tfrac{1}{2}.$$

$$\therefore x = 60°, 120°, 240°, \text{ or } 300°.$$

$$\therefore x = 60°, 90°, 120°, 240°, 270°, \text{ or } 300°.$$

140. Solve $\sec(x + 120°) + \sec(x - 120°) = 2 \cos x$.

$$\sec(x + 120°) + \sec(x - 120°) = 2 \cos x.$$

$$\frac{1}{\cos(x + 120°)} + \frac{1}{\cos(x - 120°)} = 2 \cos x.$$

$$\frac{\cos(x - 120°) + \cos(x + 120°)}{\cos(x + 120°)\cos(x - 120°)} = 2 \cos x.$$

By [22],
$$\frac{2\cos x\cos 120^\circ}{\cos(x+120^\circ)\cos(x-120^\circ)}=2\cos x.$$

By [5] and [9],
$$\frac{2\cos x\cos 120^\circ}{(\cos x\cos 120^\circ-\sin x\sin 120^\circ)(\cos x\cos 120^\circ+\sin x\sin 120^\circ)}=2\cos x.$$
$$\frac{2\cos x\cos 120^\circ}{\cos^2 x\cos^2 120^\circ-\sin^2 x\sin^2 120^\circ}=2\cos x.$$

Now
$$\sin 120^\circ=\tfrac{1}{2}\sqrt{3}\text{ and }\cos 120^\circ=-\tfrac{1}{2}.$$
$$\therefore\ \frac{-\cos x}{\frac{1}{4}\cos^2 x-\frac{3}{4}\sin^2 x}=2\cos x.$$
$$-\cos x=\tfrac{1}{2}\cos x(\cos^2 x-3\sin^2 x).$$
$$-2\cos x=\cos x(4\cos^2 x-3).$$
$$\cos x(4\cos^2 x-1)=0.$$

(i)
$$\cos x=0.$$
$$\therefore\ x=90^\circ\text{ or }270^\circ.$$

(ii)
$$4\cos^2 x-1=0.$$
$$4\cos^2 x=1.$$
$$\cos^2 x=\tfrac{1}{4}.$$
$$\cos x=\pm\tfrac{1}{2}.$$
$$\therefore\ x=60^\circ,\ 120^\circ,\ 240^\circ,\text{ or }300^\circ.$$
$$\therefore\ x=60^\circ,\ 90^\circ,\ 120^\circ,\ 240^\circ,\ 270^\circ,\text{ or }300^\circ.$$

141. Solve $\csc x=\cot x+\sqrt{3}$.

$$\csc x=\cot x+\sqrt{3}.$$
$$\csc^2 x=\cot^2 x+2\sqrt{3}\cot x+3.$$

By Prob. 3, Ex. V,
$$1+\cot^2 x=\cot^2 x+2\sqrt{3}\cot x+3.$$
$$2\sqrt{3}\cot x=-2.$$
$$\cot x=-\frac{1}{\sqrt{3}}=-\tfrac{1}{3}\sqrt{3}.$$
$$\therefore\ x=120^\circ\text{ or }300^\circ.$$

But the value $x=300^\circ$ does not satisfy the given equation.
$$\therefore\ x=120^\circ.$$

142. Solve $4\cos 2x+6\sin x=5$.

$$4\cos 2x+6\sin x=5.$$

By [13],
$$4(1-2\sin^2 x)+6\sin x=5.$$
$$4-8\sin^2 x+6\sin x=5.$$
$$8\sin^2 x-6\sin x+1=0.$$
$$(2\sin x-1)(4\sin x-1)=0.$$

(i)
$$2\sin x - 1 = 0.$$
$$2\sin x = 1.$$
$$\sin x = \tfrac{1}{2}.$$
$$\therefore x = 30^\circ \text{ or } 150^\circ.$$

(ii)
$$4\sin x - 1 = 0.$$
$$4\sin x = 1.$$
$$\sin x = \tfrac{1}{4} = 0.25.$$
$$\therefore x = 14^\circ\,29' \text{ or } 165^\circ\,31'.$$
$$\therefore x = 14^\circ\,29',\ 30^\circ,\ 150^\circ, \text{ or } 165^\circ\,31'.$$

143. Solve $\cos x - \cos 2x = 1$.

$$\cos x - \cos 2x = 1.$$

By [13],
$$\cos x - \cos^2 x + \sin^2 x = 1.$$
$$\cos x - \cos^2 x + 1 - \cos^2 x = 1.$$
$$2\cos^2 x - \cos x = 0.$$
$$\cos x(2\cos x - 1) = 0.$$

(i)
$$\cos x = 0.$$
$$\therefore x = 90^\circ \text{ or } 270^\circ.$$

(ii)
$$2\cos x - 1 = 0.$$
$$2\cos x = 1.$$
$$\cos x = \tfrac{1}{2}.$$
$$\therefore x = 60^\circ \text{ or } 300^\circ.$$
$$\therefore x = 60^\circ,\ 90^\circ,\ 270^\circ, \text{ or } 300^\circ.$$

144. Solve $\sin 4x - \sin 2x = \sin x$.

$$\sin 4x - \sin 2x = \sin x.$$

By [21],
$$2\cos 3x \sin x = \sin x.$$
$$\sin x(2\cos 3x - 1) = 0.$$

(i)
$$\sin x = 0.$$
$$\therefore x = 0^\circ \text{ or } 180^\circ.$$

(ii)
$$2\cos 3x - 1 = 0.$$
$$2\cos 3x = 1.$$
$$\cos 3x = \tfrac{1}{2}.$$
$$\therefore 3x = 60^\circ + n\,360^\circ \text{ or } 300^\circ + n\,360^\circ.$$
$$\therefore x = 20^\circ,\ 140^\circ, \text{ or } 260^\circ;\ \text{or } 100^\circ,\ 220^\circ, \text{ or } 340^\circ.$$
$$\therefore x = 0^\circ,\ 20^\circ,\ 100^\circ,\ 140^\circ,\ 180^\circ,\ 220^\circ,\ 260^\circ, \text{ or } 340^\circ.$$

145. Solve $2\sin^2 x + \sin^2 2x = 2$.

$$2\sin^2 x + \sin^2 2x = 2.$$

By [12],
$$2\sin^2 x + 4\sin^2 x\cos^2 x = 2.$$
$$\sin^2 x + 2\sin^2 x(1 - \sin^2 x) = 1.$$

$$\sin^2 x + 2\sin^2 x - 2\sin^4 x = 1.$$
$$2\sin^4 x - 3\sin^2 x + 1 = 0.$$
$$(\sin^2 x - 1)(2\sin^2 x - 1) = 0.$$

(i) $$\sin^2 x - 1 = 0.$$
$$\sin^2 x = 1.$$
$$\sin x = \pm 1.$$
$$\therefore x = 0^\circ, 90^\circ, 180^\circ, \text{ or } 270^\circ.$$

(ii) $$2\sin^2 x - 1 = 0.$$
$$2\sin^2 x = 1.$$
$$\sin^2 x = \tfrac{1}{2}.$$
$$\sin x = \pm\sqrt{\tfrac{1}{2}} = \pm\tfrac{1}{2}\sqrt{2}.$$
$$\therefore x = 45^\circ, 135^\circ, 225^\circ, \text{ or } 315^\circ.$$
$$\therefore x = 0^\circ, 45^\circ, 90^\circ, 135^\circ, 180^\circ, 225^\circ, 270^\circ, \text{ or } 315^\circ.$$

But the values $x = 0^\circ$ and $x = 180^\circ$ do not satisfy the given equation.

$$\therefore x = 45^\circ, 90^\circ, 135^\circ, 225^\circ, 270^\circ, \text{ or } 315^\circ.$$

146. Solve $\cos 5x + \cos 3x + \cos x = 0$.

$$\cos 5x + \cos 3x + \cos x = 0.$$

By [22], $$2\cos 4x \cos x + \cos x = 0.$$
$$\cos x(2\cos 4x + 1) = 0.$$

(i) $$\cos x = 0.$$
$$\therefore x = 90^\circ \text{ or } 270^\circ.$$

(ii) $$2\cos 4x + 1 = 0.$$
$$2\cos 4x = -1.$$
$$\cos 4x = -\tfrac{1}{2}.$$
$$\therefore 4x = 120^\circ + n\,360^\circ \text{ or } 240^\circ + n\,360^\circ.$$
$$\therefore x = 30^\circ, 120^\circ, 210^\circ, \text{ or } 300^\circ;\ \text{or } 60^\circ, 150^\circ, 240^\circ, \text{ or } 330^\circ.$$
$$\therefore x = 30^\circ, 60^\circ, 90^\circ, 120^\circ, 150^\circ, 210^\circ, 240^\circ, 270^\circ, 300^\circ, \text{ or } 330^\circ.$$

147. Solve $\sec x - \cot x = \csc x - \tan x$.

$$\sec x - \cot x = \csc x - \tan x.$$
$$\frac{1}{\cos x} - \frac{\cos x}{\sin x} = \frac{1}{\sin x} - \frac{\sin x}{\cos x}.$$
$$\sin x - \cos^2 x = \cos x - \sin^2 x.$$
$$(\sin x - \cos x) + (\sin^2 x - \cos^2 x) = 0.$$
$$(\sin x - \cos x)(1 + \sin x + \cos x) = 0.$$

(i) $$\sin x - \cos x = 0.$$
$$\sin x = \cos x.$$
$$\therefore x = 45^\circ \text{ or } 225^\circ.$$

(ii)

$$\begin{aligned}
1 + \sin x + \cos x &= 0.\\
\sin x + \cos x &= -1.\\
\sin^2 x + 2\sin x\cos x + \cos^2 x &= 1.\\
1 + 2\sin x\cos x &= 1.\\
2\sin x\cos x &= 0.
\end{aligned}$$

By [12],

$$\begin{aligned}
\sin 2x &= 0.\\
\therefore 2x &= 0^\circ,\ 180^\circ,\ 360^\circ,\text{ or }540^\circ.\\
\therefore x &= 0^\circ,\ 90^\circ,\ 180^\circ,\text{ or }270^\circ.\\
\therefore x &= 0^\circ,\ 45^\circ,\ 90^\circ,\ 180^\circ,\ 225^\circ,\text{ or }270^\circ.
\end{aligned}$$

148. Solve $\tan^2 x + \cot^2 x = \frac{10}{3}$.

$$\begin{aligned}
\tan^2 x + \cot^2 x &= \tfrac{10}{3}.\\
\tan^2 x + \frac{1}{\tan^2 x} &= \frac{10}{3}.\\
3\tan^4 x + 3 &= 10\tan^2 x.\\
3\tan^4 x - 10\tan^2 x + 3 &= 0.\\
(\tan^2 x - 3)(3\tan^2 x - 1) &= 0.
\end{aligned}$$

(i)

$$\begin{aligned}
\tan^2 x - 3 &= 0.\\
\tan^2 x &= 3.\\
\tan x &= \pm\sqrt{3}.\\
\therefore x &= 60^\circ,\ 120^\circ,\ 240^\circ,\text{ or }300^\circ.
\end{aligned}$$

(ii)

$$\begin{aligned}
3\tan^2 x - 1 &= 0.\\
3\tan^2 x &= 1.\\
\tan^2 x &= \tfrac{1}{3}.\\
\tan x &= \pm\sqrt{\tfrac{1}{3}} = \pm\tfrac{1}{3}\sqrt{3}.\\
\therefore x &= 30^\circ,\ 150^\circ,\ 210^\circ,\text{ or }330^\circ.
\end{aligned}$$

$$\therefore x = 30^\circ,\ 60^\circ,\ 120^\circ,\ 150^\circ,\ 210^\circ,\ 240^\circ,\ 300^\circ,\text{ or }330^\circ.$$

149. Solve $\sin 4x - \cos 3x = \sin 2x$.

$$\begin{aligned}
\sin 4x - \cos 3x &= \sin 2x.\\
\sin 4x - \sin 2x - \cos 3x &= 0.
\end{aligned}$$

By [21],

$$\begin{aligned}
2\cos 3x\sin x - \cos 3x &= 0.\\
\cos 3x\,(2\sin x - 1) &= 0.
\end{aligned}$$

(i)

$$\begin{aligned}
\cos 3x &= 0.\\
\therefore 3x &= 90^\circ + n\,360^\circ\text{ or }270^\circ + n\,360^\circ.\\
\therefore x &= 30^\circ,\ 150^\circ,\text{ or }270^\circ;\text{ or }90^\circ,\ 210^\circ,\text{ or }330^\circ.
\end{aligned}$$

(ii)

$$\begin{aligned}
2\sin x - 1 &= 0.\\
2\sin x &= 1.\\
\sin x &= \tfrac{1}{2}.\\
\therefore x &= 30^\circ\text{ or }150^\circ.
\end{aligned}$$

$$\therefore x = 30^\circ,\ 90^\circ,\ 150^\circ,\ 210^\circ,\ 270^\circ,\text{ or }330^\circ.$$

150. Solve $\sin x + \cos x = \sec x$.

$$\sin x + \cos x = \sec x.$$

$$\sin x + \cos x = \frac{1}{\cos x}.$$

$$\sin x \cos x + \cos^2 x = 1.$$

$$\sin x \cos x + (\cos^2 x - 1) = 0.$$

$$\sin x \cos x - \sin^2 x = 0.$$

$$\sin x (\cos x - \sin x) = 0.$$

(i) $$\sin x = 0.$$

$$\therefore x = 0^\circ \text{ or } 180^\circ.$$

(ii) $$\cos x - \sin x = 0.$$

$$\cos x = \sin x.$$

$$\therefore x = 45^\circ \text{ or } 225^\circ.$$

$$\therefore x = 0^\circ,\ 45^\circ,\ 180^\circ, \text{ or } 225^\circ.$$

151. Solve $2 \cos x \cos 3x + 1 = 0$.

$$2 \cos x \cos 3x + 1 = 0.$$

By Prob. 19, Ex. XIV,

$$2 \cos x (4 \cos^3 x - 3 \cos x) + 1 = 0.$$

$$8 \cos^4 x - 6 \cos^2 x + 1 = 0.$$

$$(4 \cos^2 x - 1)(2 \cos^2 x - 1) = 0.$$

(i) $$4 \cos^2 x - 1 = 0.$$

$$4 \cos^2 x = 1.$$

$$\cos^2 x = \tfrac{1}{4}.$$

$$\cos x = \pm \tfrac{1}{2}.$$

$$\therefore x = 60^\circ,\ 120^\circ,\ 240^\circ, \text{ or } 300^\circ.$$

(ii) $$2 \cos^2 x - 1 = 0.$$

$$2 \cos^2 x = 1.$$

$$\cos^2 x = \tfrac{1}{2}.$$

$$\cos x = \pm \sqrt{\tfrac{1}{2}} = \pm \tfrac{1}{2} \sqrt{2}.$$

$$\therefore x = 45^\circ,\ 135^\circ,\ 225^\circ, \text{ or } 315^\circ.$$

$$\therefore x = 45^\circ,\ 60^\circ,\ 120^\circ,\ 135^\circ,\ 225^\circ,\ 240^\circ,\ 300^\circ, \text{ or } 315^\circ.$$

152. Solve $\cos 3x - 2 \cos 2x + \cos x = 0$.

$$\cos 3x - 2 \cos 2x + \cos x = 0.$$

$$(\cos 3x + \cos x) - 2 \cos 2x = 0.$$

By [22], $$2 \cos 2x \cos x - 2 \cos 2x = 0.$$

$$\cos 2x (\cos x - 1) = 0.$$

(i) $$\cos 2x = 0.$$

$$\therefore 2x = 90^\circ,\ 270^\circ,\ 450^\circ, \text{ or } 630^\circ.$$

$$\therefore x = 45^\circ,\ 135^\circ,\ 225^\circ, \text{ or } 315^\circ.$$

(ii)

$$\cos x - 1 = 0.$$
$$\cos x = 1.$$
$$\therefore x = 0^\circ.$$
$$\therefore x = 0^\circ, 45^\circ, 135^\circ, 225^\circ, \text{ or } 315^\circ.$$

Solve $\tan 2x \tan x = 1$.

$$\tan 2x \tan x = 1.$$
$$\tan 2x = \frac{1}{\tan x}.$$
$$\tan 2x = \cot x.$$
$$\therefore 2x = 90^\circ - x \text{ or } 270^\circ - x.$$
$$3x = 90^\circ + n\,360^\circ \text{ or } 270^\circ + n\,360^\circ.$$
$$\therefore x = 30^\circ, 150^\circ, \text{ or } 270^\circ; \text{ or } 90^\circ, 210^\circ, \text{ or } 330^\circ.$$
$$\therefore x = 30^\circ, 90^\circ, 150^\circ, 210^\circ, 270^\circ, \text{ or } 330^\circ.$$

154. Solve $\sin(x + 12^\circ) + \sin(x - 8^\circ) = \sin 20^\circ$.

$$\sin(x + 12^\circ) + \sin(x - 8^\circ) = \sin 20^\circ.$$

By [20],

$$2\sin(x + 2^\circ)\cos 10^\circ = \sin 20^\circ.$$
$$2\sin(x + 2^\circ)\cos 10^\circ = 2\sin 10^\circ \cos 10^\circ.$$
$$2\sin(x + 2^\circ) = 2\sin 10^\circ.$$
$$\sin(x + 2^\circ) = \sin 10^\circ.$$
$$\therefore x + 2^\circ = 10^\circ \text{ or } 170^\circ.$$
$$\therefore x = 8^\circ \text{ or } 168^\circ.$$

155. Solve $\tan(60^\circ + x)\tan(60^\circ - x) = -2$.

$$\tan(60^\circ + x)\tan(60^\circ - x) = -2.$$

By [6] and [10],

$$\frac{\tan 60^\circ + \tan x}{1 - \tan 60^\circ \tan x} \times \frac{\tan 60^\circ - \tan x}{1 + \tan 60^\circ \tan x} = -2.$$
$$\frac{\tan^2 60^\circ - \tan^2 x}{1 - \tan^2 60^\circ \tan^2 x} = -2.$$
$$\frac{3 - \tan^2 x}{1 - 3\tan^2 x} = -2.$$
$$3 - \tan^2 x = -2 + 6\tan^2 x.$$
$$7\tan^2 x = 5.$$
$$\tan^2 x = \tfrac{5}{7} = 0.71428571.$$
$$\tan x = \pm 0.8451.$$
$$\therefore x = 40^\circ\, 12', 139^\circ\, 48', 220^\circ\, 12', \text{ or } 319^\circ\, 48'.$$

156. Solve $\sin(x + 120^\circ) + \sin(x + 60^\circ) = \frac{3}{2}$.

$$\sin(x + 120^\circ) + \sin(x + 60^\circ) = \tfrac{3}{2}.$$

By [20], $$2\sin(x + 90^\circ)\cos 30^\circ = \tfrac{3}{2}.$$

$$2\cos x \cos 30^\circ = \tfrac{3}{2}.$$

$$2\cos x\,(\tfrac{1}{2}\sqrt{3}) = \tfrac{3}{2}.$$

$$\cos x = \frac{3}{2\sqrt{3}} = \tfrac{1}{2}\sqrt{3}.$$

$$\therefore x = 30^\circ \text{ or } 330^\circ.$$

157. Solve $\sin(x + 30^\circ)\sin(x - 30^\circ) = \frac{1}{2}$.

$$\sin(x + 30^\circ)\sin(x - 30^\circ) = \tfrac{1}{2}.$$

By [23], $$-\tfrac{1}{2}(\cos 2x - \cos 60^\circ) = \tfrac{1}{2}.$$

$$\cos 2x - \cos 60^\circ = -1.$$

$$\cos 2x - \tfrac{1}{2} = -1.$$

$$\cos 2x = -\tfrac{1}{2}.$$

$$\therefore 2x = 120^\circ,\ 240^\circ,\ 480^\circ, \text{ or } 600^\circ.$$

$$\therefore x = 60^\circ,\ 120^\circ,\ 240^\circ, \text{ or } 300^\circ.$$

158. Solve $\sin^4 x + \cos^4 x = \frac{5}{8}$.

$$\sin^4 x + \cos^4 x = \tfrac{5}{8}.$$

$$\sin^4 x + 2\sin^2 x\cos^2 x + \cos^4 x = 2\sin^2 x\cos^2 x + \tfrac{5}{8}.$$

$$(\sin^2 x + \cos^2 x)^2 = 2\sin^2 x\cos^2 x + \tfrac{5}{8}.$$

$$1 = 2\sin^2 x\cos^2 x + \tfrac{5}{8}.$$

$$2\sin^2 x\cos^2 x = \tfrac{3}{8}.$$

$$4\sin^2 x\cos^2 x = \tfrac{3}{4}.$$

By [12], $$\sin^2 2x = \tfrac{3}{4}.$$

$$\sin 2x = \pm\tfrac{1}{2}\sqrt{3}.$$

$$\therefore 2x = 60^\circ,\ 120^\circ,\ 240^\circ,\ 300^\circ,\ 420^\circ,\ 480^\circ,\ 600^\circ, \text{ or } 660^\circ.$$

$$\therefore x = 30^\circ,\ 60^\circ,\ 120^\circ,\ 150^\circ,\ 210^\circ,\ 240^\circ,\ 300^\circ, \text{ or } 330^\circ.$$

159. Solve $\sin^4 x - \cos^4 x = \frac{7}{25}$.

$$\sin^4 x - \cos^4 x = \tfrac{7}{25}.$$

$$(\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x) = \tfrac{7}{25}.$$

$$\sin^2 x - \cos^2 x = \tfrac{7}{25}.$$

$$\cos^2 x - \sin^2 x = -\tfrac{7}{25}.$$

By [13], $$\cos 2x = -\tfrac{7}{25}.$$

$$\therefore 2x = 106^\circ\,16',\ 253^\circ\,44',\ 466^\circ\,16', \text{ or } 613^\circ\,44'.$$

$$\therefore x = 53^\circ\,8',\ 126^\circ\,52',\ 233^\circ\,8', \text{ or } 306^\circ\,52'.$$

160. Solve $\tan(x+30^\circ)=2\cos x$.

$$\tan(x+30^\circ)=2\cos x.$$

Let $$x+30^\circ=y.$$

Then $$x=y-30^\circ.$$

Substitute, $$\tan y=2\cos(y-30^\circ).$$

By [9], $$\tan y=2\cos y\cos 30^\circ+2\sin y\sin 30^\circ.$$

$$\frac{\sin y}{\cos y}=\sqrt{3}\cos y+\sin y.$$

$$\sin y=\sqrt{3}\cos^2 y+\sin y\cos y.$$

$$\sin y(1-\cos y)=\sqrt{3}\cos^2 y.$$

$$\sin^2 y(1-\cos y)^2=3\cos^4 y.$$

$$(1-\cos^2 y)(1-2\cos y+\cos^2 y)=3\cos^4 y.$$

$$1-2\cos y+2\cos^3 y-\cos^4 y=3\cos^4 y.$$

$$4\cos^4 y-2\cos^3 y+2\cos y-1=0.$$

$$(2\cos y-1)(2\cos^3 y+1)=0.$$

(i) $$2\cos y-1=0.$$

$$2\cos y=1.$$

$$\cos y=\tfrac{1}{2}.$$

$$\therefore y=60^\circ \text{ or } 300^\circ.$$

$$\therefore x=30^\circ \text{ or } 270^\circ.$$

(ii) $$2\cos^3 y+1=0.$$

$$2\cos^3 y=-1.$$

$$\cos^3 y=-\tfrac{1}{2}.$$

$$\cos y=-\sqrt[3]{\tfrac{1}{2}}.$$

The two other roots of equation (ii) are complex.

$$\cos y=-0.7937.$$

$$\therefore y=127^\circ\,28' \text{ or } 232^\circ\,32'.$$

$$\therefore x=97^\circ\,28' \text{ or } 202^\circ\,32'.$$

$$\therefore x=30^\circ,\ 97^\circ\,28',\ 202^\circ\,32', \text{ or } 270^\circ.$$

But the values $x=97^\circ\,28'$, $x=202^\circ\,32'$, and $x=270^\circ$ do not satisfy the given equation.

$$\therefore x=30^\circ.$$

161. Solve $\sec x=2\tan x+\frac{1}{4}$.

$$\sec x=2\tan x+\tfrac{1}{4}.$$

$$\sec^2 x=4\tan^2 x+\tan x+\tfrac{1}{16}.$$

By Prob. 2, Ex. V, $$1+\tan^2 x=4\tan^2 x+\tan x+\tfrac{1}{16}.$$

$$3\tan^2 x+\tan x-\tfrac{15}{16}=0.$$

$$48\tan^2 x+16\tan x-15=0.$$

$$(4\tan x+3)(12\tan x-5)=0.$$

(i) $$4\tan x + 3 = 0.$$
$$4\tan x = -3.$$
$$\tan x = -\tfrac{3}{4} = -0.75.$$
$$\therefore x = 143^\circ\ 8' \text{ or } 323^\circ\ 8'.$$

(ii) $$12\tan x - 5 = 0.$$
$$12\tan x = 5.$$
$$\tan x = \frac{5}{12} = 0.4167.$$
$$\therefore x = 22^\circ\ 37' \text{ or } 202^\circ\ 37'.$$
$$\therefore x = 22^\circ\ 37',\ 143^\circ\ 8',\ 202^\circ\ 37',\text{ or } 323^\circ\ 8'.$$

But the values $x = 202^\circ\ 37'$ and $x = 323^\circ\ 8'$ do not satisfy the given equation.
$$\therefore x = 22^\circ\ 37' \text{ or } 143^\circ\ 8'.$$

162. Solve $\sin 11x \sin 4x + \sin 5x \sin 2x = 0.$

$$\sin 11x \sin 4x + \sin 5x \sin 2x = 0.$$

By [23], $$\sin 11x \sin 4x = -\tfrac{1}{2}(\cos 15x - \cos 7x),$$
and $$\sin 5x \sin 2x = -\tfrac{1}{2}(\cos 7x - \cos 3x).$$
$$\therefore -\tfrac{1}{2}(\cos 15x - \cos 7x) - \tfrac{1}{2}(\cos 7x - \cos 3x) = 0.$$
$$\cos 15x - \cos 7x + \cos 7x - \cos 3x = 0.$$
$$\cos 15x - \cos 3x = 0.$$
By [23], $$-2\sin\tfrac{1}{2}(15x + 3x)\sin\tfrac{1}{2}(15x - 3x) = 0.$$
$$-\sin 9x \sin 6x = 0.$$
$$\sin 9x \sin 6x = 0.$$
By Prob. 18, Ex. XIV, $$\sin 9x = 3\sin 3x - 4\sin^3 3x,$$
and by [16], $$\sin 6x = \pm\sqrt{\frac{1-\cos 3x}{2}}.$$
Substitute, $$(3\sin 3x - 4\sin^3 3x)\left(\pm\sqrt{\frac{1-\cos 3x}{2}}\right) = 0.$$
$$(\sin 3x)(3 - 4\sin^2 3x)\left(\pm\sqrt{\frac{1-\cos 3x}{2}}\right) = 0.$$

(i) $$\sin 3x = 0.$$
$$\therefore 3x = 0^\circ,\ 180^\circ,\ 360^\circ,\ 540^\circ,\ 720^\circ,\text{ or } 900^\circ.$$
$$\therefore x = 0^\circ,\ 60^\circ,\ 120^\circ,\ 180^\circ,\ 240^\circ,\text{ or } 300^\circ.$$

(ii) $$3 - 4\sin^2 3x = 0.$$
$$4\sin^2 3x = 3.$$
$$2\sin 3x = \pm\sqrt{3}.$$
$$\sin 3x = \pm\tfrac{1}{2}\sqrt{3}.$$
$$\therefore 3x = 60^\circ,\ 120^\circ,\ 240^\circ,\ 300^\circ,\ 420^\circ,\ 480^\circ,\ 600^\circ,\ 660^\circ,\ 780^\circ,\ 840^\circ,\ 960^\circ,\text{ or } 1020^\circ.$$

$$\therefore x = 20^\circ,\ 40^\circ,\ 80^\circ,\ 100^\circ,\ 140^\circ,\ 160^\circ,\ 200^\circ,\ 220^\circ,\ 260^\circ,\ 280^\circ,\ 320^\circ, \text{ or } 340^\circ.$$

(iii)

$$\pm\sqrt{\frac{1-\cos 3x}{2}} = 0.$$

$$\frac{1-\cos 3x}{2} = 0.$$

$$1 - \cos 3x = 0.$$

$$\cos 3x = 1.$$

$$\therefore 3x = 0^\circ,\ 360^\circ, \text{ or } 720^\circ.$$

$$\therefore x = 0^\circ,\ 120^\circ, \text{ or } 240^\circ.$$

$$\therefore x = 0^\circ,\ 20^\circ,\ 40^\circ,\ 60^\circ,\ 80^\circ,\ 100^\circ,\ 120^\circ,\ 140^\circ,\ 160^\circ,\ 180^\circ,\ 200^\circ,\ 220^\circ,\ 240^\circ,\ 260^\circ,\ 280^\circ,\ 300^\circ,\ 320^\circ, \text{ or } 340^\circ.$$

163. Solve $\cos x + \cos 3x + \cos 5x + \cos 7x = 0.$

$$\cos x + \cos 3x + \cos 5x + \cos 7x = 0.$$

$$(\cos 7x + \cos 5x) + (\cos 3x + \cos x) = 0.$$

By [22],

$$2\cos 6x \cos x + 2\cos 2x \cos x = 0.$$

$$\cos x(\cos 6x + \cos 2x) = 0.$$

By [22],

$$\cos x \times 2\cos 4x \cos 2x = 0.$$

$$\cos x \cos 2x \cos 4x = 0.$$

(i)

$$\cos x = 0.$$

$$\therefore x = 90^\circ \text{ or } 270^\circ.$$

(ii)

$$\cos 2x = 0.$$

$$\therefore 2x = 90^\circ,\ 270^\circ,\ 450^\circ, \text{ or } 630^\circ.$$

$$\therefore x = 45^\circ,\ 135^\circ,\ 225^\circ, \text{ or } 315^\circ.$$

(iii)

$$\cos 4x = 0.$$

$$\therefore 4x = 90^\circ,\ 270^\circ,\ 450^\circ,\ 630^\circ,\ 810^\circ,\ 990^\circ,\ 1170^\circ, \text{ or } 1350^\circ.$$

$$\therefore x = 22\tfrac{1}{2}^\circ,\ 67\tfrac{1}{2}^\circ,\ 112\tfrac{1}{2}^\circ,\ 157\tfrac{1}{2}^\circ,\ 202\tfrac{1}{2}^\circ,\ 247\tfrac{1}{2}^\circ,\ 292\tfrac{1}{2}^\circ, \text{ or } 337\tfrac{1}{2}^\circ.$$

$$\therefore x = 22\tfrac{1}{2}^\circ,\ 45^\circ,\ 67\tfrac{1}{2}^\circ,\ 90^\circ,\ 112\tfrac{1}{2}^\circ,\ 135^\circ,\ 157\tfrac{1}{2}^\circ,\ 202\tfrac{1}{2}^\circ,\ 225^\circ,\ 247\tfrac{1}{2}^\circ,\ 270^\circ,\ 292\tfrac{1}{2}^\circ,\ 315^\circ, \text{ or } 337\tfrac{1}{2}^\circ.$$

164. Solve $\sin(x + 12^\circ)\cos(x - 12^\circ) = \cos 33^\circ \sin 57^\circ.$

$$\sin(x + 12^\circ)\cos(x - 12^\circ) = \cos 33^\circ \sin 57^\circ.$$

By [20],

$$\tfrac{1}{2}(\sin 2x + \sin 24^\circ) = \tfrac{1}{2}(\sin 90^\circ + \sin 24^\circ).$$

$$\sin 2x + \sin 24 = \sin 90^\circ + \sin 24^\circ.$$

$$\sin 2x = \sin 90^\circ.$$

$$\therefore 2x = 90^\circ + n\,360^\circ.$$

$$\therefore x = 45^\circ \text{ or } 225^\circ.$$

165. Solve $\sin^{-1}x + \sin^{-1}\frac{1}{2}x = 120°$.

$$\sin^{-1}x + \sin^{-1}\tfrac{1}{2}x = 120°.$$

$$\sin(\sin^{-1}x + \sin^{-1}\tfrac{1}{2}x) = \sin 120° = \cos 30° = \tfrac{1}{2}\sqrt{3}.$$

By [4], $\sin(\sin^{-1}x)\cos(\sin^{-1}\frac{1}{2}x) + \cos(\sin^{-1}x)\sin(\sin^{-1}\frac{1}{2}x) = \frac{1}{2}\sqrt{3}.$

$$x\sqrt{1-\tfrac{1}{4}x^2} + \tfrac{1}{2}x\sqrt{1-x^2} = \tfrac{1}{2}\sqrt{3}.$$

$$x\sqrt{4-x^2} + x\sqrt{1-x^2} = \sqrt{3}.$$

$$x\sqrt{4-x^2} = \sqrt{3} - x\sqrt{1-x^2}.$$

$$x^2(4-x^2) = 3 - 2\sqrt{3}\,x\sqrt{1-x^2} + x^2(1-x^2).$$

$$4x^2 - x^4 = 3 - 2\sqrt{3}\,x\sqrt{1-x^2} + x^2 - x^4.$$

$$3x^2 - 3 = -2\sqrt{3}\,x\sqrt{1-x^2}.$$

Square, $9x^4 - 18x^2 + 9 = 12x^2 - 12x^4.$

$$21x^4 - 30x^2 + 9 = 0.$$

$$7x^4 - 10x^2 + 3 = 0.$$

$$(x^2-1)(7x^2-3) = 0.$$

(i) $x^2 - 1 = 0.$

$$x^2 = 1.$$

$$\therefore x = \pm 1.$$

(ii) $7x^2 - 3 = 0.$

$$7x^2 = 3.$$

$$x^2 = \tfrac{3}{7}.$$

$$\therefore x = \pm\sqrt{\tfrac{3}{7}} = \pm\tfrac{1}{7}\sqrt{21}.$$

$$\therefore x = \pm 1 \text{ or } \pm\tfrac{1}{7}\sqrt{21}.$$

166. Solve $\tan^{-1}x + \tan^{-1}2x = \tan^{-1}3\sqrt{3}$.

$$\tan^{-1}x + \tan^{-1}2x = \tan^{-1}3\sqrt{3}.$$

$$\tan(\tan^{-1}x + \tan^{-1}2x) = 3\sqrt{3}.$$

By [6], $\dfrac{x+2x}{1-2x^2} = 3\sqrt{3}.$

$$\frac{3x}{1-2x^2} = 3\sqrt{3}.$$

$$\frac{x}{1-2x^2} = \sqrt{3}.$$

$$x = \sqrt{3} - 2\sqrt{3}\,x^2.$$

$$2\sqrt{3}\,x^2 + x - \sqrt{3} = 0.$$

$$x^2 + \tfrac{1}{6}\sqrt{3}\,x - \tfrac{1}{2} = 0.$$

$$(x - \tfrac{1}{3}\sqrt{3})(x + \tfrac{1}{2}\sqrt{3}) = 0.$$

$$\therefore x = \tfrac{1}{3}\sqrt{3} \text{ or } -\tfrac{1}{2}\sqrt{3}.$$

167. Solve $\sin^{-1}x + 2\cos^{-1}x = \frac{2}{3}\pi$.

$$\sin^{-1}x + 2\cos^{-1}x = \tfrac{2}{3}\pi.$$

Now $$\sin^{-1}x + \cos^{-1}x = \tfrac{1}{2}\pi.$$

Subtract, $$\cos^{-1}x = \tfrac{1}{6}\pi.$$

$$\therefore x = \tfrac{1}{2}\sqrt{3}.$$

168. Solve $\sin^{-1}x + 3\cos^{-1}x = 210°$.

$$\sin^{-1}x + 3\cos^{-1}x = 210°.$$

But $$\sin^{-1}x + \cos^{-1}x = 90°.$$

Subtract, $$2\cos^{-1}x = 120°.$$

$$\cos^{-1}x = 60°.$$

$$\therefore x = \tfrac{1}{2}.$$

169. Solve $\tan^{-1}x + 2\cot^{-1}x = 135°$.

$$\tan^{-1}x + 2\cot^{-1}x = 135°.$$

But $$\tan^{-1}x + \cot^{-1}x = 90°.$$

Subtract, $$\cot^{-1}x = 45°.$$

$$\therefore x = 1.$$

170. Solve $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}2x$.

$$\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}2x.$$

$$\tan\left[\tan^{-1}(x+1) + \tan^{-1}(x-1)\right] = 2x.$$

By [6], $$\frac{x+1+x-1}{1-(x+1)(x-1)} = 2x.$$

$$\frac{2x}{2-x^2} = 2x.$$

$$x = 2x - x^3.$$

$$x^3 - x = 0.$$

$$x(x^2-1) = 0.$$

$$\therefore x = 0 \text{ or } \pm 1.$$

171. Solve $\tan^{-1}\dfrac{x+2}{x+1} + \tan^{-1}\dfrac{x-2}{x-1} = \dfrac{3}{4}\pi$.

$$\tan^{-1}\frac{x+2}{x+1} + \tan^{-1}\frac{x-2}{x-1} = \frac{3}{4}\pi.$$

$$\tan\left[\tan^{-1}\frac{x+2}{x+1} + \tan^{-1}\frac{x-2}{x-1}\right] = \tan\tfrac{3}{4}\pi = -1.$$

By [6], $$\frac{\dfrac{x+2}{x+1} + \dfrac{x-2}{x-1}}{1 - \dfrac{x+2}{x+1} \times \dfrac{x-2}{x-1}} = -1.$$

$$\frac{(x-1)(x+2)+(x+1)(x-2)}{(x+1)(x-1)-(x+2)(x-2)}=-1.$$

$$\frac{2x^2-4}{3}=-1.$$

$$2x^2-4=-3.$$

$$2x^2=1.$$

$$x^2=\tfrac{1}{2}.$$

$$\therefore x=\pm\tfrac{1}{2}\sqrt{2}.$$

172. Solve $\tan^{-1}\dfrac{2x}{1-x^2}=60°$.

$$\tan^{-1}\frac{2x}{1-x^2}=60°.$$

By [14],

$$2\tan^{-1}x=60°.$$

$$\tan^{-1}x=30° \text{ or } 210°.$$

$$\therefore x=\tan 30° \text{ or } \tan 210°.$$

$$\therefore x=\tfrac{1}{3}\sqrt{3}.$$

173. Solve $\cos 2\theta\sec\theta+\sec\theta+1=0$.

$$\cos 2\theta\sec\theta+\sec\theta+1=0.$$

By [18],

$$\frac{\cos^2\theta-\sin^2\theta}{\cos\theta}+\sec\theta+1=0.$$

$$\frac{\cos^2\theta-\sin^2\theta}{\cos\theta}+\frac{1}{\cos\theta}+1=0.$$

$$\frac{2\cos^2\theta-1}{\cos\theta}+\frac{1}{\cos\theta}+1=0.$$

$$2\cos^2\theta-1+1+\cos\theta=0.$$

$$2\cos^2\theta+\cos\theta=0.$$

$$\cos\theta(2\cos\theta+1)=0.$$

(i)

$$\cos\theta=0.$$

$$\therefore \theta=90° \text{ or } 270°.$$

(ii)

$$2\cos\theta+1=0.$$

$$2\cos\theta=-1.$$

$$\cos\theta=-\tfrac{1}{2}.$$

$$\therefore \theta=120° \text{ or } 240°.$$

$$\therefore \theta=90°,\ 120°,\ 240°, \text{ or } 270°.$$

But the values $\theta=90°$ and $\theta=270°$ do not satisfy the given equation.

$$\therefore \theta=120° \text{ or } 240°.$$

174. Solve $\sin x \cos 2x \tan x \cot 2x \sec x \csc 2x = 1$.

$$\sin x \cos 2x \tan x \cot 2x \sec x \csc 2x = 1.$$

By [13], $$\cos 2x = \cos^2 x - \sin^2 x.$$

By [15], $$\cot 2x = \frac{\cot^2 x - 1}{2 \cot x}$$

$$= \frac{\dfrac{\cos^2 x}{\sin^2 x} - 1}{2\dfrac{\cos x}{\sin x}} = \frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x}.$$

By [8] and [13], $$\csc 2x = \frac{1}{\sin 2x} = \frac{1}{2 \sin x \cos x}.$$

Substitute,

$$\sin x (\cos^2 x - \sin^2 x)\left(\frac{\sin x}{\cos x}\right)\left(\frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x}\right)\left(\frac{1}{\cos x}\right)\left(\frac{1}{2 \sin x \cos x}\right) = 1.$$

$$\frac{(\cos^2 x - \sin^2 x)^2}{4 \cos^4 x} = 1.$$

$$\frac{(2 \cos^2 x - 1)^2}{4 \cos^4 x} = 1.$$

$$\frac{2 \cos^2 x - 1}{2 \cos^2 x} = \pm 1.$$

$$\therefore 2 \cos^2 x - 1 = \pm 2 \cos^2 x.$$

$$4 \cos^2 x = 1.$$

$$\cos^2 x = \tfrac{1}{4}.$$

$$\cos x = \pm \tfrac{1}{2}.$$

$$\therefore x = 60°, 120°, 240°, \text{ or } 300°.$$

175. Solve $\sin \frac{1}{2} x (\cos 2x - 2)(1 - \tan^2 x) = 0$.

$$\sin \tfrac{1}{2} x (\cos 2x - 2)(1 - \tan^2 x) = 0.$$

(i) $$\sin \tfrac{1}{2} x = 0.$$

$$\therefore \tfrac{1}{2} x = 0° \text{ or } 180°.$$

$$\therefore x = 0°.$$

(ii) $$\cos 2x - 2 = 0.$$

$$\cos 2x = 2.$$

$\therefore x$ is impossible.

(iii) $$1 - \tan^2 x = 0.$$

$$\tan^2 x = 1.$$

$$\tan x = \pm 1.$$

$$\therefore x = 45°, 135°, 225°, \text{ or } 315°.$$

$$\therefore x = 0°, 45°, 135°, 225°, \text{ or } 315°.$$

176. Solve $\sin 3x = \cos 2x - 1$.

$$\sin 3x = \cos 2x - 1.$$

By Prob. 18, Ex. XIV, and [13],

$$3\sin x - 4\sin^3 x = 1 - 2\sin^2 x - 1.$$

$$4\sin^3 x - 2\sin^2 x - 3\sin x = 0.$$

$$\sin x(4\sin^2 x - 2\sin x - 3) = 0.$$

(i) $$\sin x = 0.$$

$$\therefore x = 0° \text{ or } 180°.$$

(ii) $$4\sin^2 x - 2\sin x - 3 = 0.$$

$$16\sin^2 x - 8\sin x + 1 = 13.$$

$$4\sin x - 1 = \pm\sqrt{13}.$$

$$4\sin x = 1 \pm \sqrt{13}.$$

$$\sin x = \tfrac{1}{4}(1 \pm \sqrt{13}).$$

$$\sin x = 1.1514 \text{ or } -0.6514.$$

$$\therefore x = 220° 39' \text{ or } 319° 21'.$$

$$\therefore x = 0°, 180°, 220° 39', \text{ or } 319° 21'.$$

177. Solve $\tan x + \tan 2x = 0$.

$$\tan x + \tan 2x = 0.$$

By [14], $$\tan x + \frac{2\tan x}{1 - \tan^2 x} = 0.$$

$$\tan x - \tan^3 x + 2\tan x = 0.$$

$$\tan x(\tan^2 x - 3) = 0.$$

(i) $$\tan x = 0.$$

$$\therefore x = 0° \text{ or } 180°.$$

(ii) $$\tan^2 x - 3 = 0.$$

$$\tan^2 x = 3.$$

$$\tan x = \pm\sqrt{3}.$$

$$\therefore x = 60°, 120°, 240°, \text{ or } 300°.$$

$$\therefore x = 0°, 60°, 120°, 180°, 240°, \text{ or } 300°.$$

178. Solve $\sin 2\theta = \cos 3\theta$.

$$\sin 2\theta = \cos 3\theta.$$

By [12], and Prob. 19, Ex. XIV,

$$2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta.$$

$$4\cos^3\theta - 3\cos\theta - 2\sin\theta\cos\theta = 0.$$

$$\cos\theta(4\cos^2\theta - 3 - 2\sin\theta) = 0.$$

(i) $$\cos\theta = 0.$$

$$\therefore \theta = 90° \text{ or } 270°.$$

(ii)

$$\begin{aligned}
4\cos^2\theta - 3 - 2\sin\theta &= 0.\\
4 - 4\sin^2\theta - 3 - 2\sin\theta &= 0\\
4\sin^2\theta + 2\sin\theta &= 1.\\
16\sin^2\theta + 8\sin\theta + 1 &= 5.\\
4\sin\theta + 1 &= \pm\sqrt{5}.\\
4\sin\theta &= -1 \pm \sqrt{5}.\\
\sin\theta &= \tfrac{1}{4}(-1 \pm \sqrt{5})\\
&= \tfrac{1}{4}(-1 \pm 2.23606)\\
&= 0.3090 \text{ or } -0.8090.\\
\therefore \theta &= 18^\circ \text{ or } 162^\circ; \text{ or } 234^\circ \text{ or } 306^\circ.\\
\therefore \theta &= 18^\circ, 90^\circ, 162^\circ, 234^\circ, 270^\circ, \text{ or } 306^\circ.
\end{aligned}$$

179. Solve $(3 - 4\cos^2 x)\sin 2x = 0$.

$$(3 - 4\cos^2 x)\sin 2x = 0.$$

(i)

$$\begin{aligned}
3 - 4\cos^2 x &= 0.\\
4\cos^2 x &= 3.\\
2\cos x &= \pm\sqrt{3}.\\
\cos x &= \pm\tfrac{1}{2}\sqrt{3}.\\
\therefore x &= 30^\circ, 150^\circ, 210^\circ, \text{ or } 330^\circ.
\end{aligned}$$

(ii)

$$\begin{aligned}
\sin 2x &= 0.\\
\therefore 2x &= 0^\circ, 180^\circ, 360^\circ, \text{ or } 540^\circ\\
\therefore x &= 0^\circ, 90^\circ, 180^\circ, \text{ or } 270^\circ.\\
\therefore x &= 0^\circ, 30^\circ, 90^\circ, 150^\circ, 180^\circ, 210^\circ, 270^\circ, \text{ or } 330^\circ.
\end{aligned}$$

180. Solve $\sin x + \sin 2x + \sin 3x = 0$.

$$\begin{aligned}
\sin x + \sin 2x + \sin 3x &= 0.\\
(\sin 3x + \sin x) + \sin 2x &= 0.
\end{aligned}$$

By [20], $2\sin 2x\cos x + \sin 2x = 0.$

$$\sin 2x(2\cos x + 1) = 0.$$

(i)

$$\begin{aligned}
\sin 2x &= 0.\\
\therefore 2x &= 0^\circ, 180^\circ, 360^\circ, \text{ or } 540^\circ.\\
\therefore x &= 0^\circ, 90^\circ, 180^\circ, \text{ or } 270^\circ.
\end{aligned}$$

(ii)

$$\begin{aligned}
2\cos x + 1 &= 0.\\
2\cos x &= -1.\\
\cos x &= -\tfrac{1}{2}.\\
\therefore x &= 60^\circ, 120^\circ, 240^\circ, \text{ or } 300^\circ.\\
\therefore x &= 0^\circ, 60^\circ, 90^\circ, 120^\circ, 180^\circ, 240^\circ, 270^\circ, \text{ or } 300^\circ.
\end{aligned}$$

But the values $x = 60^\circ$ and $x = 300^\circ$ do not satisfy the given equation.

$$\therefore x = 0^\circ, 90^\circ, 120^\circ, 180^\circ, 240^\circ, \text{ or } 270^\circ.$$

181. Solve $\sin\theta + 2\sin 2\theta + 3\sin 3\theta = 0$.

$$\sin\theta + 2\sin 2\theta + 3\sin 3\theta = 0.$$

By [13], and Prob. 18, Ex. XIV,

$$\sin\theta + 4\sin\theta\cos\theta + 9\sin\theta - 12\sin^3\theta = 0.$$

$$10\sin\theta + 4\sin\theta\cos\theta - 12\sin^3\theta = 0.$$

$$\sin\theta(5 + 2\cos\theta - 6\sin^2\theta) = 0.$$

(i) $$\sin\theta = 0.$$

$$\therefore \theta = 0^\circ \text{ or } 180^\circ.$$

(ii) $$5 + 2\cos\theta - 6\sin^2\theta = 0.$$

$$5 + 2\cos\theta - 6 + 6\cos^2\theta = 0.$$

$$6\cos^2\theta + 2\cos\theta = 1.$$

$$36\cos^2\theta + 12\cos\theta + 1 = 7.$$

$$6\cos\theta + 1 = \pm\sqrt{7}.$$

$$6\cos\theta = -1 \pm \sqrt{7}.$$

$$\begin{aligned}\cos\theta &= \tfrac{1}{6}(-1 \pm \sqrt{7})\\ &= \tfrac{1}{6}(-1 \pm 2.64575)\\ &= 0.2743 \text{ or } -0.6076.\end{aligned}$$

$$\therefore \theta = 74^\circ\ 5' \text{ or } 285^\circ\ 55';\ \text{or } 127^\circ\ 25' \text{ or } 232^\circ\ 35'.$$

$$\therefore \theta = 0^\circ,\ 74^\circ\ 5',\ 127^\circ\ 25',\ 180^\circ,\ 232^\circ\ 35',\ \text{or } 285^\circ\ 55'.$$

182. Solve $\sin^2 x\cos^2 x - \cos^2 x - \sin^2 x + 1 = 0$.

$$\sin^2 x\cos^2 x - \cos^2 x - \sin^2 x + 1 = 0.$$

$$\sin^2 x\cos^2 x - (\cos^2 x + \sin^2 x) + 1 = 0.$$

$$\sin^2 x\cos^2 x - 1 + 1 = 0.$$

$$\sin^2 x\cos^2 x = 0.$$

$$\sin x\cos x = 0.$$

$$2\sin x\cos x = 0.$$

$$\sin 2x = 0.$$

$$\therefore 2x = 0^\circ,\ 180^\circ,\ 360^\circ,\ \text{or } 540^\circ.$$

$$\therefore x = 0^\circ,\ 90^\circ,\ 180^\circ,\ \text{or } 270^\circ.$$

183. Solve $\sin x + \sin 3x = \cos x - \cos 3x$.

$$\sin x + \sin 3x = \cos x - \cos 3x.$$

By Probs. 18 and 19, Ex. XIV,

$$\sin x + 3\sin x - 4\sin^3 x = \cos x - 4\cos^3 x + 3\cos x.$$

$$4\sin^3 x - 4\cos^3 x - 4\sin x + 4\cos x = 0.$$

$$(\sin^3 x - \cos^3 x) - (\sin x - \cos x) = 0.$$

$$(\sin x - \cos x)(\sin^2 x + \sin x\cos x + \cos^2 x) - (\sin x - \cos x) = 0.$$

$$(\sin x - \cos x)(1 + \sin x\cos x) - (\sin x - \cos x) = 0.$$

$$(\sin x - \cos x)(1 + \sin x\cos x - 1) = 0.$$

$$(\sin x - \cos x)(\sin x\cos x) = 0.$$

(i)
$$\sin x - \cos x = 0.$$
$$\sin x = \cos x.$$
$$\therefore x = 45° \text{ or } 225°.$$

(ii)
$$\sin x = 0.$$
$$\therefore x = 0° \text{ or } 180°.$$

(iii)
$$\cos x = 0.$$
$$\therefore x = 90° \text{ or } 270°.$$
$$\therefore x = 0°, 45°, 90°, 180°, 225°, \text{ or } 270°.$$

184. Solve $(1 - \sqrt{1 - \tan^2 x}) \cos 2x \operatorname{vers} 3x = 0.$

$$(1 - \sqrt{1 - \tan^2 x}) \cos 2x \operatorname{vers} 3x = 0.$$

(i)
$$1 - \sqrt{1 - \tan^2 x} = 0.$$
$$1 = \sqrt{1 - \tan^2 x}.$$
$$1 = 1 - \tan^2 x.$$
$$\tan^2 x = 0.$$
$$\tan x = 0.$$
$$\therefore x = 0° \text{ or } 180°.$$

(ii)
$$\cos 2x = 0.$$
$$\therefore 2x = 90°, 270°, 450°, \text{ or } 630°.$$
$$\therefore x = 45°, 135°, 225°, \text{ or } 315°.$$

(iii)
$$\operatorname{vers} 3x = 0.$$
$$1 - \cos 3x = 0.$$
$$\cos 3x = 1.$$
$$\therefore 3x = 0°, 360°, \text{ or } 720°.$$
$$\therefore x = 0°, 120°, \text{ or } 240°.$$
$$\therefore x = 0°, 45°, 120°, 135°, 180°, 225°, 240°, \text{ or } 315°.$$

185. Solve $\tan(\theta + 45°) = 8 \tan \theta.$

$$\tan(\theta + 45°) = 8 \tan \theta.$$

By [6],
$$\frac{\tan \theta + \tan 45°}{1 - \tan \theta \tan 45°} = 8 \tan \theta.$$
$$\frac{\tan \theta + 1}{1 - \tan \theta} = 8 \tan \theta.$$
$$\tan \theta + 1 = 8 \tan \theta - 8 \tan^2 \theta.$$
$$8 \tan^2 \theta - 7 \tan \theta = -1.$$
$$256 \tan^2 \theta - (\,) + 49 = 17.$$
$$16 \tan \theta - 7 = \pm \sqrt{17}.$$
$$16 \tan \theta = 7 \pm \sqrt{17}.$$

$$\tan\theta = \tfrac{1}{16}(7 \pm \sqrt{17})$$
$$= \tfrac{1}{16}(7 \pm 4.1231)$$
$$= 0.6952 \text{ or } 0.1798.$$
$$\therefore \theta = 34^\circ\, 48' \text{ or } 214^\circ\, 48';\ \text{or } 10^\circ\, 12' \text{ or } 190^\circ\, 12'.$$
$$\therefore \theta = 10^\circ\, 12',\ 34^\circ\, 48',\ 190^\circ\, 12', \text{ or } 214^\circ\, 48'.$$

186. Solve $\sin(x - 30^\circ) = \frac{1}{2}\sqrt{3}\sin x$.

$$\sin(x - 30^\circ) = \tfrac{1}{2}\sqrt{3}\sin x.$$

By [8], $$\sin x\cos 30^\circ - \cos x\sin 30^\circ = \tfrac{1}{2}\sqrt{3}\sin x.$$
$$\tfrac{1}{2}\sqrt{3}\sin x - \tfrac{1}{2}\cos x = \tfrac{1}{2}\sqrt{3}\sin x.$$
$$-\tfrac{1}{2}\cos x = 0.$$
$$\cos x = 0.$$
$$\therefore x = 90^\circ \text{ or } 270^\circ.$$

187. Solve $\tan(\theta + 45^\circ)\tan\theta = 2$.

$$\tan(\theta + 45^\circ)\tan\theta = 2.$$

By [6], $$\frac{\tan\theta + \tan 45^\circ}{1 - \tan\theta\tan 45^\circ} \times \tan\theta = 2.$$
$$\frac{(\tan\theta + 1)\tan\theta}{1 - \tan\theta} = 2.$$
$$\frac{\tan^2\theta + \tan\theta}{1 - \tan\theta} = 2.$$
$$\tan^2\theta + \tan\theta = 2 - 2\tan\theta.$$
$$\tan^2\theta + 3\tan\theta = 2.$$
$$4\tan^2\theta + (\ ) + 9 = 17.$$
$$2\tan\theta + 3 = \pm\sqrt{17}.$$
$$2\tan\theta = -3 \pm \sqrt{17}.$$
$$\tan\theta = \tfrac{1}{2}(-3 \pm \sqrt{17})$$
$$= \tfrac{1}{2}(-3 \pm 4.1231)$$
$$= 0.5616 \text{ or } -3.5616.$$
$$\therefore \theta = 29^\circ\, 19' \text{ or } 209^\circ\, 19';\ \text{or } 105^\circ\, 41' \text{ or } 285^\circ\, 41'.$$
$$\therefore \theta = 29^\circ\, 19',\ 105^\circ\, 41',\ 209^\circ\, 19', \text{ or } 285^\circ\, 41'.$$

188. Solve $\sin^{-1}\frac{1}{2}x = 30^\circ$.

$$\sin^{-1}\tfrac{1}{2}x = 30^\circ.$$
$$\sin(\sin^{-1}\tfrac{1}{2}x) = \sin 30^\circ.$$
$$\tfrac{1}{2}x = \tfrac{1}{2}.$$
$$\therefore x = 1.$$

189. Solve for x and y the system

$$x\sin\alpha + y\sin\beta = a, \quad (1)$$
$$x\cos\alpha + y\cos\beta = b. \quad (2)$$

Multiply (1) by $\cos\alpha$, $x\sin\alpha\cos\alpha + y\sin\beta\cos\alpha = a\cos\alpha.$ (3)
Multiply (2) by $\sin\alpha$, $x\sin\alpha\cos\alpha + y\cos\beta\sin\alpha = b\sin\alpha.$ (4)
Subtract (4) from (3), $y(\sin\beta\cos\alpha - \cos\beta\sin\alpha) = a\cos\alpha - b\sin\alpha.$
By [8], $y\sin(\beta-\alpha) = a\cos\alpha - b\sin\alpha.$

$$\therefore y = \frac{a\cos\alpha - b\sin\alpha}{\sin(\beta-\alpha)}.$$

Multiply (1) by $\cos\beta$, $x\sin\alpha\cos\beta + y\sin\beta\cos\beta = a\cos\beta.$ (5)
Multiply (2) by $\sin\beta$, $x\cos\alpha\sin\beta + y\sin\beta\cos\beta = b\sin\beta.$ (6)
Subtract (5) from (6), $x(\cos\alpha\sin\beta - \sin\alpha\cos\beta) = b\sin\beta - a\cos\beta.$
By [8], $x\sin(\beta-\alpha) = b\sin\beta - a\cos\beta.$

$$\therefore x = \frac{b\sin\beta - a\cos\beta}{\sin(\beta-\alpha)}.$$

190. Solve for x and y the system

$$\sin x + \sin y = a, \qquad (1)$$
$$\cos x + \cos y = b. \qquad (2)$$

Transform (1) by [20], $2\sin\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y) = a.$ (3)
Transform (1) by [22], $2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y) = b.$ (4)

Divide (3) by (4), $\tan\tfrac{1}{2}(x+y) = \dfrac{a}{b}.$ (5)

By [2], $\dfrac{\sin\frac{1}{2}(x+y)}{\cos\frac{1}{2}(x+y)} = \dfrac{a}{b}.$ (6)

$$\frac{\sin\frac{1}{2}(x+y)}{\sqrt{1-\sin^2\frac{1}{2}(x+y)}} = \frac{a}{b}.$$

Square, $\dfrac{\sin^2\frac{1}{2}(x+y)}{1-\sin^2\frac{1}{2}(x+y)} = \dfrac{a^2}{b^2}.$

$$b^2\sin^2\tfrac{1}{2}(x+y) = a^2 - a^2\sin^2\tfrac{1}{2}(x+y).$$
$$(a^2+b^2)\sin^2\tfrac{1}{2}(x+y) = a^2.$$
$$\sin^2\tfrac{1}{2}(x+y) = \frac{a^2}{a^2+b^2}.$$
$$\sin\tfrac{1}{2}(x+y) = \frac{a}{\sqrt{a^2+b^2}}. \qquad (7)$$

Substitute in (3) the value of $\sin\tfrac{1}{2}(x+y)$,

$$\frac{2a}{\sqrt{a^2+b^2}}\cos\tfrac{1}{2}(x-y) = a.$$
$$\cos\tfrac{1}{2}(x-y) = \tfrac{1}{2}\sqrt{a^2+b^2}. \qquad (8)$$

(5), $x+y = 2\tan^{-1}\dfrac{a}{b}.$ (9)

$$x-y = 2\cos^{-1}\tfrac{1}{2}\sqrt{a^2+b^2}. \qquad (10)$$

Add (9) and (10), and divide by 2,

$$x = \tan^{-1}\frac{a}{b} + \cos^{-1}\tfrac{1}{2}\sqrt{a^2+b^2}.$$

Subtract (10) from (9), and divide by 2,

$$y = \tan^{-1}\frac{a}{b} - \cos^{-1}\tfrac{1}{2}\sqrt{a^2+b^2}.$$

191. Solve for r and θ the system

$$r\sin\theta = a, \qquad (1)$$

$$r\cos\theta = b. \qquad (2)$$

Divide (1) by (2), $\tan\theta = \frac{a}{b}$. (3)

$$\therefore \theta = \tan^{-1}\frac{a}{b}. \qquad (4)$$

Square (1), $r^2\sin^2\theta = a^2$. (5)

Square (2), $r^2\cos^2\theta = b^2$. (6)

Add (5) and (6), $r^2(\sin^2\theta + \cos^2\theta) = a^2 + b^2$.

$$\therefore r^2 = a^2 + b^2.$$

$$\therefore r = \sqrt{a^2+b^2}.$$

192. Solve for r and θ the system

$$r\sin(\theta+\alpha) = a, \qquad (1)$$

$$r\cos(\theta+\beta) = b. \qquad (2)$$

Expand (1) by [4],

$$r\sin\theta\cos\alpha + r\cos\theta\sin\alpha = a. \qquad (3)$$

Expand (2) by [5],

$$r\cos\theta\cos\beta - r\sin\theta\sin\beta = b. \qquad (4)$$

Multiply (3) by $\cos\beta$,

$$r\sin\theta\cos\alpha\cos\beta + r\cos\theta\sin\alpha\cos\beta = a\cos\beta. \qquad (5)$$

Multiply (4) by $\sin\alpha$,

$$-r\sin\theta\sin\alpha\sin\beta + r\cos\theta\sin\alpha\cos\beta = b\sin\alpha. \qquad (6)$$

Subtract (6) from (5),

$$r\sin\theta(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = a\cos\beta - b\sin\alpha.$$

By [9], $r\sin\theta\cos(\alpha-\beta) = a\cos\beta - b\sin\alpha$.

$$\therefore r\sin\theta = \frac{a\cos\beta - b\sin\alpha}{\cos(\alpha-\beta)}. \qquad (7)$$

Multiply (3) by $\sin\beta$,

$$r\sin\theta\cos\alpha\sin\beta + r\cos\theta\sin\alpha\sin\beta = a\sin\beta. \qquad (8)$$

Multiply (4) by $\cos\alpha$,

$$-r\sin\theta\cos\alpha\sin\beta + r\cos\theta\cos\alpha\cos\beta = b\cos\alpha. \qquad (9)$$

Add (8) and (9),

$$r\cos\theta(\sin\alpha\sin\beta+\cos\alpha\cos\beta)=a\sin\beta+b\cos\alpha.$$

By [9],

$$r\cos\theta\cos(\alpha-\beta)=a\sin\beta+b\cos\alpha.$$

$$\therefore r\cos\theta=\frac{a\sin\beta+b\cos\alpha}{\cos(\alpha-\beta)}. \qquad (10)$$

Divide (7) by (10),

$$\tan\theta=\frac{a\cos\beta-b\sin\alpha}{a\sin\beta+b\cos\alpha}. \qquad (11)$$

$$\therefore \theta=\tan^{-1}\frac{a\cos\beta-b\sin\alpha}{a\sin\beta+b\cos\alpha}. \qquad (12)$$

From (1) and (2),

$$r=\frac{a}{\sin(\theta+\alpha)}=\frac{b}{\cos(\theta+\beta)}. \qquad (13)$$

Substitute in (13) the value of θ found in (12),

$$r=\frac{a}{\sin\left(\tan^{-1}\frac{a\cos\beta-b\sin\alpha}{a\sin\beta+b\cos\alpha}+\alpha\right)}=\frac{b}{\cos\left(\tan^{-1}\frac{a\cos\beta-b\sin\alpha}{a\sin\beta+b\cos\alpha}+\beta\right)}.$$

193. Solve for r, θ, and ϕ the system

$$r\cos\phi\sin\theta=a, \qquad (1)$$

$$r\cos\phi\cos\theta=b, \qquad (2)$$

$$r\sin\phi=c. \qquad (3)$$

Divide (1) by (2),

$$\tan\theta=\frac{a}{b}. \qquad (4)$$

$$\therefore \theta=\tan^{-1}\frac{a}{b}. \qquad (5)$$

Square (1), $$r^2\cos^2\phi\sin^2\theta=a^2. \qquad (6)$$

Square (2), $$r^2\cos^2\phi\cos^2\theta=b^2. \qquad (7)$$

Add (6) and (7), $$r^2\cos^2\phi(\sin^2\theta+\cos^2\theta)=a^2+b^2.$$

$$\therefore r^2\cos^2\phi=a^2+b^2. \qquad (8)$$

$$\therefore r\cos\phi=\sqrt{a^2+b^2}. \qquad (9)$$

Divide (3) by (9),

$$\tan\phi=\frac{c}{\sqrt{a^2+b^2}}.$$

$$\therefore \phi=\tan^{-1}\frac{c}{\sqrt{a^2+b^2}}. \qquad (10)$$

Square (3), $$r^2\sin^2\phi=c^2. \qquad (11)$$

Add (11) and (8), $$r^2(\sin^2\phi+\cos^2\phi)=a^2+b^2+c^2.$$

$$\therefore r^2=a^2+b^2+c^2.$$

$$\therefore r=\sqrt{a^2+b^2+c^2}.$$

194. Solve for x and y the system

$$x\sin 21^\circ + y\cos 44^\circ = 179.70, \tag{1}$$

$$x\cos 21^\circ + y\sin 44^\circ = 232.30. \tag{2}$$

$$0.3584\,x + 0.7193\,y = 179.70. \tag{3}$$

$$0.9336\,x + 0.6947\,y = 232.30. \tag{4}$$

$$3584\,x + 7193\,y = 1797000. \tag{5}$$

$$9336\,x + 6947\,y = 2323000. \tag{6}$$

Multiply (5) by 1167, $4182528\,x + 8394231\,y = 2097099000.$ (7)

Multiply (6) by 448, $4182528\,x + 3112256\,y = 1040704000.$ (8)

Subtract (8) from (7), $5281975\,y = 1056395000.$

$$\therefore y = 200.$$

Substitute the value of y in (5),

$$3584\,x + 1438600 = 1797000.$$

$$3584\,x = 358400.$$

$$\therefore x = 100.$$

195. Solve for x and y the system

$$\sin x - \sin y = 0.7038, \tag{1}$$

$$\cos x - \cos y = -\,0.7245. \tag{2}$$

Expand (1) by [21],

$$2\cos\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y) = 0.7038. \tag{3}$$

Expand (2) by [23],

$$-\,2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y) = -\,0.7245. \tag{4}$$

Divide (4) by (3), $\tan\tfrac{1}{2}(x+y) = \dfrac{0.7245}{0.7038}.$ (5)

$$\frac{\sin\frac{1}{2}(x+y)}{\cos\frac{1}{2}(x+y)} = \frac{0.7245}{0.7038}.$$

$$\frac{\sin\frac{1}{2}(x+y)}{\sqrt{1-\sin^2\frac{1}{2}(x+y)}} = \frac{0.7245}{0.7038}.$$

$$\frac{\sin^2\frac{1}{2}(x+y)}{1-\sin^2\frac{1}{2}(x+y)} = \frac{0.7245^2}{0.7038^2}.$$

$$0.7038^2\sin^2\tfrac{1}{2}(x+y) = 0.7245^2 - 0.7245^2\sin^2\tfrac{1}{2}(x+y).$$

$$(0.7038^2 + 0.7245^2)\sin^2\tfrac{1}{2}(x+y) = 0.7245^2.$$

$$\sqrt{0.7038^2 + 0.7245^2}\,\sin\tfrac{1}{2}(x+y) = \pm 0.7245.$$

$$\sin\tfrac{1}{2}(x+y) = \frac{\pm\,0.7245}{\sqrt{0.7038^2 + 0.7245^2}}.$$

$$\sin\tfrac{1}{2}(x+y) = \frac{\pm\,0.7245}{1.0100667}.$$

$$\sin\tfrac{1}{2}(x+y) = \pm\,0.7172. \tag{6}$$

Substitute in (4) the value of $\sin \frac{1}{2}(x+y)$,

$$2(\pm 0.7172)\sin \tfrac{1}{2}(x-y) = 0.7245.$$

$$\sin \tfrac{1}{2}(x-y) = \frac{0.7245}{\pm 1.4344}.$$

$$\sin \tfrac{1}{2}(x-y) = \pm 0.5051. \tag{7}$$

From (6), $\frac{1}{2}(x+y) = 45° 50', 134° 10', 225° 50',$ or $314° 10'$. (8)

From (7), $\frac{1}{2}(x-y) = 30° 20', 149° 40', 210° 20',$ or $329° 40'$. (9)

Add (8) and (9), $x = 76° 10', 283° 50', 436° 10',$ or $643° 50'$.

Subtract (9) from (8), $y = 15° 30', 344° 30', 15° 30',$ or $344° 30'$.

$$\therefore x = 76° 10' \text{ or } 283° 50',$$
$$y = 15° 30' \text{ or } 344° 30'.$$

But the values $x = 283° 50'$, $y = 344° 30'$ do not satisfy the given system of equations.

$$\therefore x = 76° 10'; \; y = 15° 30'.$$

196. Solve for r and θ the system

$$r \sin \theta = 92.344, \tag{1}$$
$$r \cos \theta = 205.309. \tag{2}$$

Divide (1) by (2),

$$\tan \theta = \frac{92.344}{205.309} = 0.4498.$$

$$\therefore \theta = 24° 13' \text{ or } 204° 13'. \tag{3}$$

From (1),

$$r = \frac{92.344}{\sin \theta} = \frac{92.344}{\pm 0.4102} = \pm 225.12.$$

$$\therefore \theta = 24° 13', \quad r = 225.12;$$

or

$$\theta = 204° 13', \; r = -225.12.$$

197. Solve for r and θ the system

$$r \sin(\theta - 19° 18') = 59.4034, \tag{1}$$
$$r \cos(\theta - 30° 54') = 147.9347. \tag{2}$$

Expand (1) by [8],

$$r \sin \theta \cos 19° 18' - r \cos \theta \sin 19° 18' = 59.4034. \tag{3}$$

Expand (2) by [9],

$$r \cos \theta \cos 30° 54' + r \sin \theta \sin 30° 54' = 147.9347. \tag{4}$$

Multiply (3) by $\sin 30° 54'$,

$$r \sin \theta \sin 30° 54' \cos 19° 18' - r \cos \theta \sin 19° 18' \sin 30° 54'$$
$$= 59.4034 \sin 30° 54'. \tag{5}$$

Multiply (4) by $\cos 19° 18'$,

$$r \sin \theta \sin 30° 54' \cos 19° 18' + r \cos \theta \cos 19° 18' \cos 30° 54'$$
$$= 147.9347 \cos 19° 18'. \tag{6}$$

Subtract (5) from (6),

$$r\cos\theta(\cos 19^\circ 18'\cos 30^\circ 54' + \sin 19^\circ 18'\sin 30^\circ 54')$$
$$= 147.9347\cos 19^\circ 18' - 59.4034\sin 30^\circ 54'.$$

By [9], $r\cos\theta\cos(30^\circ 54' - 19^\circ 18')$

$$= 147.9347\cos 19^\circ 18' - 59.4034\sin 30^\circ 54'.$$

$$\therefore r\cos\theta = \frac{147.9347\cos 19^\circ 18' - 59.4034\sin 30^\circ 54'}{\cos 10^\circ 36'}. \qquad (7)$$

Multiply (3) by $\cos 30^\circ 54'$,

$$r\sin\theta\cos 30^\circ 54'\cos 19^\circ 18' - r\cos\theta\cos 30^\circ 54'\sin 19^\circ 18'$$
$$= 59.4034\cos 30^\circ 54'. \qquad (8)$$

Multiply (4) by $\sin 19^\circ 18'$,

$$r\sin\theta\sin 30^\circ 54'\sin 19^\circ 18' + r\cos\theta\cos 30^\circ 54'\sin 19^\circ 18'$$
$$= 147.9347\sin 19^\circ 18'. \qquad (9)$$

Add (8) and (9), $r\sin\theta(\cos 30^\circ 54'\cos 19^\circ 18' + \sin 30^\circ 54'\sin 19^\circ 18')$

$$= 147.9347\sin 19^\circ 18' + 59.4034\cos 30^\circ 54'.$$

By [9], $r\sin\theta\cos(30^\circ 54' - 19^\circ 18')$

$$= 147.9347\sin 19^\circ 18' + 59.4034\cos 30^\circ 54'.$$

$$\therefore r\sin\theta = \frac{147.9347\sin 19^\circ 18' + 59.4034\cos 30^\circ 54'}{\cos 10^\circ 36'}. \qquad (10)$$

Divide (10) by (7), $\tan\theta = \dfrac{147.9347\sin 19^\circ 18' + 59.4034\cos 30^\circ 54'}{147.9347\cos 19^\circ 18' - 59.4034\sin 30^\circ 54'}$

$$= \frac{147.9347 \times 0.3305 + 59.4034 \times 0.8581}{147.9347 \times 0.9438 - 59.4034 \times 0.5135}$$
$$= \frac{48.89241835 + 50.97405754}{139.62076986 - 30.50364590}$$
$$= \frac{99.86647589}{109.11712396}$$
$$= 0.9152.$$
$$\therefore \theta = 42^\circ 28' \text{ or } 222^\circ 28'. \qquad (11)$$

Substitute in (1) the value of θ found in (11),

$$r\sin(42^\circ 28' - 19^\circ 18') = 59.4034 \text{ or } r\sin(222^\circ 28' - 19^\circ 18') = 59.4034.$$
$$r\sin 23^\circ 10' = 59.4034 \text{ or } r\sin 203^\circ 10' = 59.4034.$$
$$r = \frac{59.4034}{\sin 23^\circ 10'} \text{ or } r = \frac{59.4034}{\sin 203^\circ 10'}.$$
$$r = \frac{59.4034}{0.3934} \quad \text{or } r = \frac{59.4034}{-0.3934}.$$
$$r = 151 \quad \text{or } r = -151.$$
$$\therefore \theta = 42^\circ 28', \quad r = 151;$$

or

$$\theta = 222^\circ 28', \quad r = -151.$$

198. Solve for r, θ, and ϕ the system

$$r\cos\phi\cos\theta = -46.7654, \tag{1}$$
$$r\sin\phi\cos\theta = 81, \tag{2}$$
$$r\sin\theta = -54. \tag{3}$$

Divide (2) by (1), $\quad \tan\phi = \dfrac{81}{-46.7654} = -1.7320.$

$$\therefore \phi = 120^\circ \text{ or } 300^\circ.$$
$$\therefore \cos\phi = \mp\tfrac{1}{2}. \tag{4}$$

Substitute in (1) the value of $\cos\phi$ found in (4),

$$r(\mp\tfrac{1}{2})\cos\theta = -46.7654.$$
$$r\cos\theta = \pm 93.5308. \tag{5}$$

Divide (3) by (5), $\quad \tan\theta = \dfrac{-54}{\pm 93.5308} = \mp 0.5773.$

$$\therefore \theta = 150^\circ \text{ or } 330^\circ;\ \text{or } 30^\circ \text{ or } 210^\circ.$$
$$\therefore \sin\theta = \tfrac{1}{2} \text{ or } -\tfrac{1}{2};\ \tfrac{1}{2} \text{ or } -\tfrac{1}{2}. \tag{6}$$

Substitute in (3) the value of $\sin\theta$ found in (6),

$$r(\pm\tfrac{1}{2}) = -54.$$
$$\therefore r = \mp 108.$$

$\therefore r = \quad 108,\ \phi = 120^\circ,\ \theta = 330^\circ;\quad r = \quad 108,\ \phi = 300^\circ,\ \theta = 210^\circ;$
$r = -108,\ \phi = 120^\circ,\ \theta = 150^\circ;$ or $r = -108,\ \phi = 300^\circ,\ \theta = 30^\circ.$

199. Eliminate θ from the system

$$x = r(\theta - \sin\theta), \tag{1}$$
$$y = r(1 - \cos\theta). \tag{2}$$

Now $\quad 1 - \cos\theta = \text{vers}\,\theta. \quad (3)$

Substitute in (2) the value of $1 - \cos\theta$ found in (3),

$$y = r\,\text{vers}\,\theta.$$
$$\text{vers}\,\theta = \frac{y}{r}.$$
$$\therefore \theta = \text{vers}^{-1}\frac{y}{r}. \tag{4}$$

From (2), $\quad 1 - \cos\theta = \dfrac{y}{r}.$

$$1 - \sqrt{1 - \sin^2\theta} = \frac{y}{r}.$$
$$1 - \frac{y}{r} = \sqrt{1 - \sin^2\theta}.$$
$$1 - \frac{2y}{r} + \frac{y^2}{r^2} = 1 - \sin^2\theta.$$

$$\sin^2\theta = \frac{2y}{r} - \frac{y^2}{r^2} = \frac{2ry - y^2}{r^2}.$$

$$\therefore \sin\theta = \pm\frac{1}{r}\sqrt{2ry - y^2}. \qquad (5)$$

Substitute in (1) the value of θ found in (4), and the value of $\sin\theta$ found in (5),

$$x = r \operatorname{vers}^{-1}\frac{y}{r} \mp \sqrt{2ry - y^2},$$

$$\text{or } x = \pm\sqrt{2ry - y^2} + r\operatorname{vers}^{-1}\frac{y}{r}.$$

EXERCISE XXV. PAGE 121.

1. Given $\log_{10}2 = 0.30103$, $\log_{10}3 = 0.47712$, $\log_{10}7 = 0.84510$; find $\log_{10}6$, $\log_{10}14$, $\log_{10}21$, $\log_{10}4$, $\log_{10}12$, $\log_{10}5$, $\log_{10}\frac{1}{2}$, $\log_{10}\frac{1}{4}$, $\log_{10}\frac{7}{9}$, $\log_{10}\frac{21}{20}$.

$$\begin{aligned} \log_{10}6 &= \log_{10}2 + \log_{10}3 \\ \log_{10}2 &= 0.30103 \\ \log_{10}3 &= \underline{0.47712} \\ \therefore \log_{10}6 &= 0.77815 \end{aligned}$$

$$\begin{aligned} \log_{10}14 &= \log_{10}2 + \log_{10}7 \\ \log_{10}2 &= 0.30103 \\ \log_{10}7 &= \underline{0.84510} \\ \therefore \log_{10}14 &= 1.14613 \end{aligned}$$

$$\begin{aligned} \log_{10}21 &= \log_{10}3 + \log_{10}7 \\ \log_{10}3 &= 0.47712 \\ \log_{10}7 &= \underline{0.84510} \\ \therefore \log_{10}21 &= 1.32222 \end{aligned}$$

$$\begin{aligned} \log_{10}4 &= 2\log_{10}2 \\ \log_{10}2 &= 0.30103 \\ &\underline{2} \\ \therefore \log_{10}4 &= 0.60206 \end{aligned}$$

$$\begin{aligned} \log_{10}12 &= \log_{10}3 + \log_{10}4 \\ \log_{10}3 &= 0.47712 \\ \log_{10}4 &= \underline{0.60206} \\ \therefore \log_{10}12 &= 1.07918 \end{aligned}$$

$$\begin{aligned} \log_{10}5 &= \log_{10}10 - \log_{10}2 \\ \log_{10}10 &= 1.00000 \\ \log_{10}2 &= \underline{0.30103} \\ \therefore \log_{10}5 &= 0.69897 \end{aligned}$$

$$\begin{aligned} \log_{10}\tfrac{1}{2} &= \log_{10}1 - \log_{10}2 \\ \log_{10}1 &= 0.00000 \\ \log_{10}2 &= \underline{0.30103} \\ \therefore \log_{10}\tfrac{1}{2} &= \bar{1}.69897 \end{aligned}$$

$$\begin{aligned} \log_{10}\tfrac{1}{4} &= 2\log_{10}\tfrac{1}{2} \\ \log_{10}\tfrac{1}{2} &= \bar{1}.69897 \\ &\underline{2} \\ \therefore \log_{10}\tfrac{1}{4} &= \bar{1}.39794 \end{aligned}$$

$$\begin{aligned} \log_{10}\tfrac{7}{9} &= \log_{10}7 - \log_{10}9 \\ \log_{10}7 &= 0.84510 \\ \log_{10}3^2 &= \underline{0.95424} \\ \therefore \log_{10}\tfrac{7}{9} &= \bar{1}.89086 \end{aligned}$$

$$\begin{aligned} \log_{10}\tfrac{21}{20} &= \log_{10}21 - \log_{10}20 \\ \log_{10}21 &= 1.32222 \\ \log_{10}(10\times 2) &= \underline{1.30103} \\ \therefore \log_{10}\tfrac{21}{20} &= 0.02119 \end{aligned}$$

2. With the data of Example 1; find

$$\log_2 10,\ \log_2 5,\ \log_3 5,\ \log_7 \tfrac{1}{2},\ \log_5 \tfrac{9}{343}.$$

$$\log_2 10 = \frac{\log_{10} 10}{\log_{10} 2} = \frac{1}{0.30103} = 3.3219.$$

$$\log_2 5 = \frac{\log_{10} 5}{\log_{10} 2} = \frac{0.69897}{0.30103} = 2.3219.$$

$$\log_3 5 = \frac{\log_{10} 5}{\log_{10} 3} = \frac{0.69897}{0.47712} = 1.4650.$$

$$\log_7 \tfrac{1}{2} = \frac{\log_{10} \frac{1}{2}}{\log_{10} 7} = \frac{-0.30103}{0.84510} = -0.3562.$$

$$\begin{aligned}\log_5 \tfrac{9}{343} &= 2 \log_5 3 - 3 \log_5 7\\ &= \frac{2 \log_{10} 3 - 3 \log_{10} 7}{\log_{10} 5}\\ &= \frac{0.95424 - 2.53530}{0.69897}\\ &= -2.2620.\end{aligned}$$

3. Given $\log_{10} e = 0.43429$; find

$\log_e 2$, $\log_e 3$, $\log_e 5$, $\log_e 7$, $\log_e 8$, $\log_e 9$, $\log_e \frac{2}{3}$, $\log_e \frac{4}{5}$, $\log_e \frac{35}{27}$, $\log_e \frac{7}{60}$.

$$\log_e 2 = \frac{\log_{10} 2}{\log_{10} e} = \frac{0.30103}{0.43429} = 0.69315.$$

$$\log_e 3 = \frac{\log_{10} 3}{\log_{10} e} = \frac{0.47712}{0.43429} = 1.09862.$$

$$\log_e 5 = \frac{\log_{10} 5}{\log_{10} e} = \frac{0.69897}{0.43429} = 1.60945.$$

$$\log_e 7 = \frac{\log_{10} 7}{\log_{10} e} = \frac{0.84510}{0.43429} = 1.94596.$$

$$\log_e 8 = \frac{3 \times \log_{10} 2}{\log_{10} e} = \frac{0.90309}{0.43429} = 2.07943.$$

$$\log_e 9 = \frac{2 \times \log_{10} 3}{\log_{10} e} = \frac{0.95424}{0.43429} = 2.19724.$$

$$\begin{aligned}\log_e \tfrac{2}{3} &= \log_e 2 - \log_e 3 = -0.40547.\\ \log_e \tfrac{4}{5} &= 2 \log_e 2 - \log_e 5 = -0.22315.\\ \log_e \tfrac{35}{27} &= \log_e 5 + \log_e 7 - 3 \log_e 3 = 0.25952.\\ \log_e \tfrac{7}{60} &= \log_e 7 - (\log_e 5 + \log_e 3 + 2 \log_e 2)\\ &= -2.14845.\end{aligned}$$

4. **Find x from the equations $5^x = 12$, $16^x = 10$, $27^x = 4$.**

$$5^x = 12. \quad \therefore x \log_{10} 5 = \log_{10} 12.$$

$$x = \frac{\log_{10} 12}{\log_{10} 5} = \frac{1.07918}{0.69897} = 1.54396.$$

$$16^x = 10. \quad \therefore x \log_{10} 16 = \log_{10} 10.$$

$$x = \frac{\log_{10} 10}{\log_{10} 16} = \frac{1.00000}{1.20412} = 0.83048.$$

$$27^x = 4. \quad \therefore x \log_{10} 27 = \log_{10} 4.$$

$$x = \frac{\log_{10} 4}{\log_{10} 27} = \frac{0.60206}{1.43136} = 0.42062.$$

Exercise XXVI. Page 126.

1. Calculate to five places of decimals $\log_e 3$.

In the case of $\log_e 3$ the calculation is carried out below to ten places, for use in Example 4.

In the formula

$$\log_e \frac{z+1}{z} = 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \cdots\right),$$

let $z = \frac{1}{2}$.

Then $\frac{z+1}{z} = 3$, and $2z + 1 = 2$;

and $\log_e 3 = 2\left(\frac{1}{2} + \frac{1}{3 \times 2^3} + \frac{1}{5 \times 2^5} + \cdots\right)$.

$$\begin{array}{r|rcrcl}
2 & 2.00000000000 & & & & \\
4 & 1.00000000000 & \div & 1 & = & 1.00000000000 \\
4 & 0.25000000000 & \div & 3 & = & 0.08333333333 \\
4 & 0.06250000000 & \div & 5 & = & 0.01250000000 \\
4 & 0.01562500000 & \div & 7 & = & 0.00223214286 \\
4 & 0.00390625000 & \div & 9 & = & 0.00043402778 \\
4 & 0.00097656250 & \div & 11 & = & 0.00008877841 \\
4 & 0.00024414062 & \div & 13 & = & 0.00001878005 \\
4 & 0.00006103515 & \div & 15 & = & 0.00000406901 \\
4 & 0.00001525879 & \div & 17 & = & 0.00000089758 \\
4 & 0.00000381470 & \div & 19 & = & 0.00000020077 \\
4 & 0.00000095367 & \div & 21 & = & 0.00000004541 \\
4 & 0.00000023842 & \div & 23 & = & 0.00000001037 \\
4 & 0.00000005960 & \div & 25 & = & 0.00000000238 \\
4 & 0.00000001490 & \div & 27 & = & 0.00000000055 \\
4 & 0.00000000372 & \div & 29 & = & 0.00000000013 \\
4 & 0.00000000093 & \div & 31 & = & 0.00000000003 \\
 & 0.00000000023 & \div & 33 & = & \underline{0.00000000001} \\
 & & & & & 1.09861228867
\end{array}$$

$$\therefore \log_e 3 = 1.0986122886.$$

2. Calculate to five places of decimals $\log_e 5$.

For use in this example and in succeeding examples let us first calculate to eight places of decimals the value of $\log_e 2$.

Let $z = 1.$

Then $z + 1 = 2$, and $2z + 1 = 3$;

and $$\log_e 2 = 2\left(\frac{1}{3} + \frac{1}{3 \times 3^3} + \frac{1}{5 \times 3^5} + \cdots\right).$$

$$\begin{array}{r|l}
3 & \underline{2.000000000} \\
9 & \underline{0.666666667} \div 1 = 0.666666667 \\
9 & \underline{0.074074074} \div 3 = 0.024691358 \\
9 & \underline{0.008230453} \div 5 = 0.001646091 \\
9 & \underline{0.000914495} \div 7 = 0.000130642 \\
9 & \underline{0.000101611} \div 9 = 0.000011290 \\
9 & \underline{0.000011290} \div 11 = 0.000001026 \\
9 & \underline{0.000001254} \div 13 = 0.000000096 \\
9 & \underline{0.000000139} \div 15 = 0.000000009 \\
\hline
 & 0.000000015 \div 17 = \underline{0.000000001} \\
 & \qquad\qquad\qquad\quad\;\; 0.693147180
\end{array}$$

$$\therefore \log_e 2 = 0.69314718.$$

Let $z = 4.$

Then $z + 1 = 5$, and $2z + 1 = 9$;

and $$\log_e \frac{5}{4} = 2\left(\frac{1}{9} + \frac{1}{3 \times 9^3} + \frac{1}{5 \times 9^5} + \cdots\right).$$

$$\begin{array}{r|l}
9 & \underline{2.000000} \\
9 & \underline{0.222222} \div 1 = 0.222222 \\
9 & \underline{0.024691} \\
9 & \underline{0.002743} \div 3 = 0.000914 \\
9 & \underline{0.000305} \\
\hline
 & 0.000034 \div 5 = \underline{0.000007} \\
 & \qquad\qquad\quad\;\; 0.223143
\end{array}$$

$$\begin{aligned}
\therefore \log_e \tfrac{5}{4} &= 0.22314. \\
\log_e 5 &= 0.22314 + \log_e 4 \\
&= 0.22314 + 2 \times \log_e 2 \\
&= 0.22314 + 2 \times 0.69315 \\
&= 1.60944.
\end{aligned}$$

3. Calculate to five places of decimals $\log_e 7$.

Let $z = 6$.

Then $z + 1 = 7$, and $2z + 1 = 13$;

and
$$\log_e \frac{7}{6} = 2\left(\frac{1}{13} + \frac{1}{3 \times 13^3} + \frac{1}{5 \times 13^5} + \cdots\right).$$

$$\begin{array}{r|l l}
13 & 2.000000 & \\ \hline
13 & 0.153846 & \div 1 = 0.153846 \\ \hline
13 & 0.011834 & \\ \hline
13 & 0.000910 & \div 3 = 0.000303 \\ \hline
13 & 0.000070 & \\ \hline
 & 0.000005 & \div 5 = \underline{0.000001} \\
 & & 0.154150
\end{array}$$

$$\begin{aligned}
\therefore \log_e \tfrac{7}{6} &= 0.15415. \\
\log_e 7 &= 0.15415 + \log_e 6 \\
&= 0.15415 + \log_e 2 + \log_e 3 \\
&= 0.15415 + 0.693147 + 1.098612 \\
&= 1.94591.
\end{aligned}$$

4. Calculate to ten places of decimals $\log_e 10$.

Let $z = 9$.

Then $z + 1 = 10$, and $2z + 1 = 19$;

and
$$\log_e \frac{10}{9} = 2\left(\frac{1}{19} + \frac{1}{3 \times 19^3} + \frac{1}{5 \times 19^5} + \cdots\right).$$

$$\begin{array}{r|l l}
19 & 2.00000000000 & \\ \hline
19 & 0.10526315789 & \div 1 = 0.10526315789 \\ \hline
19 & 0.00554016620 & \\ \hline
19 & 0.00029158769 & \div 3 = 0.00009719590 \\ \hline
19 & 0.00001534672 & \\ \hline
19 & 0.00000080772 & \div 5 = 0.00000016154 \\ \hline
19 & 0.00000004251 & \\ \hline
 & 0.00000000224 & \div 7 = \underline{0.00000000032} \\
 & & 0.10536051565
\end{array}$$

$$\begin{aligned}
\therefore \log_e \tfrac{10}{9} &= 0.1053605156. \\
\log_e 10 &= 0.1053605156 + 2\log_e 3 \\
&= 2.3025850930.
\end{aligned}$$

5. **Calculate to five places of decimals $\log_{10} 2$, $\log_{10} e$, $\log_{10} 11$.**

$$\log_{10} 2 = \frac{\log_e 2}{\log_e 10} = \frac{0.693147}{2.302585} = 0.30103.$$

$$\log_{10} e = \frac{\log_e e}{\log_e 10} = \frac{1}{2.302585} = 0.43429.$$

To calculate $\log_{10} 11$, let $z = 10$.

Then $z + 1 = 11$, and $2z + 1 = 21$;

and $$\log_{10} \tfrac{11}{10} = 2 \log_{10} e \left(\frac{1}{21} + \frac{1}{3 \times 21^3} + \frac{1}{5 \times 21^5} + \cdots \right).$$

$$\begin{array}{r|l}
21 & 2.000000 \\ \hline
21 & 0.095238 \div 1 = 0.095238 \\ \hline
21 & 0.004535 \\ \hline
 & 0.000216 \div 3 = \underline{0.000072} \\
 & \qquad\qquad\quad\;\; 0.095310
\end{array}$$

$$\begin{aligned}
\therefore \log_{10} \tfrac{11}{10} &= 0.09531 \times \log_{10} e \\
&= 0.09531 \times 0.43429 \\
&= 0.04139. \\
\log_{10} 11 &= 0.04139 + \log_{10} 10 \\
&= 1.04139.
\end{aligned}$$

EXERCISE XXVII. PAGE 128.

1. Given $\pi = 3.141592653589$; compute $\sin 1'$, $\cos 1'$, and $\tan 1'$ to eleven places of decimals.

The circular measure of $1'$ is

$$\frac{\pi}{10800} = \frac{3.141592653589}{10800} = 0.0002908882 +,$$

the next figure being 0 or 1.

Again, taking the value of $\sin 1'$ as computed in the text-book, $0.00029088 +$, we have

$$\begin{aligned}
\cos 1' &> \sqrt{1 - (0.00029089)^2} \\
&> \sqrt{1 - 0.000000084617} \\
&> \sqrt{0.999999915383} \\
&> 0.999999957691.
\end{aligned}$$

Also $$\begin{aligned}
\cos 1' &< \sqrt{1 - (0.00029088)^2} \\
&< \sqrt{1 - 0.000000084611} \\
&< \sqrt{0.999999915389} \\
&< 0.999999957694.
\end{aligned}$$

Hence, $\cos 1' = 0.99999995769$, correct to eleven decimal places.

By Sect. XLV, $\sin x > x \cos x$.

$$\begin{aligned} \therefore \sin 1' &> 0.0002908882 \times 0.99999995769 \\ &> 0.0002908882\,(1 - 0.00000004231) \\ &> 0.0002908882 - 0.000000000012 \\ &> 0.00029088818. \end{aligned}$$

Therefore, $\sin 1'$ lies between 0.00029088818 and 0.00029088821. That is, correct to nine places of decimals,

$$\sin 1' = 0.000290888,$$

the next two figures being 18, 19, 20, or 21.

Repeating the process, beginning with the last value of $\sin 1'$, the computation can be carried still further. To eleven places,

$$\sin 1' = 0.00029088820.$$

From the values of $\sin 1'$ and $\cos 1'$ we have

$$\begin{aligned} \tan 1' &= \frac{\sin 1'}{\cos 1'} \\ &= \frac{0.00029088820}{0.99999995769} \\ &= 0.000290888212. \end{aligned}$$

2. Given $\pi = 3.141592653589$; compute $\sin 2'$ by the same method, and also by the formula $\sin 2x = 2 \sin x \cos x$. Carry the operations to nine places of decimals. Do the two results agree?

The circular measure of $2'$ is

$$\frac{\pi}{5400} = \frac{3.141592653589}{5400} = 0.0005817764 +.$$

Hence, $\sin 2'$ lies between 0 and 0.0005817765,

and

$$\begin{aligned} \cos 2' &> \sqrt{1 - (0.0005817765)^2} \\ &> \sqrt{1 - 0.0000003384638959} \\ &> \sqrt{0.9999996615361040} \\ &> 0.99999983076\!8. \end{aligned}$$

But $\sin x > x \cos x$.

$$\begin{aligned} \therefore \sin 2' &> 0.0005817764 \times 0.99999983076 \\ &> 0.0005817764\,(1 - 0.00000016924) \\ &> 0.0005817764 - 0.000000000098 \\ &> 0.0005817763. \end{aligned}$$

Hence, $\sin 2' = 0.000581776$, correct to nine decimal places.

Again,
$$\begin{aligned}\sin 2' &= 2 \sin 1' \cos 1' \\ &= 2 \times 0.00029088820 \times 0.99999995769 \\ &= 0.00058177640\,(1 - 0.00000004231) \\ &= 0.00058177640 - 0.000000000025 \\ &= 0.000581776 +.\end{aligned}$$

The two methods, therefore, agree to at least nine decimal places.

3. Given $\pi = 3.141592653589$; compute $\sin 1^\circ$ to four places of decimals.

The circular measure of 1° is

$$\frac{\pi}{180} = \frac{3.141592653589}{180} = 0.01745329.$$

Hence,

$$\begin{aligned}\cos 1^\circ &> \sqrt{1 - (0.01746)^2} \\ &> \sqrt{0.99969515} \\ &> 0.999847.\end{aligned}$$

$$\sin x > x \cos x.$$

$$\begin{aligned}\therefore \sin 1^\circ &> 0.017453 \times 0.999847 \\ &> 0.017453\,(1 - 0.000153) \\ &> 0.017453 - 0.0000027 \\ &> 0.0174503.\end{aligned}$$

Hence, to four decimal places, $\sin 1^\circ = 0.0175$.

4. From the formula

$$\cos x = 1 - 2 \sin^2 \frac{x}{2}$$

show that $\cos x > 1 - \frac{x^2}{2}$.

By Sect. XLV, $\sin x < x$.

$$\therefore \sin \frac{x}{2} < \frac{x}{2}.$$

$$\sin^2 \frac{x}{2} < \frac{x^2}{4}.$$

$$1 - 2 \sin^2 \frac{x}{2} > 1 - \frac{x^2}{2}.$$

$$\therefore \cos x > 1 - \frac{x^2}{2}.$$

5. Show by aid of a table of natural sines that $\sin x$ and x agree to four places of decimals for all angles less than $4^\circ\, 40'$.

The circular measure of $4^\circ\, 40'$, or $280'$, is

$$\begin{aligned}\frac{280\,\pi}{10800} &= \frac{7\,\pi}{270} \\ &= \frac{7 \times 3.141592653589}{270} \\ &= 0.0814487.\end{aligned}$$

The circular measure of $4^\circ\, 41'$ is $0.0814487 + 0.0002909 = 0.0817396$.

From a table,

$$\sin 4^\circ\, 40' = 0.0814.$$
$$\sin 4^\circ\, 41' = 0.0816.$$

Hence, $\sin x$ and the circular measure of x agree for $4^\circ\, 40'$, and therefore for all smaller angles to four decimal places; but they differ for larger angles.

6. If the values of $\log x$ and $\log \sin x$ agree to five decimal places, find from a table the greatest value x can have.

Let x be expressed in seconds. Then its circular measure is

$$\frac{\pi x}{648000}$$

and its logarithm is

$$\begin{aligned}&\log x'' + (\log \pi - \log 648000) \\ &= \log x'' + (0.49715 - 5.81158) \\ &= \log x'' - 5.31443 \\ &= \log x'' + 4.68557 - 10.\end{aligned}$$

But from the explanation preceding Table IV, if we remember that log sines are given in the table increased by 10, we have

$$\log \sin x + 10 = \log x'' + S.$$
$$\therefore \log \sin x = \log x'' + S - 10.$$

Hence, if, for five decimal places, $\log \sin x = \log x$, we have

$$\log x'' + 4.68557 - 10 = \log x'' + S - 10.$$
$$\therefore S = 4.68557.$$

But, the greatest angle for which this value of S can be used is given in the table as $2409''$.

Hence, the greatest angle for which $\log x$ and $\log \sin x$ agree to five decimal places is

$$2409'' = 40'\ 9''.$$

EXERCISE XXVIII. PAGE 130.

1. Compute the sine and cosine of $6'$ to seven decimal places.

From Prob. 2, Ex. XXVII,

$$\sin 2' = 0.000581776.$$

Also, from Prob. 1, Ex. XXVII,

$$\cos 1' = 0.999999958,$$
$$\text{and } \sin 1' = 0.000290888.$$

Hence

$$\begin{aligned}\sin 3' &= 2 \sin 2' \cos 1' - \sin 1' \\ &= 2 \times 0.000581776\,(1 - 0.0000001) - 0.000290888 \\ &= 2 \times 0.000581776 - 0.000290888 \\ &= 0.000872664.\end{aligned}$$

$$\begin{aligned}\cos 2' &= 2 \cos^2 1' - 1 \\ &= 2\,(0.999999958)^2 - 1 \\ &= 2 \times 0.999999916 - 1 \\ &= 0.999999832.\end{aligned}$$

$$\begin{aligned}\cos 3' &= 2 \cos 2' \cos 1' - \cos 1' \\ &= \cos 1'\,(2 \cos 2' - 1) \\ &= 0.999999958\,(2 \times 0.999999832 - 1) \\ &= (1 - 0.000000042)\,(0.999999664) \\ &= 0.999999622.\end{aligned}$$

By [12],

$$\begin{aligned}\sin 6' &= 2 \sin 3' \cos 3' \\ &= 2 \times 0.000872664 \times 0.999999622 \\ &= 0.001745327.\end{aligned}$$

By [13],

$$\begin{aligned}\cos 6' &= 2 \cos^2 3' - 1 \\ &= 2\,(0.999999622)^2 - 1 \\ &= 2 \times 0.999999244 - 1 \\ &= 0.999998488.\end{aligned}$$

2. In Formula (1) let $y = 1^\circ$. Assuming $\sin 1^\circ = 0.017454+$, $\cos 1^\circ = 0.999848+$, compute the sine and cosine of two degrees.

$$\begin{aligned}
\sin 2^\circ &= 2 \sin 1^\circ \cos 1^\circ \\
&= 2 \times 0.017454 \times 0.999848 \\
&= 0.034902. \\
\cos 2^\circ &= 2 \cos^2 1^\circ - 1 \\
&= 2 \times (0.999848)^2 - 1 \\
&= 2 \times (0.999696) - 1 \\
&= 0.999392.
\end{aligned}$$

3. In Formula (1) let $y = 1^\circ$. Assuming $\sin 1^\circ = 0.017454+$, $\cos 1^\circ = 0.999848+$, compute the sine and cosine of three degrees.

$$\begin{aligned}
\sin 3^\circ &= 2 \sin 2^\circ \cos 1^\circ - \sin 1^\circ \\
&= 2 \times 0.034902 \times 0.999848 - 0.017454 \\
&= 0.052339. \\
\cos 3^\circ &= (2 \cos 2^\circ - 1) \cos 1^\circ \\
&= 0.998784 \times 0.999848 \\
&= 0.998632.
\end{aligned}$$

4. In Formula (1) let $y = 1^\circ$. Assuming $\sin 1^\circ = 0.017454+$, $\cos 1^\circ = 0.999848+$, compute the sine and cosine of four degrees.

$$\begin{aligned}
\sin 4^\circ &= 2 \sin 3^\circ \cos 1^\circ - \sin 2^\circ \\
&= 2 \times 0.052339 \times 0.999848 - 0.034902 \\
&= 0.069760. \\
\cos 4^\circ &= 2 \cos 3^\circ \cos 1^\circ - \cos 2^\circ \\
&= 2 \times 0.998630 \times 0.999848 - 0.999392 \\
&= 1.996956 - 0.999392 \\
&= 0.997564.
\end{aligned}$$

5. In Formula (1) let $y = 1^\circ$. Assuming $\sin 1^\circ = 0.017454+$, $\cos 1^\circ = 0.999848+$, compute the sine and cosine of five degrees.

$$\begin{aligned}
\sin 5^\circ &= 2 \sin 4^\circ \cos 1^\circ - \sin 3^\circ \\
&= 2 \times 0.069760 \times 0.999848 - 0.052339 \\
&= 0.087160. \\
\cos 5^\circ &= 2 \cos 4^\circ \cos 1^\circ - \cos 3^\circ \\
&= 2 \times 0.997564 \times 0.999848 - 0.998632 \\
&= 1.994825 - 0.998632 \\
&= 0.996193.
\end{aligned}$$

EXERCISE XXIX. PAGE 135.

1. Find the six 6th roots of -1; of $+1$.

$-1 = \cos 180° + i \sin 180°.$

$+1 = \cos 0° + 1 \sin 0°.$

Hence, the six 6th roots of -1 are

$\cos 30° + i \sin 30° = \dfrac{\sqrt{3} + i}{2},$

$\cos 90° + i \sin 90° = i,$

$\cos 150° + i \sin 150° = \dfrac{-\sqrt{3} + i}{2},$

$\cos 210° + i \sin 210° = \dfrac{-\sqrt{3} - i}{2},$

$\cos 270° + i \sin 270° = -i,$

$\cos 330° + i \sin 330° = \dfrac{\sqrt{3} - i}{2}.$

The six 6th roots of $+1$ are

$\cos 0° + i \sin 0° = +1,$

$\cos 60° + i \sin 60° = \dfrac{1 + \sqrt{-3}}{2},$

$\cos 120° + i \sin 120° = \dfrac{-1 + \sqrt{-3}}{2},$

$\cos 180° + i \sin 180° = -1,$

$\cos 240° + i \sin 240° = \dfrac{-1 - \sqrt{-8}}{2},$

$\cos 300° + i \sin 300° = \dfrac{1 - \sqrt{-3}}{2}.$

2. Find the three cube roots of i.

$i = \cos 90° + i \sin 90°.$

Hence, the three cube roots of i are

$\cos 30° + i \sin 30° = \dfrac{\sqrt{3} + i}{2},$

$\cos 150° + i \sin 150° = \dfrac{-\sqrt{3} + i}{2},$

$\cos 270° + i \sin 270° = -i.$

3. Find the four 4th roots of $-i$.

$-i = \cos 270° + i \sin 270°.$

Hence, the four 4th roots of $-i$ are

$\cos 67\tfrac{1}{2}° + i \sin 67\tfrac{1}{2}°,$
$\cos 157\tfrac{1}{2}° + i \sin 157\tfrac{1}{2}°,$
$\cos 247\tfrac{1}{2}° + i \sin 247\tfrac{1}{2}°,$
$\cos 337\tfrac{1}{2}° + i \sin 337\tfrac{1}{2}°.$

4. Express $\sin 4\theta$ and $\cos 4\theta$ in terms of $\sin \theta$ and $\cos \theta$.

By Sect. XLVII,

$$\sin 4\theta = 4 \cos^3\theta \sin\theta - \frac{4 \times 3 \times 2}{\lfloor\underline{3}} \cos\theta \sin^3\theta$$
$$= 4 \cos^3\theta \sin\theta - 4 \cos\theta \sin^3\theta;$$

and

$$\cos 4\theta = \cos^4\theta - \frac{4 \times 3}{\lfloor\underline{2}} \cos^2\theta \sin^2\theta + \frac{4 \times 3 \times 2 \times 1}{\lfloor\underline{4}} \sin^4\theta$$
$$= \cos^4\theta - 6 \cos^2\theta \sin^2\theta + \sin^4\theta.$$

EXERCISE XXX. PAGE 137.

1. Verify by the series just obtained that $\sin^2 x + \cos^2 x = 1$.

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \cdots$$

$$\therefore \sin^2 x = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \cdots$$

and

$$\cos^2 x = 1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} + \frac{x^8}{315} - \cdots$$

$$\therefore \sin^2 x + \cos^2 x = 1.$$

2. Verify by the series just obtained that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$.

The series for the sine consists entirely of odd powers of x and, therefore, changes its sign with x; while the series for the cosine consists entirely of even powers, and is unchanged when x changes its sign.

3. Verify by the series just obtained that $\sin 2x = 2 \sin x \cos x$.

$$\begin{aligned}\sin 2x &= 2x - \frac{(2x)^3}{6} + \frac{(2x)^5}{120} - \frac{(2x)^7}{5040} + \cdots \\ &= 2x - \frac{4x^3}{3} + \frac{4x^5}{15} - \frac{8x^7}{315} + \cdots \\ &= 2\left(x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{4x^7}{315} + \cdots\right).\end{aligned}$$

Also

$$\sin x \cos x = x - \frac{2x^3}{3} + \frac{2x^5}{5} - \frac{4x^7}{315} + \cdots$$

$$\therefore \sin 2x = 2 \sin x \cos x.$$

4. Verify by the series just obtained that $\cos 2x = 1 - 2\sin^2 x$.

By Prob. 1,

$$\sin^2 x = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \cdots$$

$$\begin{aligned}\therefore 1 - 2\sin^2 x &= 1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \frac{2x^8}{315} - \cdots \\ &= 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{24} - \frac{(2x)^6}{720} + \frac{(2x)^8}{40320} - \cdots \\ &= \cos 2x.\end{aligned}$$

5. Find the series for $\sec x$ as far as the term containing the 6th power of x.

$$\sec x = \frac{1}{\cos x} = 1 \div \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)$$
$$= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots$$

6. Find the series for $x \cot x$, noting that $x \cot x = \frac{x}{\sin x} \cos x$.

$$x \cos x = x - \frac{x^3}{2} + \frac{x^5}{24} - \frac{x^7}{720} + \cdots$$
$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$
$$\therefore \frac{x \cos x}{\sin x} = 1 - \frac{x^2}{3} - \frac{x^4}{45} - \frac{2x^6}{945} - \cdots$$

7. Calculate $\sin 10^\circ$ and $\cos 10^\circ$ to five places of decimals.

The circular measure of 10° is $\frac{\pi}{18}$.

Hence
$$\sin 10^\circ = \frac{\pi}{18} - \frac{1}{6}\left(\frac{\pi}{18}\right)^3 + \frac{1}{120}\left(\frac{\pi}{18}\right)^5 - \cdots$$
$$\cos 10^\circ = 1 - \frac{1}{2}\left(\frac{\pi}{18}\right)^2 + \frac{1}{24}\left(\frac{\pi}{18}\right)^4 - \frac{1}{720}\left(\frac{\pi}{18}\right)^6 + \cdots$$

Taking $\pi = 3.141592653589$, we find

$$\pi = 3.141593, \qquad \frac{\pi}{18} = 0.174533,$$
$$\pi^2 = 9.869604, \qquad \frac{\pi^2}{2 \times 18^2} = 0.015231,$$
$$\pi^3 = 31.006277, \qquad \frac{\pi^3}{6 \times 18^3} = 0.000886,$$
$$\pi^4 = 97.409091, \qquad \frac{\pi^4}{24 \times 18^4} = 0.000039,$$
$$\pi^5 = 306.019685, \qquad \frac{\pi^5}{120 \times 18^5} = 0.000001.$$

$$\therefore \sin 10^\circ = 0.174533 - 0.000886 + 0.000001$$
$$= 0.173648.$$
$$\cos 10^\circ = 1 - 0.015231 + 0.000039$$
$$= 0.984808.$$

Note. The powers of π need be computed only once, and can then be used for finding the functions of all angles.

8. Calculate tan 15° to five places of decimals.

The circular measure of 15° is $\frac{\pi}{12}$.

Hence $$\tan 15° = \frac{\pi}{12} + \frac{1}{3}\left(\frac{\pi}{12}\right)^3 + \frac{2}{15}\left(\frac{\pi}{12}\right)^5 + \frac{17}{315}\left(\frac{\pi}{12}\right)^7 + \cdots$$

$$\pi = 3.141593, \qquad \frac{\pi}{12} = 0.261799,$$

$$\pi^3 = 31.006277, \qquad \frac{\pi^3}{3 \times 12^3} = 0.005981,$$

$$\pi^5 = 306.019685, \qquad \frac{2\,\pi^5}{15 \times 12^5} = 0.000164,$$

$$\pi^7 = 3020.293227, \qquad \frac{17\,\pi^7}{315 \times 12^7} = 0.000005.$$

$$\therefore \tan 15° = 0.267949.$$

9. From the exponential value of cos x show that

$$\cos 3x = 4\cos^3 x - 3\cos x.$$

By Sect. XLVIII, $$\cos x = \tfrac{1}{2}(e^{xi} + e^{-xi}).$$

$$\begin{aligned}\therefore \cos 3x &= \tfrac{1}{2}(e^{3xi} + e^{-3xi})\\ &= \tfrac{1}{2}(e^{xi} + e^{-xi})(e^{2xi} - 1 + e^{-2xi})\\ &= \cos x\left[\{4 \times \tfrac{1}{2}(e^{xi} + e^{-xi})\}^2 - 3\right]\\ &= \cos x\,(4\cos^2 x - 3)\\ &= 4\cos^3 x - 3\cos x.\end{aligned}$$

10. From the exponential value of sin x show that

$$\sin 3x = 3\sin x - 4\sin^3 x.$$

By Sect. XLVIII, $$\begin{aligned}\sin 3x &= \frac{1}{2i}(e^{3xi} - e^{-3xi})\\ &= \frac{1}{2i}(e^{ri} - e^{-xi})(e^{2xi} + 1 + e^{-2xi})\\ &= \sin x\left[(-4)\left\{\frac{1}{2i}\left(e^{xi} - e^{-xi}\right)\right\}^2 + 3\right]\\ &= \sin x\,(-4\sin^2 x + 3)\\ &= 3\sin x - 4\sin^3 x.\end{aligned}$$

SPHERICAL TRIGONOMETRY.

Exercise XXXI. Page 142.

1. The angles of a triangle are 70°, 80°, and 100°. Find the sides of the polar triangle.

Given $A = 70°$, $B = 80°$, $C = 100°$; to find a', b', c'.

$$\begin{aligned} a' &= 180° - 70° = 110°. \\ b' &= 180° - 80° = 100°. \\ c' &= 180° - 100° = 80°. \end{aligned}$$

2. The sides of a triangle are 40°, 90°, and 125°. Find the angles of the polar triangle.

Given $a = 40°$, $b = 90°$, $c = 125°$; required A', B', C'.

$$\begin{aligned} A' &= 180° - 40° = 140°. \\ B' &= 180° - 90° = 90°. \\ C' &= 180° - 125° = 55°. \end{aligned}$$

3. Show that, if a triangle has three right angles, the sides of the triangle are quadrants.

Every vertex is the pole of the opposite side. Every side is, therefore, 90°.

4. Show that, if a triangle has two right angles, the sides opposite these angles are quadrants, and the third angle is measured by the number of degrees in the opposite side.

Let ABC be the triangle, and

$$B = C = 90°.$$

Then A is the pole of a. Therefore, b and c are quadrants, and the angle A is equal to the side BC measured in degrees.

5. How can the sides of a spherical triangle, measured in degrees, be found in units of length, when the length of the radius of the sphere is known?

Since the sides of the triangle are arcs of great circles, every degree of arc is $\frac{1}{360}$ of the circumference of a great circle, or $\frac{2\pi r}{360}$, where r is the radius of the sphere. Hence, to find the length of a side, multiply its measure in degrees by $\frac{2\pi r}{360}$ or $\frac{\pi r}{180}$.

6. Find the lengths of the sides of the triangle in Example 2 if the radius of the sphere is 4 feet.

$$a = 40° = 40 \times \frac{\pi \times 4}{180} \text{ feet} = \frac{8\pi}{9} \text{ feet.}$$

$$b = 90° = 90 \times \frac{\pi \times 4}{180} \text{ feet} = 2\pi \text{ feet.}$$

$$c = 125° = 125 \times \frac{\pi \times 4}{180} \text{ feet} = \frac{25\pi}{9} \text{ feet.}$$

EXERCISE XXXII. PAGE 146.

1. Show, by aid of Formula [38], p. 144, that the hypotenuse of a right spherical triangle is *less than* or *greater than* 90°, according as the two legs are *alike* or *unlike* in kind.

By [38], $\cos c = \cos a \cos b$.

If a and b are both $< 90°$ or both $> 90°$, $\cos a$ and $\cos b$ have the same sign. Hence, $\cos c$ is positive, and $c < 90°$.

But if a and b are unlike in kind, $\cos a$ and $\cos b$ have opposite signs. Hence, $\cos c$ is negative, and $c > 90°$.

2. Show, by aid of Formula [41], that in a right spherical triangle each leg and the opposite angle are always alike in kind.

By [41], $\cos A = \cos a \sin B$.

Now $B < 180°$.

$\therefore \sin B$ is positive.

Hence, the sign of $\cos A$ is same as the sign of $\cos a$, and both must be greater than or both less than 90°; that is, alike in kind.

3. What inferences may be drawn from Formulas [38]–[43] respecting the values of the other parts: (i) if $c = 90°$; (ii) if $a = 90$; (iii) if $c = 90°$ and $a = 90°$; (iv) if $a = 90°$ and $b = 90°$?

(i) If $c = 90°$,

[38] becomes $0 = \cos a \cos b$.

$\therefore \cos a$ or $\cos b = 0$.

$\therefore a$ or $b = 90°$.

If $a = 90°$,

[41] becomes

$\cos A = 0 \times \sin B = 0$.

$\therefore A = 90°$.

Hence, from Prob. 4, Ex. XXXI,

$B = b$.

(ii) If $a = 90°$,

[41] becomes

$\cos A = 0 \times \sin B$.

$\therefore A = 90°$,

$c = 90°$,

and $B = b$.

(iii) If $c = 90°$,

and $a = 90°$,

from (i) and (ii), $A = 90°$,

and $B = b$.

(iv) If $a = 90°$,

and $b = 90°$,

from (ii), $c = 90°$,

and $B = b = 90°$.

4. Deduce from Formulas [38]–[43] and Formulas [18]–[23] the formula $\tan^2 \frac{1}{2} b = \tan \frac{1}{2}(c-a) \tan \frac{1}{2}(c+a)$.

From [38], $\cos b = \dfrac{\cos c}{\cos a}$.

By [18], $\tan^2 \frac{1}{2} b = \dfrac{1 - \cos b}{1 + \cos b}$

$$= \frac{1 - \dfrac{\cos c}{\cos a}}{1 + \dfrac{\cos c}{\cos a}}$$

$$= \frac{\cos a - \cos c}{\cos a + \cos c}$$

By [23] and [22], $= \dfrac{-2 \sin \frac{1}{2}(a+c) \sin \frac{1}{2}(a-c)}{2 \cos \frac{1}{2}(a+c) \cos \frac{1}{2}(a-c)}$

$$= -\tan \tfrac{1}{2}(a+c) \tan \tfrac{1}{2}(a-c)$$

$$= \tan \tfrac{1}{2}(c+a) \tan \tfrac{1}{2}(c-a).$$

5. Deduce from Formulas [38]–[43] and Formulas [18]–[23] the formula $\tan^2(45^\circ - \frac{1}{2}A) = \tan \frac{1}{2}(c-a) \cot \frac{1}{2}(c+a)$.

From [39], $\sin A = \dfrac{\sin a}{\sin c}$.

Now $\tan^2(45^\circ - \frac{1}{2}A) = \tan^2 \frac{1}{2}(90^\circ - A)$

$$= \cot^2 \tfrac{1}{2}(90^\circ + A)$$

By [19], $= \dfrac{1 + \cos(90^\circ + A)}{1 - \cos(90^\circ + A)}$

$$= \frac{1 - \sin A}{1 + \sin A}$$

$$= \frac{1 - \dfrac{\sin a}{\sin c}}{1 + \dfrac{\sin a}{\sin c}}$$

$$= \frac{\sin c - \sin a}{\sin c + \sin a}$$

By [21] and [20], $= \dfrac{2 \cos \frac{1}{2}(c+a) \sin \frac{1}{2}(c-a)}{2 \sin \frac{1}{2}(c+a) \cos \frac{1}{2}(c-a)}$

$$= \cot \tfrac{1}{2}(c+a) \tan \tfrac{1}{2}(c-a).$$

6. Deduce from Formulas [38]–[43] and Formulas [18]–[23] the formula $\tan^2 \frac{1}{2} B = \sin (c - a) \csc (c + a)$.

From [40], $$\cos B = \frac{\tan a}{\tan c}.$$

By [18], $$\tan^2 \tfrac{1}{2} B = \frac{1 - \cos B}{1 + \cos B}$$

$$= \frac{1 - \dfrac{\tan a}{\tan c}}{1 + \dfrac{\tan a}{\tan c}}$$

$$= \frac{\tan c - \tan a}{\tan c + \tan a}$$

$$= \frac{\dfrac{\sin c}{\cos c} - \dfrac{\sin a}{\cos a}}{\dfrac{\sin c}{\cos c} + \dfrac{\sin a}{\cos a}}$$

$$= \frac{\sin c \cos a - \cos c \sin a}{\sin c \cos a + \cos c \sin a}$$

By [8] and [4], $$= \frac{\sin (c - a)}{\sin (c + a)}$$

$$= \sin (c - a) \csc (c + a).$$

7. Deduce from Formulas [38]–[43] and Formulas [18]–[23] the formula $\tan^2 \frac{1}{2} c = - \cos (A + B) \sec (A - B)$.

From [43], $$\cos c = \frac{\cot A}{\tan B}.$$

By [18], $$\tan^2 \tfrac{1}{2} c = \frac{1 - \cos c}{1 + \cos c}$$

$$= \frac{1 - \dfrac{\cot A}{\tan B}}{1 + \dfrac{\cot A}{\tan B}}$$

$$= \frac{\tan B - \cot A}{\tan B + \cot A}$$

$$= \frac{\dfrac{\sin B}{\cos B} - \dfrac{\cos A}{\sin A}}{\dfrac{\sin B}{\cos B} + \dfrac{\cos A}{\sin A}}$$

$$= \frac{\sin A \sin B - \cos A \cos B}{\sin A \sin B + \cos A \cos B}$$

By [5] and [9],
$$= \frac{-\cos(A+B)}{\cos(A-B)}$$
$$= -\cos(A+B)\sec(A-B).$$

8. Deduce from Formulas [38]–[43] and Formulas [18]–[23] the formula $\tan^2\frac{1}{2}a = \tan[\frac{1}{2}(A+B)-45°]\tan[\frac{1}{2}(A-B)+45°]$.

From [41], $$\cos a = \frac{\cos A}{\sin B}.$$

By [18], $$\tan^2\tfrac{1}{2}a = \frac{1-\cos a}{1+\cos a}$$
$$= \frac{1-\dfrac{\cos A}{\sin B}}{1+\dfrac{\cos A}{\sin B}}$$
$$= \frac{\sin B - \cos A}{\sin B + \cos A}$$
$$= \frac{\sin B + \sin(A-90°)}{\sin B - \sin(A-90°)}$$

By [20] and [21],
$$= \frac{2\sin\frac{1}{2}(A+B-90°)\cos\frac{1}{2}(B-A+90°)}{2\cos\frac{1}{2}(A+B-90°)\sin\frac{1}{2}(B-A+90°)}$$
$$= \tan[\tfrac{1}{2}(A+B)-45°]\cot[\tfrac{1}{2}(B-A)+45°].$$

Now $$\tfrac{1}{2}(A-B)+45° = 90° - [\tfrac{1}{2}(B-A)+45°].$$
$$\therefore \cot[\tfrac{1}{2}(B-A)+45°] = \tan[\tfrac{1}{2}(A-B)+45°].$$
$$\therefore \tan^2\tfrac{1}{2}a = \tan[\tfrac{1}{2}(A+B)-45°]\tan[\tfrac{1}{2}(A-B)+45°].$$

9. Deduce from Formulas [38]–[43] and Formulas [18]–[23] the formula $\tan^2(45° - \frac{1}{2}c) = \tan\frac{1}{2}(A-a)\cot\frac{1}{2}(A+a)$.

From [39], $$\sin c = \frac{\sin a}{\sin A}.$$
$$\therefore \cos(90°-c) = \frac{\sin a}{\sin A}.$$

By [18], $$\tan^2(45°-\tfrac{1}{2}c) = \frac{1-\cos(90°-c)}{1+\cos(90°-c)}$$
$$= \frac{1-\dfrac{\sin a}{\sin A}}{1+\dfrac{\sin a}{\sin A}}$$
$$= \frac{\sin A - \sin a}{\sin A + \sin a}$$

By [21] and [20],
$$= \frac{2 \cos \frac{1}{2}(A + a) \sin \frac{1}{2}(A - a)}{2 \sin \frac{1}{2}(A + a) \cos \frac{1}{2}(A - a)}$$
$$= \cot \tfrac{1}{2}(A + a) \tan \tfrac{1}{2}(A - a).$$

10. Deduce from Formulas [38]–[43] and Formulas [18]–[23] the formula $\tan^2(45° - \frac{1}{2} b) = \sin(A - a) \csc(A + a)$.

From [42],
$$\sin b = \frac{\tan a}{\tan A}.$$
$$\therefore \cos(90° - b) = \frac{\tan a}{\tan A}.$$

By [18],
$$\tan^2(45° - \tfrac{1}{2} b) = \frac{1 - \cos(90° - b)}{1 + \cos(90° + b)}$$
$$= \frac{1 - \dfrac{\tan a}{\tan A}}{1 + \dfrac{\tan a}{\tan A}}$$
$$= \frac{\tan A - \tan a}{\tan A + \tan a}$$
$$= \frac{\dfrac{\sin A}{\cos A} - \dfrac{\sin a}{\cos a}}{\dfrac{\sin A}{\cos A} + \dfrac{\sin a}{\cos a}}$$
$$= \frac{\sin A \cos a - \cos A \sin a}{\sin A \cos a + \cos A \sin a}$$

By [8] and [4],
$$= \frac{\sin(A - a)}{\sin(A + a)}$$
$$= \sin(A - a) \csc(A + a).$$

11. Deduce from Formulas [38]–[43] and Formulas [18]–[23] the formula $\tan^2(45° - \frac{1}{2} B) = \tan \frac{1}{2}(A - a) \tan \frac{1}{2}(A + a)$.

From [41],
$$\sin B = \frac{\cos A}{\cos a}.$$
$$\therefore \cos(90° - B) = \frac{\cos A}{\cos a}.$$

By [18],
$$\tan^2(45° - \tfrac{1}{2} B) = \frac{1 - \cos(90° - B)}{1 + \cos(90° - B)}$$
$$= \frac{1 - \dfrac{\cos A}{\cos a}}{1 + \dfrac{\cos A}{\cos a}}$$

$$= \frac{\cos a - \cos A}{\cos a + \cos A}$$

By [23] and [22],
$$= \frac{-2 \sin \frac{1}{2}(a + A) \sin \frac{1}{2}(a - A)}{2 \cos \frac{1}{2}(a + A) \cos \frac{1}{2}(a - A)}$$
$$= -\tan \tfrac{1}{2}(a + A) \tan \tfrac{1}{2}(a - A)$$
$$= \tan \tfrac{1}{2}(A + a) \tan \tfrac{1}{2}(A - a).$$

EXERCISE XXXIII. PAGE 148.

1. Show that Napier's Rules lead to the equations contained in Formulas [39], [40], [41], and [42].

$$\sin a = \cos (Co.\ c) \cos (Co.\ A).$$
$$\therefore \sin a = \sin c \sin A. \qquad [39]$$
$$\sin b = \cos (Co.\ c) \cos (Co.\ B).$$
$$\therefore \sin b = \sin c \sin B. \qquad [39]$$
$$\sin (Co.\ A) = \tan b \tan (Co.\ c).$$
$$\therefore \cos A = \tan b \cot c. \qquad [40]$$
$$\sin (Co.\ B) = \tan a \tan (Co.\ c).$$
$$\therefore \cos B = \tan a \cot c. \qquad [40]$$
$$\sin (Co.\ A) = \cos a \cos (Co.\ B).$$
$$\therefore \cos A = \cos a \sin B. \qquad [41]$$
$$\sin (Co.\ B) = \cos b \cos (Co.\ A).$$
$$\therefore \cos B = \cos b \sin A. \qquad [41]$$
$$\sin b = \tan a \tan (Co.\ A).$$
$$\therefore \sin b = \tan a \cot A. \qquad [42]$$
$$\sin a = \tan b \tan (Co.\ B).$$
$$\therefore \sin a = \tan b \cot B. \qquad [42]$$

2. What will Napier's Rules become if we take as the five parts of the triangle the hypotenuse, the two oblique angles, and the *complements* of the two legs?

Each part will be replaced by its complement, and every function will be replaced by its complementary function.

Therefore, Napier's Rules become

RULE I. The cosine of any middle part is equal to the product of the cotangents of the adjacent parts.

RULE II. The cosine of any middle part is equal to the product of the sines of the opposite parts.

EXERCISE XXXIV. PAGE 153.

1. Solve the right triangle, given $a = 36^\circ\, 27'$, $b = 43^\circ\, 32'\, 31''$.

By [38], $\cos c = \cos a \cos b$.

$$\begin{aligned} \log \cos a &= 9.90546 \\ \log \cos b &= \underline{9.86026} \\ \log \cos c &= 9.76572 \end{aligned}$$

$$\therefore c = 54^\circ\, 20'.$$

By [42], $\sin b = \tan a \cot A$.

$$\therefore \tan a = \sin b \tan A.$$

$$\therefore \tan A = \tan a \csc b.$$

$$\begin{aligned} \log \tan a &= 9.86842 \\ \log \csc b &= \underline{0.16185} \\ \log \tan A &= 10.03027 \end{aligned}$$

$$A = 46^\circ\, 59'\, 43''.$$

By [42], $\sin a = \tan b \cot B$.

$$\therefore \tan b = \sin a \tan B.$$

$$\therefore \tan B = \tan b \csc a.$$

$$\begin{aligned} \log \tan b &= 9.97789 \\ \log \csc a &= \underline{0.22613} \\ \log \tan B &= 10.20402 \end{aligned}$$

$$\therefore B = 57^\circ\, 59'\, 19''.$$

2. Solve the right triangle, given $a = 86^\circ\, 40'$, $b = 32^\circ\, 40'$.

By [38], $\cos c = \cos a \cos b$.
By [42], $\tan A = \tan a \csc b$.
By [42], $\tan B = \tan b \csc a$.

$$\begin{aligned} \log \cos a &= 8.76451 \\ \log \cos b &= \underline{9.92522} \\ \log \cos c &= 8.68973 \end{aligned}$$

$$c = 87^\circ\, 11'\, 40''.$$

$$\begin{aligned} \log \tan a &= 11.23475 \\ \log \csc b &= \underline{0.26781} \\ \log \tan A &= 11.50256 \end{aligned}$$

$$A = 88^\circ\, 11'\, 58''.$$

$$\begin{aligned} \log \tan b &= 9.80697 \\ \log \csc a &= \underline{0.00074} \\ \log \tan B &= 9.80771 \end{aligned}$$

$$B = 32^\circ\, 42'\, 39''.$$

3. Solve the right triangle, given $a = 50^\circ$, $b = 36^\circ\, 54'\, 49''$.

By [38], $\cos c = \cos a \cos b$.
By [42], $\tan A = \tan a \csc b$.
By [42], $\tan B = \tan b \csc a$.

$$\begin{aligned} \log \cos a &= 9.80807 \\ \log \cos b &= \underline{9.90284} \\ \log \cos c &= 9.71091 \\ c &= 59^\circ\, 4'\, 26''. \\ \log \tan a &= 10.07619 \\ \log \csc b &= \underline{0.22141} \\ \log \tan A &= 10.29760 \\ A &= 63^\circ\, 15'\, 13''. \\ \log \tan b &= 9.87575 \\ \log \csc a &= \underline{0.11575} \\ \log \tan B &= 9.99150 \\ B &= 44^\circ\, 26'\, 22''. \end{aligned}$$

4. Solve the right triangle, given $a = 120^\circ\, 10'$, $b = 150^\circ\, 59'\, 44''$.

By [38], $\cos c = \cos a \cos b$.
By [42], $\tan A = \tan a \csc b$.
By [42], $\tan B = \tan b \csc a$.

$$\begin{aligned} \log \cos a &= 9.70115\ (n) \\ \log \cos b &= \underline{9.94180}\ (n) \\ \log \cos c &= 9.64295 \\ c &= 63^\circ\, 55'\, 43''. \\ \log \tan a &= 10.23565\ (n) \\ \log \csc b &= \underline{0.31437} \\ \log \tan A &= 10.55002\ (n) \\ A &= 105^\circ\, 44'\, 21''. \\ \log \tan b &= 9.74383\ (n) \\ \log \csc a &= \underline{0.06320} \\ \log \tan B &= 9.80703\ (n) \\ B &= 147^\circ\, 19'\, 47''. \end{aligned}$$

5. Solve the right triangle, given $c = 55^\circ\ 9'\ 32''$, $a = 22^\circ\ 15'\ 7''$.

By [38], $\cos b = \cos c \sec a$.
By [39], $\sin A = \sin a \csc c$.
By [40], $\cos B = \tan a \cot c$.

$\log \cos c = 9.75686$
$\log \sec a = \underline{0.03361}$
$\log \cos b = 9.79047$
$b = 51^\circ\ 53'$.

$\log \sin a = 9.57828$
$\log \csc c = \underline{0.08579}$
$\log \sin A = 9.66407$
$A = 27^\circ\ 28'\ 38''$.

$\log \tan a = 9.61188$
$\log \cot c = \underline{9.84266}$
$\log \cos B = 9.45454$
$B = 73^\circ\ 27'\ 11''$.

6. Solve the right triangle, given $c = 23^\circ\ 49'\ 51''$, $a = 14^\circ\ 16'\ 35''$.

By [38], $\cos b = \cos c \sec a$.
By [39], $\sin A = \sin a \csc c$.
By [40], $\cos B = \tan a \cot c$.

$\log \cos c = 9.96130$
$\log \sec a = \underline{0.01362}$
$\log \cos b = 9.97492$
$b = 19^\circ\ 17'$.

$\log \sin a = 9.39199$
$\log \csc c = \underline{0.39358}$
$\log \sin A = 9.78557$
$A = 37^\circ\ 36'\ 49''$.

$\log \tan a = 9.40562$
$\log \cot c = \underline{10.35488}$
$\log \cos B = 9.76050$
$B = 54^\circ\ 49'\ 23''$.

7. Solve the right triangle, given $c = 44^\circ\ 33'\ 17''$, $a = 32^\circ\ 9'\ 17''$.

By [38], $\cos b = \cos c \sec a$.
By [39], $\sin A = \sin a \csc c$.
By [40], $\cos B = \tan a \cot c$.

$\log \cos c = 9.85283$
$\log \sec a = \underline{0.07231}$
$\log \cos b = 9.92514$
$b = 32^\circ\ 41'$.

$\log \sin a = 9.72608$
$\log \csc c = \underline{0.15391}$
$\log \sin A = 9.87999$
$A = 49^\circ\ 20'\ 16''$.

$\log \tan a = 9.79840$
$\log \cot c = \underline{10.00675}$
$\log \cos B = 9.80515$
$B = 50^\circ\ 19'\ 16''$.

8. Solve the right triangle, given $c = 97^\circ\ 13'\ 4''$, $a = 132^\circ\ 14'\ 12''$.

By [38], $\cos b = \cos c \sec a$.
By [39], $\sin A = \sin a \csc c$.
By [40], $\cos B = \tan a \cot c$.

$\log \cos c = 9.09914\ (n)$
$\log \sec a = \underline{0.17250}\ (n)$
$\log \cos b = 9.27164$
$b = 79^\circ\ 13'\ 38''$.

$\log \sin a = 9.86945$
$\log \csc c = \underline{0.00345}$
$\log \sin A = 9.87290$
$A = 131^\circ\ 43'\ 50''$.

$\log \tan a = 10.04196\ (n)$
$\log \cot c = \underline{9.10259}\ (n)$
$\log \cos B = 9.14455$
$B = 81^\circ\ 58'\ 53''$.

9. Solve the right triangle, given $a = 77^\circ\ 21'\ 50''$, $A = 83^\circ\ 56'\ 40''$.

By [39], $\sin c = \sin a \csc A$.
By [42], $\sin b = \tan a \cot A$.
By [41], $\sin B = \sec a \cos A$.

$\log \sin a = 9.98935$
$\log \csc A = \underline{0.00243}$
$\log \sin c = 9.99178$

$c = 78^\circ\ 53'\ 20''$,

or $= 101^\circ\ 6'\ 40''$.

$\log \tan a = 10.64939$

$\log \cot A = \underline{\ 9.02565}$

$\log \sin b = \ 9.67504$

$b = \ 28^\circ\ 14'\ 31''$,

or $= 151^\circ\ 45'\ 29''$.

$\log \sec a = 0.66004$

$\log \cos A = \underline{9.02323}$

$\log \sin B = 9.68327$

$B = \ 28^\circ\ 49'\ 57''$,

or $= 151^\circ\ 10'\ \ 3''$.

10. Solve the right triangle, given $a = 77^\circ\ 21'\ 50''$, $A = 40^\circ\ 40'\ 40''$.

By [39], $\sin c = \sin a \csc A$.

But $\sin A < \sin a$.

$\therefore \sin c > 1$, which is impossible.

11. Solve the right triangle, given $a = 92^\circ\ 47'\ 32''$, $B = 50^\circ\ 2'\ 1''$.

By [40], $\tan c = \tan a \sec B$.

By [42], $\tan b = \sin a \tan B$.

By [41], $\cos A = \cos a \sin B$.

$\log \tan a = 11.31183\ (n)$

$\log \sec B = \underline{\ 0.19223}$

$\log \tan c = 11.50406\ (n)$

$c = 91^\circ\ 47'\ 40''$.

$\log \sin a = \ 9.99948$

$\log \tan B = \underline{10.07670}$

$\log \tan b = 10.07618$

$b = 49^\circ\ 59'\ 58''$.

$\log \cos a = 8.68765\ (n)$

$\log \sin B = \underline{9.88447}$

$\log \cos A = 8.57212\ (n)$

$A = 92^\circ\ 8'\ 23''$.

12. Solve the right triangle, given $a = 2^\circ\ 0'\ 55''$, $B = 12^\circ\ 40'$.

By [40], $\tan c = \tan a \sec B$.

By [42], $\tan b = \sin a \tan B$.

By [41], $\cos A = \cos a \sin B$.

$\log \tan a = 8.54639$

$\log \sec B = \underline{0.01070}$

$\log \tan c = 8.55709$

$c = 2^\circ\ 3'\ 56''$.

$\log \sin a = 8.54612$

$\log \tan B = \underline{9.35170}$

$\log \tan b = 7.89782$

$b = 0^\circ\ 27'\ 10''$.

$\log \cos a = 9.99973$

$\log \sin B = \underline{9.34100}$

$\log \cos A = 9.34073$

$A = 77^\circ\ 20'\ 28''$.

13. Solve the right triangle, given $a = 20^\circ\ 20'\ 20''$, $B = 38^\circ\ 10'\ 10''$.

By [40], $\tan c = \tan a \sec B$.

By [42], $\tan b = \sin a \tan B$.

By [41], $\cos A = \cos a \sin B$.

$\log \tan a = 9.56900$

$\log \sec B = \underline{0.10448}$

$\log \tan c = 9.67348$

$c = 25^\circ\ 14'\ 38''$.

$\log \sin a = 9.54104$

$\log \tan B = \underline{9.89545}$

$\log \tan b = 9.43649$

$b = 15^\circ\ 16'\ 50''$.

$\log \cos a = 9.97204$

$\log \sin B = \underline{9.79098}$

$\log \cos A = 9.76302$

$A = 54^\circ\ 35'\ 17''$.

14. Solve the right triangle, given $a = 54^\circ\ 30'$, $B = 35^\circ\ 30'$.

By [40], $\tan c = \tan a \sec B$.

By [42], $\tan b = \sin a \tan B$.

By [41,] $\cos A = \cos a \sin B$.

$\log \tan a = 10.14673$

$\log \sec B = \underline{\ 0.08931}$

$\log \tan c = 10.23604$

$c = 59^\circ\ 51'\ 21''$.

$\log \sin a = 9.91069$

$\log \tan B = \underline{9.85327}$

$\log \tan b = 9.76396$

$$\begin{aligned}
b &= 30^\circ\ 8'\ 39.''\\
\log\cos a &= 9.76395\\
\log\sin B &= \underline{9.76395}\\
\log\cos A &= 9.52790\\
A &= 70^\circ\ 17'\ 35''.
\end{aligned}$$

15. Solve the right triangle, given $c = 69^\circ\ 25'\ 11''$, $A = 54^\circ\ 54'\ 42''$.

By [39], $\sin a = \sin c \sin A$.

By [40], $\tan b = \tan c \cos A$.

By [43], $\cot B = \cos c \tan A$.

$$\begin{aligned}
\log\sin c &= 9.97136\\
\log\sin A &= \underline{9.91289}\\
\log\sin a &= 9.88425\\
a &= 50^\circ.\\
\log\tan c &= 10.42541\\
\log\cos A &= \underline{\ \ 9.75954}\\
\log\tan b &= 10.18495\\
b &= 56^\circ\ 50'\ 49''.\\
\log\cos c &= \ \ 9.54595\\
\log\tan A &= \underline{10.15335}\\
\log\cot B &= \ \ 9.69930\\
B &= 63^\circ\ 25'\ 4''.
\end{aligned}$$

16. Solve the right triangle, given $c = 112^\circ\ 48'$, $A = 56^\circ\ 11'\ 56''$.

By [39], $\sin a = \sin c \sin A$.

By [40], $\tan b = \tan c \cos A$.

By [43], $\cot B = \cos c \tan A$.

$$\begin{aligned}
\log\sin c &= 9.96467\\
\log\sin A &= \underline{9.91958}\\
\log\sin a &= 9.88425\\
a &= 50^\circ.\\
\log\tan c &= 10.37638\ (n)\\
\log\cos A &= \underline{\ \ 9.74532}\\
\log\tan b &= 10.12170\ (n)\\
b &= 127^\circ\ 4'\ 30''.\\
\log\cos c &= \ \ 9.58829\ (n)\\
\log\tan A &= \underline{10.17427}\\
\log\cot B &= \ \ 9.76256\ (n)\\
B &= 120^\circ\ 3'\ 50''.
\end{aligned}$$

17. Solve the right triangle, given $c = 46^\circ\ 40'\ 12''$, $A = 37^\circ\ 46'\ 9''$.

By [39], $\sin a = \sin c \sin A$.

By [40], $\tan b = \tan c \cos A$.

By [43], $\cot B = \cos c \tan A$.

$$\begin{aligned}
\log\sin c &= 9.86178\\
\log\sin A &= \underline{9.78709}\\
\log\sin a &= 9.64887\\
a &= 26^\circ\ 27'\ 24''.\\
\log\tan c &= 10.02533\\
\log\cos A &= \underline{\ \ 9.89789}\\
\log\tan b &= \ \ 9.92322\\
b &= 39^\circ\ 57'\ 42''.\\
\log\cos c &= 9.83645\\
\log\tan A &= \underline{9.88920}\\
\log\cot B &= 9.72565\\
B &= 62^\circ\ 0'\ 4''.
\end{aligned}$$

18. Solve the right triangle, given $c = 118^\circ\ 40'\ 1''$, $A = 128^\circ\ 0'\ 4''$.

By [39], $\sin a = \sin c \sin A$.

By [40], $\tan b = \tan c \cos A$.

By [43], $\cot B = \cos c \tan A$.

$$\begin{aligned}
\log\sin c &= 9.94321\\
\log\sin A &= \underline{9.89652}\\
\log\sin a &= 9.83973\\
a &= 136^\circ\ 15'\ 32''.\\
\log\tan c &= 10.26222\ (n)\\
\log\cos A &= \underline{\ \ 9.78935}\ (n)\\
\log\tan b &= 10.05157\\
b &= 48^\circ\ 23'\ 38''.\\
\log\cos c &= \ \ 9.68098\ (n)\\
\log\tan A &= \underline{10.10717}\ (n)\\
\log\cot B &= \ \ 9.78815\\
B &= 58^\circ\ 27'\ 4''.
\end{aligned}$$

19. Solve the right triangle, given $A = 63^\circ\ 15'\ 12''$, $B = 135^\circ\ 33'\ 39''$.

By [41], $\cos a = \cos A \csc B$.

By [41], $\cos b = \cos B \csc A$.

By [43], $\cos c = \cot A \cot$

$$\begin{aligned}
\log\cos A &= 9.65326\\
\log\csc B &= \underline{0.15480}\\
\log\cos a &= 9.80806\\
a &= 50^\circ\ 0'\ 4''.
\end{aligned}$$

$$\begin{aligned}
\log\cos B &= 9.85369\ (n)\\
\log\csc A &= \underline{0.04915}\\
\log\cos b &= 9.90284\ (n)\\
b &= 143^\circ\ 5'\ 12''.
\end{aligned}$$

$$\begin{aligned}
\log\cot A &= \ \ 9.70241\\
\log\cot B &= \underline{10.00850}\ (n)\\
\log\cos c &= \ \ 9.71091\ (n)\\
c &= 120^\circ\ 55'\ 34''.
\end{aligned}$$

20. Solve the right triangle, given $A = 116^\circ\ 43'\ 12''$, $B = 116^\circ\ 31'\ 25''$.

By [41], $\cos a = \cos A \csc B$.

By [41], $\cos b = \cos B \csc A$.

By [43], $\cos c = \cot A \cot B$.

$$\begin{aligned}
\log\cos A &= 9.65286\ (n)\\
\log\csc B &= \underline{0.04830}\\
\log\cos a &= 9.70116\ (n)\\
a &= 120^\circ\ 10'\ 3''.
\end{aligned}$$

$$\begin{aligned}
\log\cos B &= 9.64988\ (n)\\
\log\csc A &= \underline{0.04904}\\
\log\cos b &= 9.69892\ (n)\\
b &= 119^\circ\ 59'\ 46''.
\end{aligned}$$

$$\begin{aligned}
\log\cot A &= 9.70190\ (n)\\
\log\cot B &= \underline{9.69818}\ (n)\\
\log\cos c &= 9.40008\\
c &= 75^\circ\ 26'\ 58''.
\end{aligned}$$

21. Solve the right triangle, given $A = 46^\circ\ 59'\ 42''$, $B = 57^\circ\ 59'\ 17''$.

By [41], $\cos a = \cos A \csc B$.

By [41], $\cos b = \cos B \csc A$.

By [43], $\cos c = \cot A \cot B$.

$$\begin{aligned}
\log\cos A &= 9.83382\\
\log\csc B &= \underline{0.07164}\\
\log\cos a &= 9.90546\\
a &= 36^\circ\ 27'.
\end{aligned}$$

$$\begin{aligned}
\log\cos B &= 9.72435\\
\log\csc A &= \underline{0.13591}\\
\log\cos b &= 9.86026\\
b &= 43^\circ\ 32'\ 30''.
\end{aligned}$$

$$\begin{aligned}
\log\cot A &= 9.96973\\
\log\cot B &= \underline{9.79599}\\
\log\cos c &= 9.76572\\
c &= 54^\circ\ 20'.
\end{aligned}$$

22. Solve the right triangle, given $A = 90^\circ$, $B = 88^\circ\ 24'\ 35''$.

By [41], $\cos a = \cos A \csc B$.

By [41], $\cos b = \cos B \csc A$.

By [43], $\cos c = \cot A \cot B$.

$$\begin{aligned}
\cos A &= 0.\\
\therefore \cos a &= 0.\\
\therefore a &= 90^\circ.\\
\csc A &= 1.\\
\therefore b &= B.\\
\therefore b &= 88^\circ\ 24'\ 35''.\\
\cot A &= 0.\\
\therefore \cos c &= 0.\\
\therefore c &= 90^\circ.
\end{aligned}$$

23. Define a quadrantal triangle, and show how its solution may be reduced to that of the right triangle.

A quadrantal triangle is a triangle that has one or more of its sides equal to a quadrant.

Let $A'B'C'$ be a quadrantal triangle with side $A'B' = 90^\circ$, or a quadrant.

Let ABC be its polar triangle.

Then, since

$$A'B' + C = 180^\circ,\ C = 90^\circ.$$

Hence, ABC is a right triangle.

Therefore, all parts of the polar triangle may be found by formulas for the right triangle.

The parts of $A'B'C'$ may then be found by subtracting proper parts of ABC from 180°.

24. Solve the quadrantal triangle whose sides are $a = 174° 12' 49''$, $b = 94° 8' 20''$, $c = 90°$.

Let A' B', C', a', b' c' represent the corresponding angles and sides of the polar triangle.

Then
$$\begin{aligned} A' &= 5° 47' 11'', \\ B' &= 85° 51' 40'', \\ C' &= 90°. \end{aligned}$$

By Prob. 7, Ex. XXXII,
$$\tan^2 \tfrac{1}{2} c' = -\cos(B' + A') \sec(B' - A').$$

By Prob. 8, Ex. XXXII,
$$\begin{aligned} \tan^2 \tfrac{1}{2} b' &= \tan[\tfrac{1}{2}(B' + A') - 45°] \tan[45° + \tfrac{1}{2}(B' - A')], \\ \tan^2 \tfrac{1}{2} a' &= \tan[\tfrac{1}{2}(B' + A') - 45°] \tan[45° - \tfrac{1}{2}(B' - A')]. \end{aligned}$$

$$\begin{aligned} B' + A' &= 91° 38' 51''. \\ B' - A' &= 80° \ 4' 29''. \\ \tfrac{1}{2}(B' + A') - 45° &= 0° 49' 25.5''. \\ 45° + \tfrac{1}{2}(B' - A') &= 85° 2' 14.5''. \\ 45° - \tfrac{1}{2}(B' - A') &= 4° 57' 45.5''. \end{aligned}$$

$$\begin{aligned} \log\cos(B' + A') &= 8.45864 \\ \log\sec(B' - A') &= \underline{0.76356} \\ 2 &\overline{)\,9.22220} \\ \log\tan \tfrac{1}{2} c' &= 9.61110 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2} c' &= 22° 12' 56\tfrac{1}{2}''. \\ c' &= 44° 25' 53''. \\ C &= 135° 34' \ 7''. \end{aligned}$$

$$\begin{aligned} \log\tan[\tfrac{1}{2}(B' + A') - 45°] &= 8.15770 \\ \log\tan[45° + \tfrac{1}{2}(B' - A')] &= \underline{11.06133} \\ 2 &\overline{)\ 9.21903} \\ \log\tan \tfrac{1}{2} b' &= 9.60952 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2} b' &= 22° \ 8' 35''. \\ b' &= 44° 17' 10''. \\ B &= 135° 42' 50''. \end{aligned}$$

$$\begin{aligned} \log\tan[\tfrac{1}{2}(B' + A') - 45°] &= 8.15770 \\ \log\tan[45° - \tfrac{1}{2}(B' - A')] &= \underline{8.93867} \\ 2 &\overline{)\,7.09637} \\ \log\tan \tfrac{1}{2} a' &= 8.54819 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2} a' &= 2° \ 1' 25''. \\ a' &= 4° \ 2' 50''. \\ A &= 175° 57' 10''. \end{aligned}$$

25. Solve the quadrantal triangle in which $c = 90°$, $A = 110° \, 47' \, 50''$, $B = 135° \, 35' \, 34''$.

Let A', B', C', a', b', c' represent the corresponding angles and sides of the polar triangle.

$$\begin{aligned} \text{Then} \qquad a' &= 69° \, 12' \, 10''. \\ b' &= 44° \, 24' \, 26''. \\ C' &= 90°. \end{aligned}$$

$$\begin{aligned} &\text{By [42],} & \tan A' &= \tan a' \csc b'. \\ &\text{By [42],} & \tan B' &= \tan b' \csc a'. \\ &\text{By [38],} & \cos c' &= \cos a' \cos b'. \end{aligned}$$

$$\begin{aligned} \log \tan a' &= 10.42043 \\ \log \csc b' &= \underline{\ \ 0.15505} \\ \log \tan A' &= 10.57548 \\ A' &= \ \ 75° \ \ 6' \, 58''. \\ a &= 104° \, 53' \ \ 2''. \end{aligned}$$

$$\begin{aligned} \log \tan b' &= \ \ 9.99101 \\ \log \csc a' &= \underline{\ \ 0.02926} \\ \log \tan B' &= 10.02027 \\ B' &= \ \ 46° \, 20' \, 12''. \\ b &= 133° \, 39' \, 48''. \end{aligned}$$

$$\begin{aligned} \log \cos a' &= 9.55031 \\ \log \cos b' &= \underline{9.85394} \\ \log \cos c' &= 9.40425 \\ c' &= \ \ 75° \, 18' \, 21''. \\ C &= 104° \, 41' \, 39''. \end{aligned}$$

26. Given in a spherical triangle A, C, and c each equal to $90°$; solve the triangle.

$$\begin{aligned} \text{By [39],} \qquad \sin a &= \sin c \sin A \\ &= 1 \times 1 = 1. \\ \therefore a &= 90°. \end{aligned}$$

Then B is the pole of b, and $B = b$; but B and b are otherwise indeterminate.

27. Given $A = 60°$, $C = 90°$, and · $90°$: solve the triangle.

$$\begin{aligned} &\text{By [39],} & \sin a &= \sin c \sin A. \\ &\text{By [40],} & \tan b &= \tan c \cos A. \\ &\text{By [43],} & \cot B &= \cos c \tan A. \end{aligned}$$

$$\begin{aligned} \sin a &= \sin A. \\ \therefore a &= A = 60°. \\ \tan b &= \infty \times \tfrac{1}{2} \\ &= \infty. \\ \therefore b &= 90°. \\ \cot B &= 0 \times \sqrt{3} \\ &= 0. \\ \therefore B &= 90°. \end{aligned}$$

28. In a right spherical triangle, given $A = 42° \, 24' \, 9''$, $B = 9° \, 4' \, 11''$; solve the triangle.

$$\begin{aligned} &\text{By [43],} & \cos c &= \cot A \cot B. \\ &\text{Now} & \cot A &> 1, \\ &\text{and} & \cot B &> 1. \\ & & \therefore \cos c &> 1, \end{aligned}$$

which is impossible.

Therefore, the triangle is impossible.

29. In a right spherical triangle, given $a = 119° \ 11'$, $B = 126° \ 54'$; solve the triangle.

$$\begin{aligned} &\text{By [42],} & \tan b &= \sin a \tan B. \\ &\text{By [40],} & \tan c &= \tan a \sec B. \\ &\text{By [41],} & \cos A &= \cos a \sin B. \end{aligned}$$

$$\begin{aligned} \log \sin a &= \ \ 9.94105 \\ \log \tan B &= \underline{10.12446} \ (n) \\ \log \tan b &= 10.06551 \ (n) \\ b &= 130° \, 41' \, 42''. \end{aligned}$$

$$\begin{aligned} \log \tan a &= 10.25298 \ (n) \\ \log \sec B &= \underline{\ \ 0.22154} \ (n) \\ \log \tan c &= 10.47452 \\ c &= 71° \, 27' \, 43''. \end{aligned}$$

$$\begin{aligned} \log \cos a &= 9.68807 \ (n) \\ \log \sin B &= \underline{9.90292} \\ \log \cos A &= 9.59099 \ (n) \\ A &= 112° \, 57' \, 2''. \end{aligned}$$

30. In a right spherical triangle, given $c = 50°$, $b = 44° 18' 39''$; solve the triangle.

By [38], $\cos a = \cos c \sec b$.
By [40], $\cos A = \tan b \cot c$.
By [39], $\sin B = \sin b \csc c$.

$$\begin{aligned} \log \cos c &= 9.80807 \\ \log \sec b &= \underline{0.14535} \\ \log \cos a &= 9.95342 \\ a &= 26° 3' 51''. \end{aligned}$$

$$\begin{aligned} \log \tan b &= 9.98955 \\ \log \cot c &= \underline{9.92381} \\ \log \cos A &= 9.91336 \\ A &= 35°. \end{aligned}$$

$$\begin{aligned} \log \sin b &= 9.84419 \\ \log \csc c &= \underline{0.11575} \\ \log \sin B &= 9.95994 \\ B &= 65° 46'. \end{aligned}$$

31. In a right spherical triangle, given $A = 156° 20' 30''$, $a = 65° 15' 45''$; solve the triangle.

The triangle is impossible, because a and A are unlike in kind.

32. In a right spherical triangle, given $A = 74° 12' 31''$, $c = 64° 28' 47''$; solve the triangle.

By [39], $\sin a = \sin c \sin A$.
By [40], $\tan b = \tan c \cos A$.
By [43], $\cot B = \cos c \tan A$.

$$\begin{aligned} \log \sin c &= 9.95542 \\ \log \sin A &= \underline{9.98329} \\ \log \sin a &= 9.93871 \\ a &= 60° 16' 17''. \end{aligned}$$

$$\begin{aligned} \log \tan c &= 10.32111 \\ \log \cos A &= \underline{\ \ 9.43479} \\ \log \tan b &= \ \ 9.75590 \\ b &= 29° 41' 4''. \end{aligned}$$

$$\begin{aligned} \log \cos c &= \ \ 9.63431 \\ \log \tan A &= \underline{10.54851} \\ \log \cot B &= 10.18282 \\ B &= 33° 16' 54''. \end{aligned}$$

33. In a right spherical triangle, given $a = 112° 42' 38''$, $B = 44° 28' 44''$; solve the triangle.

By [42], $\tan b = \sin a \tan B$.
By [40], $\tan c = \tan a \sec B$.
By [41], $\cos A = \cos a \sin B$.

$$\begin{aligned} \log \sin a &= 9.96495 \\ \log \tan B &= \underline{9.99210} \\ \log \tan b &= 9.95705 \\ b &= 42° 10' 17''. \end{aligned}$$

$$\begin{aligned} \log \tan a &= 10.37828\ (n) \\ \log \sec B &= \underline{\ \ 0.14660} \\ \log \tan c &= 10.52488\ (n) \\ c &= 106° 37' 37''. \end{aligned}$$

$$\begin{aligned} \log \cos a &= 9.58667\ (n) \\ \log \sin B &= \underline{9.84550} \\ \log \cos A &= 9.43217\ (n) \\ A &= 105° 41' 39''. \end{aligned}$$

34. In a right spherical triangle, given $b = 48° 12' 48''$, $A = 108° 14' 44''$; solve the triangle.

By [42], $\tan a = \sin b \tan A$.
By [40], $\tan c = \tan b \sec A$.
By [41], $\cos B = \cos b \sin A$.

$$\begin{aligned} \log \sin b &= \ \ 9.87253 \\ \log \tan A &= \underline{10.48192}\ (n) \\ \log \tan a &= 10.35445\ (n) \\ a &= 113° 51' 5''. \end{aligned}$$

$$\begin{aligned} \log \tan b &= 10.04882 \\ \log \sec A &= \underline{\ \ 0.50433}\ (n) \\ \log \tan c &= 10.55315\ (n) \\ c &= 105° 37' 54''. \end{aligned}$$

$$\begin{aligned} \log\cos b &= 9.82371 \\ \log\sin A &= \underline{9.97760} \\ \log\cos B &= 9.80131 \\ B &= 50^\circ\ 44'\ 19''. \end{aligned}$$

35. In a right spherical triangle, given $A = 122^\circ 58' 47''$, $B = 104^\circ 17' 55''$; solve the triangle.

By [41], $\cos a = \cos A \csc B.$

By [41], $\cos b = \csc A \cos B.$

By [43], $\cos c = \cot A \cot B.$

$$\begin{aligned} \log\cos A &= 9.73587\ (n) \\ \log\csc B &= \underline{0.01367} \\ \log\cos a &= 9.74954\ (n) \\ a &= 124^\circ\ 10'\ 37''. \end{aligned}$$

$$\begin{aligned} \log\csc A &= 0.07631 \\ \log\cos B &= \underline{9.39266}\ (n) \\ \log\cos b &= 9.46897\ (n) \\ b &= 107^\circ\ 7'\ 22''. \end{aligned}$$

$$\begin{aligned} \log\cot A &= 9.81218\ (n) \\ \log\cot B &= \underline{9.40632}\ (n) \\ \log\cos c &= 9.21850 \\ c &= 80^\circ\ 28'\ 49''. \end{aligned}$$

36. If the legs a and b of a right spherical triangle are equal, show that $\cos a = \cot A = \sqrt{\cos c}$.

By [38], $\cos c = \cos a \cos b.$

But $\cos a = \cos b.$

$$\begin{aligned} \therefore \cos c &= \cos^2 a. \\ \cos^2 a &= \cos c. \\ \therefore \cos a &= \sqrt{\cos c}. \end{aligned}$$

By [42], $\sin b = \tan a \cot A.$

But $\sin a = \sin b.$

$$\begin{aligned} \therefore \sin a &= \tan a \cot A. \\ \sin a &= \frac{\sin a \cot A}{\cos a}. \\ \therefore \cos a &= \cot A. \\ \therefore \cos a &= \cot A = \sqrt{\cos c}. \end{aligned}$$

37. In a right spherical triangle show that

$$\cos^2 A \sin^2 c = \sin(c+a)\sin(c-a).$$

By [39],

$$\begin{aligned} \sin A &= \frac{\sin a}{\sin c}. \\ \sin^2 A &= \frac{\sin^2 a}{\sin^2 c}. \\ \therefore \cos^2 A &= 1 - \frac{\sin^2 a}{\sin^2 c} \\ &= \frac{\sin^2 c - \sin^2 a}{\sin^2 c}. \\ \cos^2 A \sin^2 c &= \sin^2 c - \sin^2 a. \end{aligned}$$

By [4], $\sin(c+a) = \sin c \cos a + \cos c \sin a.$

By [8], $\sin(c-a) = \sin c \cos a - \cos c \sin a.$

$$\begin{aligned} \therefore \sin(c+a)\sin(c-a) &= \sin^2 c \cos^2 a - \cos^2 c \sin^2 a \\ &= \sin^2 c(1 - \sin^2 a) - (1 - \sin^2 c)\sin^2 a \\ &= \sin^2 c - \sin^2 c \sin^2 a - \sin^2 a + \sin^2 c \sin^2 a \\ &= \sin^2 c - \sin^2 a. \\ \therefore \cos^2 A \sin^2 c &= \sin(c+a)\sin(c-a). \end{aligned}$$

38. In a right spherical triangle show that

$$\tan a \cos c = \sin b \cot B.$$

By [42], $$\sin b = \tan a \cot A.$$

$$\therefore \cot A = \frac{\sin b}{\tan a}.$$

By [43], $$\cos c = \cot A \cot B.$$

$$\therefore \cot A = \frac{\cos c}{\cot B}.$$

$$\therefore \frac{\cos c}{\cot B} = \frac{\sin b}{\tan a}.$$

$$\therefore \tan a \cos c = \sin b \cot B.$$

39. In a right spherical triangle show that

$$\sin^2 A = \cos^2 B + \sin^2 a \sin^2 B,$$

By [41], $$\cos B = \cos b \sin A.$$

$$\therefore \sin A = \frac{\cos B}{\cos b}.$$

$$\sin^2 A = \frac{\cos^2 B}{\cos^2 b}$$

$$= \cos^2 B \sec^2 b$$

By Prob. 2, Ex. V, $$= \cos^2 B (1 + \tan^2 b)$$

$$= \cos^2 B + \tan^2 b \cos^2 B.$$

By [42], $$\sin a = \tan b \cot B.$$

$$\therefore \tan b = \sin a \tan B.$$

$$\therefore \sin^2 A = \cos^2 B + \sin^2 a \tan^2 B \cos^2 B.$$

$$\therefore \sin^2 A = \cos^2 B + \sin^2 a \sin^2 B.$$

40. In a right spherical triangle show that

$$\sin (b + c) = 2 \cos^2 \tfrac{1}{2} A \cos b \sin c.$$

By [4], $$\sin (b + c) = \sin b \cos c + \cos b \sin c$$

$$= \left(\frac{\sin b \cos c}{\cos b \sin c} + 1\right) \cos b \sin c$$

$$= (\tan b \cot c + 1) \cos b \sin c.$$

By [40], $$\tan b \cot c = \cos A.$$

$$\therefore \tan b \cot c + 1 = \cos A + 1$$

By [17], $$= 2 \cos^2 \tfrac{1}{2} A.$$

$$\therefore \sin (b + c) = 2 \cos^2 \tfrac{1}{2} A \cos b \sin c.$$

41. In a right spherical triangle show that

$$\sin(c-b) = 2\sin^2 \tfrac{1}{2}A \cos b \sin c.$$

By [8], $\sin(c-b) = \sin c \cos b - \cos c \sin b$

$$= \sin c \cos b\left(1 - \frac{\cos c \sin b}{\sin c \cos b}\right)$$

$$= \sin c \cos b\,(1 - \cot c \tan b).$$

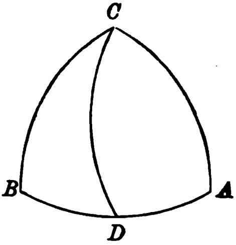
By [40], $\cot c \tan b = \cos A.$

$$\therefore 1 - \cot c \tan b = 1 - \cos A$$

By [16], $= 2\sin^2 \tfrac{1}{2}A.$

$$\therefore \sin(c-b) = 2\sin^2 \tfrac{1}{2}A \cos b \sin c.$$

42. If, in a right spherical triangle, p denotes the arc of the great circle passing through the vertex of the right angle and perpendicular to the hypotenuse, m and n, the segments of the hypotenuse made by this arc adjacent to the legs a and b, show that (i) $\tan^2 a = \tan c \tan m$, (ii) $\sin^2 p = \tan m \tan n$.



(i) In the triangle BCA, by Napier's Rules,

$$\cos B = \tan a \cot c.$$

$$\therefore \tan a = \frac{\cos B}{\cot c}. \quad (1)$$

In the triangle CBD, by Napier's Rules,

$$\cos B = \tan BD \cot BC$$

$$= \tan m \cot a.$$

$$\therefore \tan a = \frac{\tan m}{\cos B}. \quad (2)$$

Multiply (1) by (2), $\tan^2 a = \dfrac{\tan m}{\cos B} \times \dfrac{\cos B}{\cot c}$

$$= \tan m \tan c.$$

(ii) In the triangle CBD, by Napier's Rules,

$$\sin p = \tan m \cot BCD. \quad (3)$$

In the triangle CAD, by Napier's Rules,

$$\sin p = \tan n \cot DCA. \quad (4)$$

Multiply (3) by (4), $\sin^2 p = \tan m \tan n \cot BCD \cot DCA.$

But $BCD + DCA = 90°.$

$$\therefore \cot BCD \times \cot DCA = 1.$$

$$\therefore \sin^2 p = \tan m \tan n.$$

EXERCISE XXXV. PAGE 157.

1. In an isosceles spherical triangle, given the base b and the side a; find A the angle at the base, B the angle at the vertex, and h the altitude.

Let ABA' be an isosceles triangle, A and A' the equal angles, a and a' the equal sides.

Draw h, the arc of a great circle, from $B \perp$ to AA', meeting AA' in C.

Then, in the right triangle $A'BC$,

$$b = \tfrac{1}{2} b \text{ in triangle } ABA',$$
$$c = a \text{ in triangle } ABA',$$
$$B = \tfrac{1}{2} B \text{ in triangle } ABA'.$$

By [40], $\cos A = \cot a \tan \tfrac{1}{2} b.$

By [39], $\sin \tfrac{1}{2} B = \csc a \sin \tfrac{1}{2} b.$

By [38], $\cos h = \cos a \sec \tfrac{1}{2} b.$

2. In an equilateral spherical triangle, given the side a; find the angle A.

In the equilateral triangle $AA'A''$ draw arc $AC \perp$ to $A'A''$.

Then, in the right triangle $AA'C$,

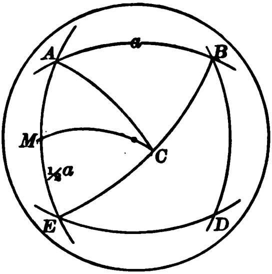
$$\sin \tfrac{1}{2} a = \sin a \sin \tfrac{1}{2} A.$$

$$\sin \tfrac{1}{2} A = \frac{\sin \tfrac{1}{2} a}{\sin a}$$

By [12], $$= \frac{\sin \tfrac{1}{2} a}{2 \sin \tfrac{1}{2} a \cos \tfrac{1}{2} a}$$
$$= \tfrac{1}{2} \sec \tfrac{1}{2} a.$$

3. Given the side a of a regular spherical polygon of n sides; find the angle A of the polygon, the distance R from the centre of the polygon to one of the vertices, and the distance r from the centre to the middle point of one of the sides.

In the regular polygon $ABDE$ draw arcs of great circles from the vertices A, B, etc., through the centre C, and from C to M, the middle of one side.



Then $$ACE = \frac{360°}{n},$$
$$ACM = \frac{180°}{n},$$
$$CAM = \tfrac{1}{2} A,$$
$$AM = \tfrac{1}{2} a,$$
$$AC = R,$$
$$MC = r.$$

By Napier's Rules,

$$\cos \frac{180°}{n} = \cos \tfrac{1}{2} a \sin \tfrac{1}{2} A,$$
$$\sin \tfrac{1}{2} a = \sin R \sin \frac{180°}{n},$$
$$\sin r = \tan \tfrac{1}{2} a \cot \frac{180°}{n}.$$

Whence,

$$\sin \tfrac{1}{2} A = \sec \tfrac{1}{2} a \cos \frac{180°}{n},$$
$$\sin R = \sin \tfrac{1}{2} a \csc \frac{180°}{n},$$
$$\sin r = \tan \tfrac{1}{2} a \cot \frac{180°}{n}.$$

4. Compute the dihedral angles made by the faces of the five regular polyhedrons.

If a sphere is described about a vertex of the polyhedron as a centre with a radius equal to an edge of the polyhedron, the adjacent vertices of the polyhedron lie on the surface of the sphere and are the vertices of a regular spherical polygon, of which the angles are required.

If a is the length of a side of this polygon, *i.e.*, one of the angles of a face of the polyhedron, and n the number of sides, *i.e.*, the number of faces of the polyhedron which meet at a vertex, we have for the different cases:

POLYHEDRON.	a	n
Tetrahedron . . .	60°	3
Hexahedron . . .	90°	3
Octahedron	60°	4
Dodecahedron . . .	108°	3
Icosahedron . . .	60°	5

But if A is an angle of the spherical polygon, we have, from Prob. 3,

$$\sin\tfrac{1}{2}A = \sec\tfrac{1}{2}a\cos\frac{180^\circ}{n}.$$

Hence, for the different cases:

POLYHEDRON.	SIN $\frac{1}{2}A$.	LOG SIN $\frac{1}{2}A$.	A.
Tetrahedron . . .	$\frac{1}{3}\sqrt{3}$	9.76144	70° 31′ 46″
Hexahedron . . .	$\frac{1}{2}\sqrt{2}$	9.84949	90°
Octahedron	$\frac{1}{3}\sqrt{6}$	9.91195	109° 28′ 14″
Dodecahedron . . .	$\frac{1}{2}\sec 54^\circ$	9.92975	116° 33′ 45″
Icosahedron . . .	$\frac{2}{3}\sqrt{3}\cos 36^\circ$	9.97043	138° 11′ 36″

5. A spherical square is a regular spherical quadrilateral. Find the angle A of the square, having given the side a.

This is a special case of Prob. 3 for which $n = 4$.

$$\therefore \sin\tfrac{1}{2}A = \sec\tfrac{1}{2}a\cos\frac{180^\circ}{4}$$

$$= \tfrac{1}{2}\sqrt{2}\sec\tfrac{1}{2}a.$$

Also $\cos\frac{1}{2}A = \sqrt{1 - \frac{1}{2}\sec^2\frac{1}{2}a}$.

$$\therefore \cot\tfrac{1}{2}A = \frac{\cos\frac{1}{2}A}{\sin\frac{1}{2}A}$$

$$= \sqrt{\frac{1 - \frac{1}{2}\sec^2\frac{1}{2}a}{\frac{1}{2}\sec^2\frac{1}{2}a}}$$

$$= \sqrt{2\cos^2\tfrac{1}{2}a - 1}$$

By [13], $= \sqrt{\cos a}$.

EXERCISE XXXVI. PAGE 160.

1. What do Formulas [44] become if $A = 90°$? if $B = 90°$? if $C = 90°$? if $a = 90°$? if $A = B = 90°$? if $a = b = 90°$?

If $A = 90°$,

$$\sin a \sin B = \sin b,$$
$$\sin a \sin C = \sin c.$$

If $B = 90°$,

$$\sin a = \sin b \sin A,$$
$$\sin b \sin C = \sin c.$$

If $C = 90°$,

$$\sin a = \sin c \sin A,$$
$$\sin b = \sin c \sin B.$$

If $a = 90°$,

$$\sin B = \sin b \sin A,$$
$$\sin C = \sin c \sin A.$$

If $A = B = 90°$,

$$\sin a = \sin b,$$
$$\sin c = \sin a \sin C = \sin b \sin C.$$

If $a = b = 90°$,

$$\sin B = \sin A,$$
$$\sin C = \sin c \sin A = \sin c \sin B.$$

2. What do Formulas [45] become if $A = 90°$? if $B = 90°$? if $C = 90°$? if $A = B = C = 90°$?

If $A = 90°$,

$$\cos a = \cos b \cos c.$$

If $B = 90°$,

$$\cos b = \cos a \cos c.$$

If $C = 90°$,

$$\cos c = \cos a \cos b.$$

If $A = B = C = 90°$,

$$\cos a = \cos b \cos c,$$
$$\cos b = \cos a \cos c,$$
$$\cos c = \cos a \cos b.$$

3. What does the first of Formulas [45] become if $A = 0°$? if $A = 90°$? if $A = 180°$?

If $A = 0°$,

$$\cos a = \cos b \cos c + \sin b \sin c = \cos (b - c).$$

If $A = 90°$,

$$\cos a = \cos b \cos c.$$

If $A = 180°$,

$$\cos a = \cos b \cos c - \sin b \sin c = \cos (b + c).$$

4. From Formulas [45] deduce Formulas [46] by means of the relations between polar triangles (Theorem 4, p. 141).

Substituting in Formulas [45] for a, b, c, and A, their equals, $180° - A'$, $180° - B'$, $180° - C'$, and $180° - a'$, we obtain

$$\cos (180° - A') = \cos (180° - B') \cos (180° - C') + \sin (180° - B') \sin (180° - C') \cos (180° - a').$$

$$\therefore -\cos A' = \cos B' \cos C' - \sin B' \sin C' \cos a'.$$

$$\cos A' = -\cos B' \cos C' + \sin B' \sin C' \cos a';$$

and similarly,

$$\cos B' = -\cos A' \cos C' + \sin A' \sin C' \cos b';$$
$$\cos C' = -\cos A' \cos B' + \sin A' \sin B' \cos c'.$$

EXERCISE XXXVII. PAGE 167.

1. What are the formulas for computing the side a when b, c, and A are given; and for computing the side b when a, c, and B are given?

(i) In Fig. 100 suppose p drawn from C, dividing c into m and n.

Then the required formulas are obtained by advancing the letters in

$$\begin{aligned}\tan m &= \tan a \cos C,\\ \cos c &= \cos a \sec m \cos (b - m).\end{aligned}$$

Hence, the required formulas are

$$\begin{aligned}\tan m &= \tan b \cos A,\\ \cos a &= \cos b \sec m \cos (c - m).\end{aligned}$$

(ii) By drawing p from A, and advancing the letters two steps,

$$\begin{aligned}\tan m &= \tan c \cos B,\\ \cos b &= \cos c \sec m \cos (a - m).\end{aligned}$$

2.

Given	find
$a = 88^\circ 12' 20''$,	$A = 63^\circ 15' 11''$,
$b = 124^\circ 7' 17''$,	$B = 132^\circ 17' 58''$,
$C = 50^\circ 2' 1''$;	$c = 59^\circ 4' 17''$.

$$\begin{aligned}\tfrac{1}{2}(b - a) &= 17^\circ 57' 28.5''.\\ \tfrac{1}{2}(b + a) &= 106^\circ 9' 48.5''.\\ \tfrac{1}{2} C &= 25^\circ 1' 0.5''.\end{aligned}$$

$$\begin{aligned}\log \cos \tfrac{1}{2}(b - a) &= 9.97831\\ \log \sec \tfrac{1}{2}(b + a) &= 0.55536\ (n)\\ \log \cot \tfrac{1}{2} C &= \underline{10.33100}\\ \log \tan \tfrac{1}{2}(B + A) &= 10.86467\ (n)\end{aligned}$$

$$\tfrac{1}{2}(B + A) = 97^\circ 46' 34.7''.$$

$$\begin{aligned}\log \sec \tfrac{1}{2}(B + A) &= 0.86868\ (n)\\ \log \cos \tfrac{1}{2}(b + a) &= 9.44464\ (n)\\ \log \sin \tfrac{1}{2} C &= \underline{9.62622}\\ \log \cos \tfrac{1}{2} c &= 9.93954\end{aligned}$$

$$\tfrac{1}{2} c = 29^\circ 32' 8.6''.$$

$$\begin{aligned}\log \sin \tfrac{1}{2}(b - a) &= 9.48900\\ \log \csc \tfrac{1}{2}(b + a) &= 0.01751\\ \log \cot \tfrac{1}{2} C &= \underline{0.33100}\\ \log \tan \tfrac{1}{2}(B - A) &= 9.83751\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}(B - A) &= 34^\circ 31' 23.6''.\\ \tfrac{1}{2}(B + A) &= 97^\circ 46' 34.7''.\\ A &= 63^\circ 15' 11''.\\ B &= 132^\circ 17' 58''.\\ c &= 59^\circ 4' 17''.\end{aligned}$$

3.

Given	find
$a = 120^\circ 55' 35''$,	$A = 129^\circ 58' 2''$,
$b = 88^\circ 12' 20''$,	$B = 63^\circ 15' 8''$,
$C = 47^\circ 42' 1''$;	$c = 55^\circ 52' 40''$.

$$\begin{aligned}\tfrac{1}{2}(a - b) &= 16^\circ 21' 37.5''.\\ \tfrac{1}{2}(a + b) &= 104^\circ 33' 57.5''.\\ \tfrac{1}{2} C &= 23^\circ 51' 0.5''.\end{aligned}$$

$$\begin{aligned}\log \cos \tfrac{1}{2}(a - b) &= 9.98205\\ \log \sec \tfrac{1}{2}(a + b) &= 0.59947\ (n)\\ \log \cot \tfrac{1}{2} C &= \underline{10.35448}\\ \log \tan \tfrac{1}{2}(A + B) &= 10.93600\ (n)\end{aligned}$$

$$\tfrac{1}{2}(A + B) = 96^\circ 36' 35.5''.$$

$$\begin{aligned}\log \sin \tfrac{1}{2}(a - b) &= 9.44976\\ \log \csc \tfrac{1}{2}(a + b) &= 0.01419\\ \log \cot \tfrac{1}{2} C &= \underline{10.35448}\\ \log \tan \tfrac{1}{2}(A - B) &= 9.81843\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}(A - B) &= 33^\circ 21' 26.7''.\\ \tfrac{1}{2}(A + B) &= 96^\circ 36' 35.5''.\\ A &= 129^\circ 58' 2''.\\ B &= 63^\circ 15' 8''.\end{aligned}$$

$$\begin{aligned}\log \sec \tfrac{1}{2}(A + B) &= 0.93890\ (n)\\ \log \cos \tfrac{1}{2}(a + b) &= 9.40053\ (n)\\ \log \sin \tfrac{1}{2} C &= \underline{9.60675}\\ \log \cos \tfrac{1}{2} c &= 9.94618\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2} c &= 27^\circ 56' 20''.\\ c &= 55^\circ 52' 40''.\end{aligned}$$

4. Given find

$$\begin{aligned} b &= 63^\circ\ 15'\ 12'', & B &= 88^\circ\ 12'\ 24'',\\ c &= 47^\circ\ 42'\ \ 1'', & C &= 55^\circ\ 52'\ 42'',\\ A &= 59^\circ\ \ 4'\ 25''; & a &= 50^\circ\ \ 1'\ 40''. \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(b+c) &= 55^\circ\ 28'\ 36.5''.\\ \tfrac{1}{2}(b-c) &= 7^\circ\ 46'\ 35.5''.\\ \tfrac{1}{2}A &= 29^\circ\ 32'\ 12.5''. \end{aligned}$$

$$\begin{aligned} \log\cos\tfrac{1}{2}(b-c) &= 9.99599\\ \log\sec\tfrac{1}{2}(b+c) &= 0.24662\\ \log\cot\tfrac{1}{2}A &= \underline{10.24671}\\ \log\tan\tfrac{1}{2}(B+C) &= 10.48932 \end{aligned}$$

$$\tfrac{1}{2}(B+C) = 72^\circ\ 2'\ 32.7''.$$

$$\begin{aligned} \log\sin\tfrac{1}{2}(b-c) &= 9.13133\\ \log\csc\tfrac{1}{2}(b+c) &= 0.08413\\ \log\cot\tfrac{1}{2}A &= \underline{10.24671}\\ \log\tan\tfrac{1}{2}(B-C) &= 9.46217 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(B-C) &= 16^\circ\ \ 9'\ 51.1''.\\ \tfrac{1}{2}(B+C) &= 72^\circ\ \ 2'\ 32.7''.\\ B &= 88^\circ\ 12'\ 24''.\\ C &= 55^\circ\ 52'\ 42''. \end{aligned}$$

$$\begin{aligned} \log\cos\tfrac{1}{2}(b+c) &= 9.75338\\ \log\sec\tfrac{1}{2}(B+C) &= 0.51101\\ \log\sin\tfrac{1}{2}A &= \underline{9.69284}\\ \log\cos\tfrac{1}{2}a &= 9.95723 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}a &= 25^\circ\ 0'\ 50''.\\ a &= 50^\circ\ 1'\ 40''. \end{aligned}$$

5. Given find

$$\begin{aligned} b &= 69^\circ\ 25'\ 11'', & B &= 56^\circ\ 11'\ 57'',\\ c &= 109^\circ\ 46'\ 19'', & C &= 123^\circ\ 21'\ 12'',\\ A &= 54^\circ\ 54'\ 42''; & a &= 67^\circ\ 11'\ 47''. \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(c-b) &= 20^\circ\ 10'\ 34''.\\ \tfrac{1}{2}(c+b) &= 89^\circ\ 35'\ 45''.\\ \tfrac{1}{2}A &= 27^\circ\ 27'\ 21''. \end{aligned}$$

$$\begin{aligned} \log\cos\tfrac{1}{2}(c-b) &= 9.97250\\ \log\sec\tfrac{1}{2}(c+b) &= 2.15157\\ \log\cot\tfrac{1}{2}A &= \underline{10.28434}\\ \log\tan\tfrac{1}{2}(C+B) &= 12.40841 \end{aligned}$$

$$\tfrac{1}{2}(C+B) = 89^\circ\ 46'\ 34.6''.$$

$$\begin{aligned} \log\sin\tfrac{1}{2}(c-b) &= 9.53770\\ \log\csc\tfrac{1}{2}(c+b) &= 0.00001\\ \log\cot\tfrac{1}{2}A &= \underline{10.28434}\\ \log\tan\tfrac{1}{2}(C-B) &= 9.82205 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(C-B) &= 33^\circ\ 34'\ 37.8''.\\ C &= 123^\circ\ 21'\ 12''.\\ B &= 56^\circ\ 11'\ 57''. \end{aligned}$$

$$\begin{aligned} \log\cos\tfrac{1}{2}(c+b) &= 7.84843\\ \log\sec\tfrac{1}{2}(C+B) &= 2.40842\\ \log\sin\tfrac{1}{2}A &= \underline{9.66376}\\ \log\cos\tfrac{1}{2}a &= 9.92061 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}a &= 33^\circ\ 35'\ 53.3''.\\ a &= 67^\circ\ 11'\ 47''. \end{aligned}$$

EXERCISE XXXVIII. PAGE 169.

1. What are the formulas for computing A when B, C, and a are given; and for computing B when A, C, and b are given?

(i) In Fig. 101 suppose p drawn from C. Then advance the letters in

$$\begin{aligned} \cot x &= \tan A \csc c,\\ \cos C &= \cos A \csc x \sin(B-x). \end{aligned}$$

The required formulas are

$$\begin{aligned} \cot x &= \tan B \csc a,\\ \cos A &= \cos B \csc x \sin(C-x). \end{aligned}$$

(ii) Suppose p drawn from A, and advance the letters two steps. The required formulas are

$$\begin{aligned} \cot x &= \tan C \csc b,\\ \cos B &= \cos C \csc x \sin(A-x). \end{aligned}$$

2.

Given	find
$A = 26^\circ\,58'\,46''$,	$a = 37^\circ\,14'\,10''$,
$B = 39^\circ\,45'\,10''$,	$b = 121^\circ\,28'\,10''$,
$c = 154^\circ\,46'\,48''$;	$C = 161^\circ\,22'\,11''$.

$\tfrac{1}{2}(B - A) = 6^\circ\,23'\,12''.$
$\tfrac{1}{2}(B + A) = 33^\circ\,21'\,58''.$
$\tfrac{1}{2}c = 77^\circ\,23'\,24''.$

$\log\cos\tfrac{1}{2}(B - A) = 9.99730$
$\log\sec\tfrac{1}{2}(B + A) = 0.07823$
$\log\tan\tfrac{1}{2}c = \underline{10.65032}$
$\log\tan\tfrac{1}{2}(b + a) = 10.72585$

$\tfrac{1}{2}(b + a) = 79^\circ\,21'\,10.3''.$

$\log\sin\tfrac{1}{2}(B - A) = 9.04625$
$\log\csc\tfrac{1}{2}(B + A) = 0.25965$
$\log\tan\tfrac{1}{2}c = \underline{10.65032}$
$\log\tan\tfrac{1}{2}(b - a) = 9.95622$

$\tfrac{1}{2}(b - a) = 42^\circ\,7'.$
$\tfrac{1}{2}(b + a) = 79^\circ\,21'\,10.3''.$
$b = 121^\circ\,28'\,10''.$
$a = 37^\circ\,14'\,10''.$

$\log\sin\tfrac{1}{2}(B + A) = 9.74035$
$\log\sec\tfrac{1}{2}(b - a) = 0.12972$
$\log\cos\tfrac{1}{2}c = \underline{9.33908}$
$\log\cos\tfrac{1}{2}C = 9.20915$

$\tfrac{1}{2}C = 80^\circ\,41'\,5.4''.$
$C = 161^\circ\,22'\,11''.$

3.

Given	find
$A = 128^\circ\,41'\,49''$,	$a = 125^\circ\,41'\,43''$,
$B = 107^\circ\,33'\,20''$,	$b = 82^\circ\,47'\,34''$,
$c = 124^\circ\,12'\,31''$;	$C = 127^\circ\,22'$.

$\tfrac{1}{2}(A - B) = 10^\circ\,34'\,14.5''.$
$\tfrac{1}{2}(A + B) = 118^\circ\,7'\,34.5''.$
$\tfrac{1}{2}c = 62^\circ\,6'\,15.5''.$

$\log\cos\tfrac{1}{2}(A - B) = 9.99257$
$\log\sec\tfrac{1}{2}(A + B) = 0.32660\ (n)$
$\log\tan\tfrac{1}{2}c = \underline{10.27624}$
$\log\tan\tfrac{1}{2}(a + b) = 10.59541\ (n)$
$\tfrac{1}{2}(a + b) = 104^\circ\,14'\,38.5''.$

$\log\sin\tfrac{1}{2}(A - B) = 9.26351$
$\log\csc\tfrac{1}{2}(A + B) = 0.05457$
$\log\tan\tfrac{1}{2}c = \underline{10.27624}$
$\log\tan\tfrac{1}{2}(a - b) = 9.59432$

$\tfrac{1}{2}(a - b) = 21^\circ\,27'\,4.9''.$
$a = 125^\circ\,41'\,43''.$
$b = 82^\circ\,47'\,34''.$

$\log\sin\tfrac{1}{2}(A + B) = 9.94543$
$\log\sec\tfrac{1}{2}(a - b) = 0.03118$
$\log\cos\tfrac{1}{2}c = \underline{9.67012}$
$\log\cos\tfrac{1}{2}C = 9.64673$

$\tfrac{1}{2}C = 63^\circ\,41'.$
$C = 127^\circ\,22'.$

4.

Given	find
$B = 153^\circ\,17'\,6''$,	$b = 152^\circ\,43'\,51''$,
$C = 78^\circ\,43'\,36''$,	$c = 83^\circ\,12'\,21''$,
$a = 86^\circ\,15'\,15''$;	$A = 78^\circ\,15'\,48''$.

$\tfrac{1}{2}(B + C) = 116^\circ\,0'\,21''.$
$\tfrac{1}{2}(B - C) = 37^\circ\,16'\,45''.$
$\tfrac{1}{2}a = 43^\circ\,7'\,37.5''.$

$\log\cos\tfrac{1}{2}(B - C) = 9.90074$
$\log\sec\tfrac{1}{2}(B + C) = 0.35807\ (n)$
$\log\tan\tfrac{1}{2}a = \underline{9.97159}$
$\log\tan\tfrac{1}{2}(b + c) = 10.23040\ (n)$

$\tfrac{1}{2}(b + c) = 120^\circ\,28'\,6.2''.$

$\log\sin\tfrac{1}{2}(B - C) = 9.78226$
$\log\csc\tfrac{1}{2}(B + C) = 0.04636$
$\log\tan\tfrac{1}{2}a = \underline{9.97159}$
$\log\tan\tfrac{1}{2}(b - c) = 9.80021$

$\tfrac{1}{2}(b - c) = 32^\circ\,15'\,45''.$
$b = 152^\circ\,43'\,51''.$
$c = 88^\circ\,12'\,21''.$

$\log\sin\tfrac{1}{2}(B + C) = 9.95364$
$\log\sec\tfrac{1}{2}(b - c) = 0.07283$
$\log\cos\tfrac{1}{2}a = \underline{9.86322}$
$\log\cos\tfrac{1}{2}A = 9.88969$

$\tfrac{1}{2}A = 39^\circ\,7'\,54''.$
$A = 78^\circ\,15'\,48''.$

5. Given find

$A = 125° 41' 44''$, $a = 128° 41' 46''$,
$C = 82° 47' 35''$, $c = 107° 33' 20''$,
$b = 52° 37' 57''$; $B = 55° 47' 40''$.

$$\begin{aligned} \tfrac{1}{2}(A + C) &= 104° 14' 39.5''. \\ \tfrac{1}{2}(A - C) &= 21° 27' \ 4.5''. \\ \tfrac{1}{2} b &= 26° 18' 58.5''. \end{aligned}$$

$$\begin{aligned} \log\cos\tfrac{1}{2}(A - C) &= 9.96883 \\ \log\sec\tfrac{1}{2}(A + C) &= 0.60896\,(n) \\ \log\tan\tfrac{1}{2} b &= \underline{9.69424} \\ \log\tan\tfrac{1}{2}(a + c) &= 10.27203\,(n) \end{aligned}$$

$$\tfrac{1}{2}(a + c) = 118° 7' 32.9''.$$

$$\begin{aligned} \log\sin\tfrac{1}{2}(A - C) &= 9.56313 \\ \log\csc\tfrac{1}{2}(A + C) &= 0.01356 \\ \log\tan\tfrac{1}{2} b &= \underline{9.69424} \\ \log\tan\tfrac{1}{2}(a - c) &= 9.27093 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(a - c) &= 10° 34' 12.9''. \\ a &= 128° 41' 46''. \\ c &= 107° 33' 20''. \end{aligned}$$

$$\begin{aligned} \log\sin\tfrac{1}{2}(A + C) &= 9.98644 \\ \log\sec\tfrac{1}{2}(a - c) &= 0.00743 \\ \log\cos\tfrac{1}{2} b &= \underline{9.95248} \\ \log\cos\tfrac{1}{2} B &= 9.94635 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2} B &= 27° 53' 50''. \\ B &= 55° 47' 40''. \end{aligned}$$

EXERCISE XXXIX. PAGE 171.

1. Given find

$a = 73° 49' 38''$, $B = 116° 42' 30''$,
$b = 120° 53' 35''$, $c = 120° 57' 27''$,
$A = 88° 52' 42''$; $C = 116° 47'$.

$$\begin{aligned} \log\sin A &= 9.99992 \\ \log\sin b &= 9.93355 \\ \log\csc a &= \underline{0.01753} \\ \log\sin B &= 9.95100 \end{aligned}$$

$$\begin{aligned} B &= [180° - (63° 17' 30'')] \\ &= 116° 42' 30''. \end{aligned}$$

(The greater side is opposite the greater angle.)

$$\begin{aligned} \tfrac{1}{2}(B + A) &= 102° 47' 36''. \\ \tfrac{1}{2}(B - A) &= 13° 54' 54''. \\ \tfrac{1}{2}(b + a) &= 97° 21' 36.5''. \\ \tfrac{1}{2}(b - a) &= 23° 31' 58.5''. \end{aligned}$$

$$\begin{aligned} \log\sin\tfrac{1}{2}(B + A) &= 9.98908 \\ \log\csc\tfrac{1}{2}(B - A) &= 0.61892 \\ \log\tan\tfrac{1}{2}(b - a) &= \underline{9.63898} \\ \log\tan\tfrac{1}{2} c &= 10.24698 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2} c &= 60° 28' 43.4''. \\ c &= 120° 57' 27''. \end{aligned}$$

$$\begin{aligned} \log\sin\tfrac{1}{2}(b + a) &= 9.99641 \\ \log\csc\tfrac{1}{2}(b - a) &= 0.39873 \\ \log\tan\tfrac{1}{2}(B - A) &= \underline{9.39402} \\ \log\cot\tfrac{1}{2} C &= 9.78916 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2} C &= 58° 23' 30''. \\ C &= 116° 47'. \end{aligned}$$

2. Given find

$a = 150° 57' \ 5''$, $B_1 = 120° 47' 45''$,
$b = 134° 15' 54''$, $c_1 = 55° 42' \ 8''$,
$A = 144° 22' 42''$; $C_1 = 97° 42' 55''$;
$c_2 = 23° 57' 17''$,
$B_2 = 59° 12' 15''$,
$C_2 = 29° \ 8' 39''$

$A > 90°$, $(a + b) > 180°$, $a > b$.
$\therefore$ two solutions.

$$\begin{aligned} \log\sin A &= 9.76524 \\ \log\sin b &= 9.85498 \\ \log\csc a &= \underline{0.31377} \\ \log\sin B &= 9.93399 \end{aligned}$$

$$\begin{aligned} B_1 &= 120° 47' 45''. \\ B_2 &= 59° 12' 15''. \end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}(A+B_1) &= 132^\circ\,35'\,13.5''.\\
\tfrac{1}{2}(A+B_2) &= 101^\circ\,47'\,28.5''.\\
\tfrac{1}{2}(A-B_1) &= 11^\circ\,47'\,28.5''.\\
\tfrac{1}{2}(A-B_2) &= 42^\circ\,35'\,13.5''.\\
\tfrac{1}{2}(a-b) &= 8^\circ\,20'\,35.5''.\\
\tfrac{1}{2}(a+b) &= 142^\circ\,36'\,29.5''.
\end{aligned}$$

$$\begin{aligned}
\log\sin\tfrac{1}{2}(a+b) &= 9.78338\\
\log\csc\tfrac{1}{2}(a-b) &= 0.83833\\
\log\tan\tfrac{1}{2}(A-B_1) &= \underline{9.31963}\\
\log\cot\tfrac{1}{2}C_1 &= 9.94134
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}C_1 &= 48^\circ\,51'\,27.7''.\\
C_1 &= 97^\circ\,42'\,55''.
\end{aligned}$$

$$\begin{aligned}
\log\sin\tfrac{1}{2}(a+b) &= 9.78338\\
\log\csc\tfrac{1}{2}(a-b) &= 0.83833\\
\log\tan\tfrac{1}{2}(A-B_2) &= \underline{9.96338}\\
\log\cot\tfrac{1}{2}C_2 &= 10.58509
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}C_2 &= 14^\circ\,34'\,19.6''.\\
C_2 &= 29^\circ\,8'\,39''.
\end{aligned}$$

$$\begin{aligned}
\log\sin\tfrac{1}{2}(A+B_1) &= 9.86703\\
\log\csc\tfrac{1}{2}(A-B_1) &= 0.68963\\
\log\tan\tfrac{1}{2}(a-b) &= \underline{9.16629}\\
\log\tan\tfrac{1}{2}c_1 &= 9.72295
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}c_1 &= 27^\circ\,51'\,4''.\\
c_1 &= 55^\circ\,42'\,8''.
\end{aligned}$$

$$\begin{aligned}
\log\sin\tfrac{1}{2}(A+B_2) &= 9.99074\\
\log\csc\tfrac{1}{2}(A-B_2) &= 0.16960\\
\log\tan\tfrac{1}{2}(a-b) &= \underline{9.16629}\\
\log\tan\tfrac{1}{2}c_2 &= 9.32663
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}c_2 &= 11^\circ\,58'\,38.7''.\\
c_2 &= 23^\circ\,57'\,17''.
\end{aligned}$$

3. Given	find
$a = 79^\circ\,0'\,54''$,	$B = 90^\circ$,
$b = 82^\circ\,17'\,4''$,	$c = 45^\circ\,12'\,19''$,
$A = 82^\circ\,9'\,26''$;	$C = 45^\circ\,44'\,5''$.

$$\begin{aligned}
\log\sin A &= 9.99592\\
\log\sin b &= 9.99605\\
\operatorname{colog}\sin a &= \underline{0.00803}\\
\log\sin B &= 0.00000
\end{aligned}$$

$$B = 90^\circ.$$

$$\begin{aligned}
\tan c &= \cos A\tan b.\\
\cot C &= \tan A\cos b.
\end{aligned}$$

$$\begin{aligned}
\log\cos A &= 9.13499\\
\log\tan b &= \underline{10.86812}\\
\log\tan c &= 10.00311
\end{aligned}$$

$$c = 45^\circ\,12'\,19''.$$

$$\begin{aligned}
\log\tan A &= 0.86093\\
\log\cos b &= \underline{9.12793}\\
\log\cot C &= 9.98886
\end{aligned}$$

$$C = 45^\circ\,44'\,5''.$$

4. Given $a = 30^\circ\,52'\,37''$, $b = 31^\circ\,9'\,16''$, $A = 87^\circ\,34'\,12''$; show that the triangle is impossible.

By [44], $\sin B = \sin A\sin b\csc a$.

$$\begin{aligned}
\log\sin A &= 9.99961\\
\log\sin b &= 9.71378\\
\log\csc a &= \underline{0.28972}\\
\log\sin B &= 0.00311
\end{aligned}$$

$$\sin B = 1.0072.$$

Therefore, the triangle is impossible, since $\sin B > 1$.

EXERCISE XL. PAGE 173.

1. Given	find
$A = 110^\circ\,10'$,	$b = 155^\circ\,5'\,18''$,
$B = 133^\circ\,18'$,	$c = 33^\circ\,1'\,37''$,
$a = 147^\circ\,5'\,32''$;	$C = 70^\circ\,20'\,40''$.

$$\begin{aligned}
\log\sin a &= 9.73503\\
\log\sin B &= 9.86200\\
\log\csc A &= \underline{0.02748}\\
\log\sin b &= 9.62451
\end{aligned}$$

$$\begin{aligned}
b &= 155^\circ\ 5'\ 18''.\\
\tfrac{1}{2}(B+A) &= 121^\circ\ 44'.\\
\tfrac{1}{2}(B-A) &= 11^\circ\ 34'.\\
\tfrac{1}{2}(b-a) &= 3^\circ\ 59'\ 53''.\\
\tfrac{1}{2}(b+a) &= 151^\circ\ 5'\ 25''.
\end{aligned}$$

$$\begin{aligned}
\log\sin\tfrac{1}{2}(B+A) &= 9.92968\\
\log\csc\tfrac{1}{2}(B-A) &= 0.69787\\
\log\tan\tfrac{1}{2}(b-a) &= \underline{8.84443}\\
\log\tan\tfrac{1}{2}c &= 9.47198\\
\tfrac{1}{2}c &= 16^\circ\ 30'\ 48.5''.\\
c &= 33^\circ\ 1'\ 37''.
\end{aligned}$$

$$\begin{aligned}
\log\csc\tfrac{1}{2}(b-a) &= 0.15663\\
\log\sin\tfrac{1}{2}(b+a) &= 9.68433\\
\log\tan\tfrac{1}{2}(B-A) &= \underline{9.31104}\\
\log\cot\tfrac{1}{2}C &= 9.15200\\
\tfrac{1}{2}C &= 35^\circ\ 10'\ 20''.\\
C &= 70^\circ\ 20'\ 40''.
\end{aligned}$$

2. Given	find
$A = 113^\circ\ 39'\ 21''$,	$b = 124^\circ\ 7'\ 20''$,
$B = 123^\circ\ 40'\ 18''$,	$c = 159^\circ\ 50'\ 15''$,
$a = 65^\circ\ 39'\ 46''$;	$C = 159^\circ\ 43'\ 34''$.

$$\begin{aligned}
\log\sin a &= 9.95959\\
\log\sin B &= 9.92024\\
\log\csc A &= \underline{0.03812}\\
\log\sin b &= 9.91795\\
b &= 124^\circ\ 7'\ 20''.\\
\tfrac{1}{2}(B+A) &= 118^\circ\ 39'\ 49.5''.\\
\tfrac{1}{2}(B-A) &= 5^\circ\ 0'\ 28.5''.\\
\tfrac{1}{2}(b-a) &= 29^\circ\ 13'\ 47''.\\
\tfrac{1}{2}(b+a) &= 94^\circ\ 53'\ 33''.
\end{aligned}$$

$$\begin{aligned}
\log\sin\tfrac{1}{2}(B+A) &= 9.94322\\
\log\csc\tfrac{1}{2}(B-A) &= 1.05902\\
\log\tan\tfrac{1}{2}(b-a) &= \underline{9.74785}\\
\log\tan\tfrac{1}{2}c &= 10.75009\\
\tfrac{1}{2}c &= 79^\circ\ 55'\ 7.3''.\\
c &= 159^\circ\ 50'\ 15''.
\end{aligned}$$

$$\begin{aligned}
\log\sin\tfrac{1}{2}(b+a) &= 9.99841\\
\log\csc\tfrac{1}{2}(b-a) &= 0.31130\\
\log\tan\tfrac{1}{2}(B-A) &= \underline{8.94264}\\
\log\cot\tfrac{1}{2}C &= 9.25235\\
\tfrac{1}{2}C &= 79^\circ\ 51'\ 46.8''.\\
C &= 159^\circ\ 43'\ 34''.
\end{aligned}$$

3. Given	find
$A = 100^\circ\ 2'\ 11''$,	$b = 90^\circ$,
$B = 98^\circ\ 30'\ 28''$,	$c = 147^\circ\ 41'\ 50''$,
$a = 95^\circ\ 20'\ 39''$;	$C = 148^\circ\ 5'\ 40''$.

$$\begin{aligned}
\log\sin a &= 9.99811\\
\log\sin B &= 9.99519\\
\log\csc A &= \underline{0.00670}\\
\log\sin b &= 10.00000\\
b &= 90^\circ.\\
\tfrac{1}{2}(A+B) &= 99^\circ\ 16'\ 19.5''.\\
\tfrac{1}{2}(A-B) &= 0^\circ\ 45'\ 51.5''.\\
\tfrac{1}{2}(a-b) &= 2^\circ\ 40'\ 19.5''.\\
\tfrac{1}{2}(a+b) &= 92^\circ\ 40'\ 19.5''.
\end{aligned}$$

$$\begin{aligned}
\log\sin\tfrac{1}{2}(A+B) &= 9.99428\\
\log\csc\tfrac{1}{2}(A-B) &= 1.87487\\
\log\tan\tfrac{1}{2}(a-b) &= \underline{8.66904}\\
\log\tan\tfrac{1}{2}c &= 10.53819\\
\tfrac{1}{2}c &= 73^\circ\ 50'\ 54.9''.\\
c &= 147^\circ\ 41'\ 50''.
\end{aligned}$$

$$\begin{aligned}
\log\sin\tfrac{1}{2}(a+b) &= 9.99953\\
\log\csc\tfrac{1}{2}(a-b) &= 1.33144\\
\log\tan\tfrac{1}{2}(A-B) &= \underline{8.12517}\\
\log\cot\tfrac{1}{2}C &= 9.45614\\
\tfrac{1}{2}C &= 74^\circ\ 2'\ 50''.\\
C &= 148^\circ\ 5'\ 40''.
\end{aligned}$$

4. Given $A = 24^\circ\ 33'\ 9''$, $B = 38^\circ\ 0'\ 12''$, $a = 65^\circ\ 20'\ 13''$; show that the triangle is impossible.

$$\begin{aligned}
\log\sin a &= 9.95845\\
\log\sin B &= 9.78937\\
\log\csc A &= \underline{0.38140}\\
\log\sin b &= 10.12922
\end{aligned}$$

$$\therefore \sin b > 1.$$

Therefore, the triangle is impossible.

EXERCISE XLI. PAGE 174,

1. Given

Given	find
$a = 120^\circ 55' 35''$,	$A = 116^\circ 44' 50''$,
$b = 59^\circ 4' 25''$,	$B = 63^\circ 15' 10''$,
$c = 106^\circ 10' 22''$;	$C = 91^\circ 7' 22''$.

$$\begin{aligned}
a &= 120^\circ\ 55'\ 35'' \\
b &= 59^\circ\ 4'\ 25'' \\
c &= \underline{106^\circ\ 10'\ 22''} \\
2s &= 286^\circ\ 10'\ 22''
\end{aligned}$$

$$\begin{aligned}
s &= 143^\circ\ 5'\ 11''. \\
s - a &= 22^\circ\ 9'\ 36''. \\
s - b &= 84^\circ\ 0'\ 46''. \\
s - c &= 36^\circ\ 54'\ 49''.
\end{aligned}$$

$$\begin{aligned}
\log\sin(s-a) &= 9.57657 \\
\log\sin(s-b) &= 9.99763 \\
\log\sin(s-c) &= 9.77859 \\
\log\csc s &= \underline{0.22141} \\
\log\tan^2 r &= 9.57420
\end{aligned}$$

$$\log\tan r = 9.78710.$$

$$\begin{aligned}
\log\tan\tfrac{1}{2}A &= 10.21053 \\
\log\tan\tfrac{1}{2}B &= 9.78947 \\
\log\tan\tfrac{1}{2}C &= \underline{10.00851}
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}A &= 58^\circ\ 22'\ 24.8''. \\
\tfrac{1}{2}B &= 31^\circ\ 37'\ 35.2''. \\
\tfrac{1}{2}C &= 45^\circ\ 33'\ 40.8''. \\
A &= 116^\circ\ 44'\ 50''. \\
B &= 63^\circ\ 15'\ 10''. \\
C &= 91^\circ\ 7'\ 22''.
\end{aligned}$$

2. Given

Given	find
$a = 50^\circ 12' 4''$,	$A = 59^\circ 4' 28''$,
$b = 116^\circ 44' 48''$,	$B = 94^\circ 23' 12''$,
$c = 129^\circ 11' 42''$;	$C = 120^\circ 4' 52''$.

$$\begin{aligned}
a &= 50^\circ\ 12'\ 4'' \\
b &= 116^\circ\ 44'\ 48'' \\
c &= \underline{129^\circ\ 11'\ 42''} \\
2s &= 296^\circ\ 8'\ 34''
\end{aligned}$$

$$\begin{aligned}
s &= 148^\circ\ 4'\ 17''. \\
s - a &= 97^\circ\ 52'\ 13''. \\
s - b &= 31^\circ\ 19'\ 29''. \\
s - c &= 18^\circ\ 52'\ 35''.
\end{aligned}$$

$$\begin{aligned}
\log\sin(s-a) &= 9.99589 \\
\log\sin(s-b) &= 9.71591 \\
\log\sin(s-c) &= 9.50992 \\
\log\csc s &= \underline{0.27666} \\
\log\tan^2 r &= 9.49838
\end{aligned}$$

$$\log\tan r = 9.74919.$$

$$\begin{aligned}
\log\tan\tfrac{1}{2}A &= 9.75330 \\
\log\tan\tfrac{1}{2}B &= 10.03328 \\
\log\tan\tfrac{1}{2}C &= \underline{10.23927}
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}A &= 29^\circ\ 32'\ 14''. \\
\tfrac{1}{2}B &= 47^\circ\ 11'\ 36''. \\
\tfrac{1}{2}C &= 60^\circ\ 2'\ 26''. \\
A &= 59^\circ\ 4'\ 28''. \\
B &= 94^\circ\ 23'\ 12''. \\
C &= 120^\circ\ 4'\ 52''.
\end{aligned}$$

3. Given

Given	find
$a = 131^\circ 35' 4''$,	$A = 132^\circ 14' 21''$.
$b = 108^\circ 30' 14''$,	$B = 110^\circ 10' 40''$,
$c = 84^\circ 46' 34''$;	$C = 99^\circ 42' 24''$.

$$\begin{aligned}
a &= 131^\circ\ 35'\ 4'' \\
b &= 108^\circ\ 30'\ 14'' \\
c &= \underline{84^\circ\ 46'\ 34''} \\
2s &= 324^\circ\ 51'\ 52''
\end{aligned}$$

$$\begin{aligned}
s &= 162^\circ\ 25'\ 56''. \\
s - a &= 30^\circ\ 50'\ 52''. \\
s - b &= 53^\circ\ 55'\ 42''. \\
s - c &= 77^\circ\ 39'\ 22''.
\end{aligned}$$

$$\begin{aligned}
\log\sin(s-a) &= 9.70991 \\
\log\sin(s-b) &= 9.90756 \\
\log\sin(s-c) &= 9.98984 \\
\log\csc s &= \underline{0.52023} \\
\log\tan^2 r &= 10.12754
\end{aligned}$$

$$\begin{aligned}
\log\tan r &= 10.06377.\\
\log\tan\tfrac{1}{2}A &= 10.35386\\
\log\tan\tfrac{1}{2}B &= 10.15621\\
\log\tan\tfrac{1}{2}C &= \underline{10.07393}\\
\tfrac{1}{2}A &= 66^\circ\ \ 7'\ 10.6''.\\
\tfrac{1}{2}B &= 55^\circ\ \ 5'\ 20''.\\
\tfrac{1}{2}C &= 49^\circ\ 51'\ 12''.\\
A &= 132^\circ\ 14'\ 21''.\\
B &= 110^\circ\ 10'\ 40''.\\
C &= 99^\circ\ 42'\ 24''.
\end{aligned}$$

4. Given	find
$a = 20^\circ\ 16'\ 38''$,	$A = 20^\circ\ \ 9'\ 55''$,
$b = 56^\circ\ 19'\ 40''$,	$B = 55^\circ\ 52'\ 35''$,
$c = 66^\circ\ 20'\ 44''$;	$C = 114^\circ\ 20'\ 21''$.

$$\begin{aligned}
a &= 20^\circ\ 16'\ 38''\\
b &= 56^\circ\ 19'\ 40''\\
c &= \underline{66^\circ\ 20'\ 44''}\\
2s &= 142^\circ\ 57'\ \ 2''\\
s &= 71^\circ\ 28'\ 31''.\\
s-a &= 51^\circ\ 11'\ 53''.\\
s-b &= 15^\circ\ \ 8'\ 51''.\\
s-c &= 5^\circ\ \ 7'\ 47''.\\
\log\sin(s-a) &= 9.89172\\
\log\sin(s-b) &= 9.41715\\
\log\sin(s-c) &= 8.95139\\
\log\csc s &= \underline{0.02311}\\
\log\tan^2 r &= 8.28337\\
\log\tan r &= 9.14169.\\
\log\tan\tfrac{1}{2}A &= 9.24997\\
\log\tan\tfrac{1}{2}B &= 9.72454\\
\log\tan\tfrac{1}{2}C &= \underline{10.19030}\\
\tfrac{1}{2}A &= 10^\circ\ \ 4'\ 57.6''.\\
\tfrac{1}{2}B &= 27^\circ\ 56'\ 17.4''.\\
\tfrac{1}{2}C &= 57^\circ\ 10'\ 10.7''.\\
A &= 20^\circ\ \ 9'\ 55''.\\
B &= 55^\circ\ 52'\ 35''.\\
C &= 114^\circ\ 20'\ 21''.
\end{aligned}$$

EXERCISE XLII. PAGE 176.

1. Given	find
$A = 130^\circ$,	$a = 139^\circ\ 21'\ 22''$,
$B = 110^\circ$,	$b = 126^\circ\ 57'\ 52''$,
$C = 80^\circ$;	$c = 56^\circ\ 51'\ 48''$.

$$\begin{aligned}
A &= 130^\circ\\
B &= 110^\circ\\
C &= \underline{80^\circ}\\
2S &= 320^\circ\\
S &= 160^\circ.\\
S-A &= 30^\circ.\\
S-B &= 50^\circ.\\
S-C &= 80^\circ.\\
\log\cos S &= 9.97299\ (n)\\
\log\sec(S-A) &= 0.06247\\
\log\sec(S-B) &= 0.19193\\
\log\sec(S-C) &= \underline{0.76033}\\
\log\tan^2 R &= 10.98772\\
\log\tan R &= 10.49386.\\
\log\tan\tfrac{1}{2}a &= 10.43139\\
\log\tan\tfrac{1}{2}b &= 10.30193\\
\log\tan\tfrac{1}{2}c &= \underline{9.73353}\\
\tfrac{1}{2}a &= 69^\circ\ 40'\ 41''.\\
\tfrac{1}{2}b &= 63^\circ\ 28'\ 56.2''.\\
\tfrac{1}{2}c &= 28^\circ\ 25'\ 54''.\\
a &= 139^\circ\ 21'\ 22''.\\
b &= 126^\circ\ 57'\ 52''.\\
c &= 56^\circ\ 51'\ 48''.
\end{aligned}$$

2. Given	find
$A = 59^\circ\ 55'\ 10''$,	$a = 51^\circ\ 17'\ 31''$,
$B = 85^\circ\ 36'\ 50''$,	$b = 64^\circ\ \ 2'\ 47''$,
$C = 59^\circ\ 55'\ 10''$;	$c = 51^\circ\ 17'\ 31''$.

$$\begin{aligned}
A &= 59^\circ\ 55'\ 10''\\
B &= 85^\circ\ 36'\ 50''\\
C &= \underline{59^\circ\ 55'\ 10''}\\
2S &= 205^\circ\ 27'\ 10''
\end{aligned}$$

$$\begin{aligned}
S &= 102^\circ\ 43'\ 35''.\\
S-A &= 42^\circ\ 48'\ 25''.\\
S-B &= 17^\circ\ \ 6'\ 45''.\\
S-C &= 42^\circ\ 48'\ 25''.
\end{aligned}$$

$$\begin{aligned}
\log\cos S &= 9.34301\ (n)\\
\log\sec(S-A) &= 0.13451\\
\log\sec(S-B) &= 0.01967\\
\log\sec(S-C) &= \underline{0.13451}\\
\log\tan^2 R &= 9.63170
\end{aligned}$$

$$\log\tan R = 9.81585.$$

$$\begin{aligned}
\log\tan\tfrac{1}{2}a &= 9.68134\\
\log\tan\tfrac{1}{2}b &= 9.79618\\
\log\tan\tfrac{1}{2}c &= \underline{9.68134}
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}a &= 25^\circ\ 38'\ 45.5''.\\
\tfrac{1}{2}b &= 32^\circ\ \ 1'\ 23.6''.\\
\tfrac{1}{2}c &= 25^\circ\ 38'\ 45.5''.\\
a &= 51^\circ\ 17'\ 31''.\\
b &= 64^\circ\ \ 2'\ 47''.\\
c &= 51^\circ\ 17'\ 31''.
\end{aligned}$$

3.

Given	find
$A = 102^\circ 14' 12''$,	$a = 104^\circ 25'\ 9''$,
$B = 54^\circ 32' 24''$,	$b = 53^\circ 49' 25''$,
$C = 89^\circ\ 5' 46''$;	$c = 97^\circ 44' 19''$.

$$\begin{aligned}
A &= 102^\circ\ 14'\ 12''\\
B &= 54^\circ\ 32'\ 24''\\
C &= \underline{89^\circ\ \ 5'\ 46''}\\
2S &= 245^\circ\ 52'\ 22''
\end{aligned}$$

$$\begin{aligned}
S &= 122^\circ\ 56'\ 11''.\\
S-A &= 20^\circ\ 41'\ 59''.\\
S-B &= 68^\circ\ 23'\ 47''.\\
S-C &= 33^\circ\ 50'\ 25''.
\end{aligned}$$

$$\begin{aligned}
\log\cos S &= 9.73536\ (n)\\
\log\sec(S-A) &= 0.02898\\
\log\sec(S-B) &= 0.43394\\
\log\sec(S-C) &= \underline{0.08061}\\
\log\tan^2 R &= 10.27889
\end{aligned}$$

$$\log\tan R = 10.13945.$$

$$\begin{aligned}
\log\tan\tfrac{1}{2}a &= 10.11047\\
\log\tan\tfrac{1}{2}b &= 9.70551\\
\log\tan\tfrac{1}{2}c &= \underline{10.05884}
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}a &= 52^\circ\ 12'\ 34.6''.\\
\tfrac{1}{2}b &= 26^\circ\ 54'\ 42.6''.\\
\tfrac{1}{2}c &= 48^\circ\ 52'\ 9.6''.\\
a &= 104^\circ\ 25'\ 9''.\\
b &= 53^\circ\ 49'\ 25''.\\
c &= 97^\circ\ 44'\ 19''.
\end{aligned}$$

4.

Given	find
$A = 4^\circ 23' 35''$,	$a = 31^\circ\ 9' 13''$,
$B = 8^\circ 28' 20''$,	$b = 84^\circ 18' 28''$,
$C = 172^\circ 17' 56''$;	$c = 115^\circ 10'$.

$$\begin{aligned}
A &= 4^\circ\ 23'\ 35''\\
B &= 8^\circ\ 28'\ 20''\\
C &= \underline{172^\circ\ 17'\ 56''}\\
2S &= 185^\circ\ \ 9'\ 51''
\end{aligned}$$

$$\begin{aligned}
S &= 92^\circ\ 34'\ 55.5''.\\
S-A &= 88^\circ\ 11'\ 20.5''.\\
S-B &= 84^\circ\ \ 6'\ 35.5''.\\
S-C &= -(79^\circ\ 43'\ \ 0.5'').
\end{aligned}$$

$$\begin{aligned}
\log\cos S &= 8.65370\ (n)\\
\log\sec(S-A) &= 1.50029\\
\log\sec(S-B) &= 0.98876\\
\log\sec(S-C) &= \underline{0.74833}\\
\log\tan^2 R &= 11.89108
\end{aligned}$$

$$\log\tan R = 10.94554.$$

$$\begin{aligned}
\log\tan\tfrac{1}{2}a &= 9.44525\\
\log\tan\tfrac{1}{2}b &= 9.95678\\
\log\tan\tfrac{1}{2}c &= \underline{10.19721}
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}a &= 15^\circ\ 34'\ 36.7''.\\
\tfrac{1}{2}b &= 42^\circ\ \ 9'\ 13.8''.\\
\tfrac{1}{2}c &= 57^\circ\ 35'.\\
a &= 31^\circ\ \ 9'\ 13''.\\
b &= 84^\circ\ 18'\ 28''.\\
c &= 115^\circ\ 10'.
\end{aligned}$$

EXERCISE XLIII. PAGE 180.

1. Given $\qquad\qquad$ find

$A = 84° 20' 19''$, $\quad E = 26159''$,
$B = 27° 22' 40''$, $\quad F = 0.12682\,R^2$.
$C = 75° 33'$;

$$\begin{aligned} A &= 84° 20' 19'' \\ B &= 27° 22' 40'' \\ C &= \underline{75° 33'} \\ A + B + C &= 187° 15' 59'' \\ \therefore E &= 7° 15' 59'' \\ &= 26159''. \end{aligned}$$

$$\begin{aligned} \log R^2 &= \log R^2 \\ \log 26159 &= 4.41762 \\ \log \frac{\pi}{648000} &= 4.68557 - 10 \\ \log F &= \overline{9.10319 - 10 + \log R^2} \\ F &= 0.12682\,R^2. \end{aligned}$$

2. Given $\qquad\qquad$ find

$a = 69° 15' 6''$, $\quad E = 216° 40' 18''$.
$b = 120° 42' 47''$,
$c = 159° 18' 33''$;

$$\begin{aligned} a &= 69° 15' 6'' \\ b &= 120° 42' 47'' \\ c &= \underline{159° 18' 33''} \\ 2s &= 349° 16' 26'' \\ s &= 174° 38' 13''. \\ s - a &= 105° 23' 7''. \\ s - b &= 53° 55' 26''. \\ s - c &= 15° 19' 40''. \\ \tfrac{1}{2} s &= 87° 19' 6.5''. \\ \tfrac{1}{2}(s - a) &= 52° 41' 33.5''. \\ \tfrac{1}{2}(s - b) &= 26° 57' 43''. \\ \tfrac{1}{2}(s - c) &= 7° 39' 50''. \end{aligned}$$

$$\begin{aligned} \log \tan \tfrac{1}{2} s &= 11.32942 \\ \log \tan \tfrac{1}{2}(s - a) &= 10.11805 \\ \log \tan \tfrac{1}{2}(s - b) &= 9.70645 \\ \log \tan \tfrac{1}{2}(s - c) &= \underline{9.12893} \\ \log \tan^2 \tfrac{1}{4} E &= 10.28285 \\ \log \tan \tfrac{1}{4} E &= 10.14142. \\ \tfrac{1}{4} E &= 54° 10' 4.6''. \\ E &= 216° 40' 18''. \end{aligned}$$

3. Given $\qquad\qquad$ find

$a = 33° 1° 45''$, $\quad E = 133° 48' 53''$.
$b = 155° 5' 18''$,
$C = 110° 10'$;

By Sect. LVII,

$$\begin{aligned} \tan m &= \tan a \cos C, \\ \cos c &= \cos a \sec m \cos (b - m). \end{aligned}$$

$$\begin{aligned} \log \tan a &= 9.81300 \\ \log \cos C &= \underline{9.53751} \\ \log \tan m &= 9.35051 \end{aligned}$$

$$\begin{aligned} m &= 167° 22'. \\ b - m &= -(12° 16' 42''). \end{aligned}$$

$$\begin{aligned} \log \cos a &= 9.92345 \\ \log \sec m &= 0.01064\ (n) \\ \log \cos (b - m) &= \underline{9.98995} \\ \log \cos c &= 9.92404\ (n) \end{aligned}$$

$$c = 147° 5' 30''.$$

$$\begin{aligned} a &= 33° 1' 45'' \\ b &= 155° 5' 18'' \\ c &= \underline{147° 5' 30''} \\ 2s &= 335° 12' 33'' \\ s &= 167° 36' 16.5''. \\ s - a &= 134° 34' 31.5''. \\ s - b &= 12° 30' 58.5''. \\ s - c &= 20° 30' 46.5''. \\ \tfrac{1}{2} s &= 83° 48' 8.25''. \\ \tfrac{1}{2}(s - a) &= 67° 17' 15.75''. \\ \tfrac{1}{2}(s - b) &= 6° 15' 29.25''. \\ \tfrac{1}{2}(s - c) &= 10° 15' 23.25''. \end{aligned}$$

$$\begin{aligned}
\log\tan\tfrac{1}{2}s &= 10.96419\\
\log\tan\tfrac{1}{2}(s-a) &= 10.37824\\
\log\tan\tfrac{1}{2}(s-b) &= 9.04005\\
\log\tan\tfrac{1}{2}(s-c) &= \underline{9.25755}\\
\log\tan^2\tfrac{1}{4}E &= 9.64003\\
\log\tan\tfrac{1}{4}E &= 9.82002.\\
\tfrac{1}{4}E &= 33^\circ\,27'\,13.3''.\\
E &= 133^\circ\,48'\,53''.
\end{aligned}$$

4. Given $c = 114^\circ\,27'\,57''$, $A = 78^\circ\,42'\,33''$, $B = 127^\circ\,13'\,7''$; find the area.

$$\begin{aligned}
\tfrac{1}{2}(B-A) &= 24^\circ\,15'\,17''.\\
\tfrac{1}{2}(B+A) &= 102^\circ\,57'\,50''.\\
\tfrac{1}{2}c &= 57^\circ\,13'\,58.5''.
\end{aligned}$$

$$\begin{aligned}
\log\sin\tfrac{1}{2}(B-A) &= 9.61362\\
\log\csc\tfrac{1}{2}(B+A) &= 0.01121\\
\log\tan\tfrac{1}{2}c &= \underline{10.19128}\\
\log\tan\tfrac{1}{2}(b-a) &= 9.81611
\end{aligned}$$

$$\tfrac{1}{2}(b-a) = 33^\circ\,13'.$$

$$\begin{aligned}
\log\sin\tfrac{1}{2}(B+A) &= 9.98879\\
\log\sec\tfrac{1}{2}(b-a) &= 0.07748\\
\log\cos\tfrac{1}{2}c &= \underline{9.73338}\\
\log\cos\tfrac{1}{2}C &= 9.79965
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}C &= 50^\circ\,55'.\\
C &= 101^\circ\,50'.
\end{aligned}$$

$$\begin{aligned}
A &= 78^\circ\,42'\,33''\\
B &= 127^\circ\,13'\,7''\\
C &= \underline{101^\circ\,50'}\\
A+B+C &= 307^\circ\,45'\,40''\\
\therefore E &= 127^\circ\,45'\,40''\\
&= 459940''.
\end{aligned}$$

$$\begin{aligned}
\log R^2 &= \log R^2\\
\log 459940 &= 5.66270\\
\log\frac{\pi}{648000} &= \underline{4.68557 - 10}\\
\log F &= 0.34827 + \log R^2\\
F &= 2.2298\,R^2.
\end{aligned}$$

5. Given $a = 76^\circ\,14'\,47''$, $b = 82^\circ\,40'\,15''$, $A = 60^\circ\,22'\,44''$; find the area.

Here a and A are alike in kind and $\sin b > \sin a > \sin A \sin b$. Hence, there are two solutions.

$$\begin{aligned}
\log\sin A &= 9.93918\\
\log\sin b &= 9.99643\\
\log\csc a &= \underline{0.01264}\\
\log\sin B &= 9.94825
\end{aligned}$$

$$\begin{aligned}
B &= 62^\circ\,34'\,51'',\\
\text{and}\qquad B_1 &= 117^\circ\,25'\,9''.
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}(b+a) &= 79^\circ\,27'\,31''.\\
\tfrac{1}{2}(b-a) &= 3^\circ\,12'\,44''.\\
\tfrac{1}{2}(B+A) &= 61^\circ\,28'\,47.5''.\\
\tfrac{1}{2}(B-A) &= 1^\circ\,6'\,3.5''.
\end{aligned}$$

$$\begin{aligned}
\log\sin\tfrac{1}{2}(B+A) &= 9.94382\\
\log\csc\tfrac{1}{2}(B-A) &= 1.71637\\
\log\tan\tfrac{1}{2}(b-a) &= \underline{8.74914}\\
\log\tan\tfrac{1}{2}c &= 10.40933
\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}c &= 68^\circ\,42'\,42.6''.\\ c &= 137^\circ\,25'\,25''.\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}(B_1+A) &= 88^\circ\,53'\,56.5''.\\ \tfrac{1}{2}(B_1-A) &= 28^\circ\,31'\,12.5''.\end{aligned}$$

$$\begin{aligned}\log\sin\tfrac{1}{2}(B_1+A) &= 9.99992\\ \log\csc\tfrac{1}{2}(B_1-A) &= 0.32105\\ \log\tan\tfrac{1}{2}(b-a) &= \underline{8.74914}\\ \log\tan\tfrac{1}{2}c_1 &= 9.07011\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}c_1 &= 6^\circ\,42'\,\ 9.4''.\\ c_1 &= 13^\circ\,24'\,19''.\end{aligned}$$

$$\begin{array}{rlcl} a = & 76^\circ\,14'\,47'' & & 76^\circ\,14'\,47''\\ b = & 82^\circ\,40'\,15'' & & 82^\circ\,40'\,15''\\ c = & \underline{137^\circ\,25'\,25''} & \text{or} & \underline{13^\circ\,24'\,19''}\\ 2s = & 296^\circ\,20'\,27'' & \text{or} & 172^\circ\,19'\,21''\end{array}$$

$$\begin{array}{rlcl} s = & 148^\circ\,10'\,13.5'' & \text{or} & 86^\circ\ \ 9'\,40.5''.\\ s-a = & 71^\circ\,55'\,26.5'' & \text{or} & 9^\circ\,54'\,53.5''.\\ s-b = & 65^\circ\,29'\,58.5'' & \text{or} & 3^\circ\,29'\,25.5''.\\ s-c = & 10^\circ\,44'\,48.5'' & \text{or} & 72^\circ\,45'\,21.5''.\\ \tfrac{1}{2}s = & 74^\circ\ \ 5'\ \ 6.75'' & \text{or} & 43^\circ\ \ 4'\,50.25''.\\ \tfrac{1}{2}(s-a) = & 35^\circ\,57'\,43.25'' & \text{or} & 4^\circ\,57'\,26.75''.\\ \tfrac{1}{2}(s-b) = & 32^\circ\,44'\,59.25'' & \text{or} & 1^\circ\,44'\,42.75''.\\ \tfrac{1}{2}(s-c) = & 5^\circ\,22'\,24.25'' & \text{or} & 36^\circ\,22'\,40.75''.\end{array}$$

$$\begin{array}{rlcl} \log\tan\tfrac{1}{2}s = & 10.54494 & \text{or} & 9.97088\\ \log\tan\tfrac{1}{2}(s-a) = & 9.86065 & \text{or} & 8.93822\\ \log\tan\tfrac{1}{2}(s-b) = & 9.80836 & \text{or} & 8.48386\\ \log\tan\tfrac{1}{2}(s-c) = & \underline{8.97340} & \text{or} & \underline{9.86727}\\ \log\tan^2\tfrac{1}{2}E = & 9.18735 & \text{or} & 7.26023\end{array}$$

$$\log\tan\tfrac{1}{2}E = 9.59368 \quad\text{or}\quad 8.63012.$$

$$\begin{array}{rlcl} \tfrac{1}{2}E = & 21^\circ\,25'\,22.7'' & \text{or} & 2^\circ\,26'\,36.0''.\\ E = & 85^\circ\,41'\,31'' & \text{or} & 9^\circ\,46'\,24''.\\ E = & 308491'' & \text{or} & 35184''.\end{array}$$

$$\begin{array}{rlcl} \log R^2 = & \log R^2 & & \log R^2\\ \log E = & 5.48925 & \text{or} & 4.54635\\ \log\dfrac{\pi}{648000} = & 4.68557-10 & & 4.68557-10\\ \log F = & \overline{0.17482+\log R^2} & \text{or} & \overline{9.23192-10+\log R^2}\end{array}$$

$$F = 1.4956\,R^2 \qquad \text{or}\quad 0.17085\,R^2.$$

6. Given $A = 80° 12' 35''$, $B = 77° 38' 22''$, $a = 76° 42' 28''$; find the area.

By Sect. LX, Note 2,

$$\begin{aligned} \cot x &= \cos a \tan B, \\ \sin(C - x) &= \cos A \sec B \sin x. \end{aligned}$$

$$\begin{aligned} \log\cos a &= 9.36157 \\ \log\tan B &= \underline{10.65927} \\ \log\cot x &= 10.02084 \\ x &= 43° 37' 34''. \end{aligned}$$

$$\begin{aligned} \log\cos A &= 9.23055 \\ \log\sec B &= 0.66946 \\ \log\sin x &= \underline{9.83881} \\ \log\sin(C - x) &= 9.73882 \\ C - x &= 33° 14'. \\ C &= 76° 51' 34''. \end{aligned}$$

$$\begin{aligned} A &= 80° 12' 35'' \\ B &= 77° 38' 22'' \\ C &= \underline{76° 51' 34''} \\ A + B + C &= 234° 42' 31'' \\ \therefore E &= 54° 42' 31'' \\ &= 196951''. \end{aligned}$$

$$\begin{aligned} \log R^2 &= \log R^2 \\ \log E &= 5.29436 \\ \log\frac{\pi}{648000} &= 4.68557 - 10 \\ \log F &= \overline{9.97993 - 10 + \log R^2} \\ F &= 0.95484\, R^2. \end{aligned}$$

7. Given $b = 44° 27' 40''$, $c = 15° 22' 44''$, $A = 167° 42' 27''$; find the area.

$$\begin{aligned} \tfrac{1}{2}(b - c) &= 14° 32' 28''. \\ \tfrac{1}{2}(b + c) &= 29° 55' 12''. \\ \tfrac{1}{2}A &= 83° 51' 13.5''. \end{aligned}$$

$$\begin{aligned} \log\cos\tfrac{1}{2}(b - c) &= 9.98586 \\ \log\sec\tfrac{1}{2}(b + c) &= 0.06212 \\ \log\cot\tfrac{1}{2}A &= \underline{9.03215} \\ \log\tan\tfrac{1}{2}(B + C) &= 9.08013 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(B + C) &= 6° 51' 27.5''. \\ B + C &= 13° 42' 55'' \\ A &= 167° 42' 27'' \\ B + C &= \underline{13° 42' 55''} \\ A + B + C &= 181° 25' 22'' \\ \therefore E &= 1° 25' 22'' \\ &= 5122''. \end{aligned}$$

$$\begin{aligned} \log R^2 &= \log R^2 \\ \log E &= 3.70944 \\ \log\frac{\pi}{648000} &= 4.68557 - 10 \\ \log F &= \overline{8.39501 - 10 + \log R^2} \\ F &= 0.024832\, R^2. \end{aligned}$$

8. Given $b = 67° 15' 42''$, $A = 84° 55' 8''$, $C = 96° 18' 49''$; find the area.

By Sect. LVIII,

$$\begin{aligned} \cot x &= \tan C \cos b, \\ \cos B &= \cos C \csc x \sin(A - x). \end{aligned}$$

$$\begin{aligned} \log\tan C &= 10.95608\,(n) \\ \log\cos b &= \underline{9.58718} \\ \log\cot x &= 10.54326\,(n) \\ x &= 164° 1' 35''. \end{aligned}$$

$$\begin{aligned} \log\cos C &= 9.04128\,(n) \\ \log\csc x &= 0.56036 \\ \log\sin(A - x) &= \underline{9.99210}\,(n) \\ \log\cos B &= 9.59374 \\ B &= 66° 53' 44''. \end{aligned}$$

$$\begin{aligned} A &= 84° 55' 8'' \\ B &= 66° 53' 44'' \\ C &= \underline{96° 18' 49''} \\ A + B + C &= 248° 7' 41'' \\ \therefore E &= 68° 7' 41'' \\ &= 245261''. \end{aligned}$$

$$\begin{aligned} \log R^2 &= \log R^2 \\ \log E &= 5.38963 \\ \log\frac{\pi}{648000} &= 4.68557 - 10 \\ \log F &= \overline{0.07520 + \log R^2} \\ F &= 1.1891\, R^2. \end{aligned}$$

9. Given $b = 72° \; 19' \; 38''$, $c = 54° \; 58' \; 52''$, $B = 77° \; 15' \; 14''$; find the area.

By Sect. LIX, Note 2,

$$\begin{aligned} \tan m &= \cos B \tan c, \\ \cos(a - m) &= \cos b \sec c \cos m. \end{aligned}$$

$$\begin{aligned} \log \cos B &= 9.34367 \\ \log \tan c &= \underline{10.15447} \\ \log \tan m &= 9.49814 \end{aligned}$$

$$m = 17° \; 28' \; 41''.$$

$$\begin{aligned} \log \cos b &= 9.48227 \\ \log \sec c &= 0.24121 \\ \log \cos m &= \underline{9.97947} \\ \log \cos(a - m) &= 9.70295 \end{aligned}$$

$$\begin{aligned} a - m &= 59° \; 41' \; 41''. \\ a &= 77° \; 10' \; 22''. \end{aligned}$$

$$\begin{aligned} a &= 77° \; 10' \; 22'' \\ b &= 72° \; 19' \; 38'' \\ c &= \underline{54° \; 58' \; 52''} \\ 2s &= 204° \; 28' \; 52'' \end{aligned}$$

$$\begin{aligned} s &= 102° \; 14' \; 26''. \\ s - a &= 25° \; 4' \; 4''. \\ s - b &= 29° \; 54' \; 48''. \\ s - c &= 47° \; 15' \; 34''. \\ \tfrac{1}{2} s &= 51° \; 7' \; 13''. \\ \tfrac{1}{2}(s - a) &= 12° \; 32' \; 2''. \\ \tfrac{1}{2}(s - b) &= 14° \; 57' \; 24''. \\ \tfrac{1}{2}(s - c) &= 23° \; 37' \; 47''. \end{aligned}$$

$$\begin{aligned} \log \tan \tfrac{1}{2} s &= 10.09350 \\ \log \tan \tfrac{1}{2}(s - a) &= 9.34697 \\ \log \tan \tfrac{1}{2}(s - b) &= 9.42673 \\ \log \tan \tfrac{1}{2}(s - c) &= \underline{9.64099} \\ \log \tan^2 \tfrac{1}{4} E &= 8.50819 \end{aligned}$$

$$\log \tan \tfrac{1}{4} E = 9.25410.$$

$$\begin{aligned} \tfrac{1}{4} E &= 10° \; 10' \; 37.5''. \\ E &= 40° \; 42' \; 30'' \\ &= 146550''. \end{aligned}$$

$$\begin{aligned} \log R^2 &= \log R^2 \\ \log E &= 5.16599 \\ \log \frac{\pi}{648000} &= \underline{4.68557 - 10} \\ \log F &= 9.85156 - 10 + \log R^2 \\ F &= 0.7105 \, R^2. \end{aligned}$$

10. Given $B = 127° \; 16' \; 4''$, $C = 42° \; 34' \; 19''$, $b = 54° \; 47' \; 55''$; find the area.

$$\sin c = \sin b \sin C \csc B.$$

$$\begin{aligned} \log \sin b &= 9.91229 \\ \log \sin C &= 9.83027 \\ \log \csc B &= \underline{0.09919} \\ \log \sin c &= 9.84175 \end{aligned}$$

$$\begin{aligned} c &= 43° \; 59' \; 51''. \\ \tfrac{1}{2}(B + C) &= 84° \; 55' \; 11.5''. \\ \tfrac{1}{2}(B - C) &= 42° \; 20' \; 52.5''. \\ \tfrac{1}{2}(b + c) &= 49° \; 23' \; 53''. \\ \tfrac{1}{2}(b - c) &= 5° \; 24' \; 2''. \end{aligned}$$

$$\begin{aligned} \log \sin \tfrac{1}{2}(b + c) &= 9.88039 \\ \log \csc \tfrac{1}{2}(b - c) &= 1.02633 \\ \log \tan \tfrac{1}{2}(B - C) &= \underline{9.95974} \\ \log \cot \tfrac{1}{2} A &= 10.86646 \\ \tfrac{1}{2} A &= 7° \; 44' \; 41.1''. \\ A &= 15° \; 29' \; 22''. \end{aligned}$$

$$\begin{aligned} A &= 15° \; 29' \; 22'' \\ B &= 127° \; 16' \; 4'' \\ C &= \underline{42° \; 34' \; 19''} \\ A + B + C &= 185° \; 19' \; 45'' \\ \therefore E &= 5° \; 19' \; 45'' \\ &= 19185''. \end{aligned}$$

$$\begin{aligned} \log R^2 &= \log R^2 \\ \log E &= 4.28296 \\ \log \frac{\pi}{648000} &= \underline{4.86557 - 10} \\ \log F &= 8.96853 - 10 + \log R^2 \\ F &= 0.09301 \, R^2. \end{aligned}$$

11. Given $a = 128° \; 42' \; 56''$, $b = 107° \; 13' \; 48''$, $c = 88° \; 37' \; 51''$; find the area.

$$\begin{aligned}
a &= 128^\circ\ 42'\ 56'' \\
b &= 107^\circ\ 13'\ 48'' \\
c &= \underline{\ 88^\circ\ 37'\ 51''} \\
2\,s &= 324^\circ\ 34'\ 35'' \\
s &= 162^\circ\ 17'\ 17.5''. \\
s - a &= \ 33^\circ\ 34'\ 21.5''. \\
s - b &= \ 55^\circ\ \ 3'\ 29.5''. \\
s - c &= \ 73^\circ\ 39'\ 26.5''. \\
\tfrac{1}{2}\,s &= \ 81^\circ\ \ 8'\ 38.75''. \\
\tfrac{1}{2}(s - a) &= \ 16^\circ\ 47'\ 10.75''. \\
\tfrac{1}{2}(s - b) &= \ 27^\circ\ 31'\ 44.75''. \\
\tfrac{1}{2}(s - c) &= \ 36^\circ\ 49'\ 43.25''. \\
\log \tan \tfrac{1}{2}\,s &= 10.80742 \\
\log \tan \tfrac{1}{2}(s - a) &= \ 9.47951 \\
\log \tan \tfrac{1}{2}(s - b) &= \ 9.71701 \\
\log \tan \tfrac{1}{2}(s - c) &= \ \underline{9.87441} \\
\log \tan^2 \tfrac{1}{2} E &= \ 9.87835 \\
\log \tan \tfrac{1}{2} E &= \ 9.93918. \\
\tfrac{1}{2} E &= \ 41^\circ\ 0'\ 4.6''. \\
E &= 164^\circ\ 0'\ 18'' \\
&= 590418''.
\end{aligned}$$

$$\begin{aligned}
\log R^2 &= \log R^2 \\
\log E &= 5.77116 \\
\log \frac{\pi}{648000} &= \underline{4.68557 - 10} \\
\log F &= 0.45673 + \log R^2 \\
F &= 2.8624\,R^2.
\end{aligned}$$

12. Given $A = 127^\circ\ 22'\ 28''$, $B = 131^\circ\ 45'\ 27''$, $C = 100^\circ\ 52'\ 16''$; find the area.

$$\begin{aligned}
A &= 127^\circ\ 22'\ 28'' \\
B &= 131^\circ\ 45'\ 27'' \\
C &= \underline{100^\circ\ 52'\ 16''} \\
A + B + C &= 360^\circ\ \ 0'\ 11'' \\
\therefore E &= 180^\circ\ \ 0'\ 11'' \\
&= 648011''.
\end{aligned}$$

$$\begin{aligned}
\log R^2 &= \log R^2 \\
\log E &= 5.81159 \\
\log \frac{\pi}{648000} &= \underline{4.68557 - 10} \\
\log F &= 0.49716 + \log R^2 \\
F &= 3.1416\,R^2.
\end{aligned}$$

13. Given $a = 116^\circ\ 19'\ 45''$, $A = 160^\circ\ 42'\ 24''$, $C = 171^\circ\ 27'\ 15''$; find the area.

By Sect. LX, Note 2,

$$\begin{aligned}
\cot x &= \cos a \tan C, \\
\sin (B - x) &= \cos A \sec C \sin x.
\end{aligned}$$

$$\begin{aligned}
\log \cos a &= 9.64692\ (n) \\
\log \tan C &= \underline{9.17687}\ (n) \\
\log \cot x &= 8.82379 \\
x &= 86^\circ\ 11'\ 13''. \\
\log \cos A &= 9.97490\ (n) \\
\log \sec C &= 0.00485\ (n) \\
\log \sin x &= \underline{9.99904} \\
\log \sin (B - x) &= 9.97879 \\
B - x &= \ 72^\circ\ 14'\ 15''. \\
B &= 158^\circ\ 25'\ 28''. \\
A &= 160^\circ\ 42'\ 24'' \\
B &= 158^\circ\ 25'\ 28'' \\
C &= \underline{171^\circ\ 27'\ 15''} \\
A + B + C &= 490^\circ\ 35'\ \ 7'' \\
\therefore E &= 310^\circ\ 35'\ \ 7'' \\
&= 1118107''.
\end{aligned}$$

$$\begin{aligned}
\log R^2 &= \log R^2 \\
\log E &= 6.04848 \\
\log \frac{\pi}{648000} &= \underline{4.68557 - 10} \\
\log F &= 0.73405 + \log R^2 \\
F &= 5.4206\,R^2.
\end{aligned}$$

14. Find the area of a triangle on the earth's surface (regarded as spherical) if each side of the triangle is equal to 1°. (Radius of earth = 3958 miles.)

$$\begin{aligned}
a &= 1^\circ \\
b &= 1^\circ \\
c &= \underline{1^\circ} \\
2\,s &= 3^\circ \\
s &= 1^\circ\ 30'. \\
s - a &= 0^\circ\ 30'. \\
s - b &= 0^\circ\ 30'.
\end{aligned}$$

$$s - c = 0° \, 30'.$$
$$\tfrac{1}{2}s = 0° \, 45'.$$
$$\tfrac{1}{2}(s - a) = 0° \, 15'.$$
$$\tfrac{1}{2}(s - b) = 0° \, 15'.$$
$$\tfrac{1}{2}(s - c) = 0° \, 15'.$$
$$\log \tan \tfrac{1}{2}s = 8.11696$$
$$\log \tan \tfrac{1}{2}(s - a) = 7.63982$$
$$\log \tan \tfrac{1}{2}(s - b) = 7.63982$$
$$\log \tan \tfrac{1}{2}(s - c) = \underline{7.63982}$$
$$\log \tan^2 \tfrac{1}{4}E = 1.03642$$
$$\log \tan \tfrac{1}{4}E = 5.51821.$$

$$\tfrac{1}{4}E = 0° \, 0' \, 6.8139''.$$
$$E = 0° \, 0' \, 27.2556''.$$
$$\log R^2 = 7.19496$$
$$\log E = 1.43546$$
$$\log \frac{\pi}{648000} = \underline{4.68557 - 10}$$
$$\log F = 3.31599$$
$$F = 2070.1.$$

Therefore, the area is 2070.1 square miles.

EXERCISE LXIV. PAGE 197.

1. Find the dihedral angle made by adjacent lateral faces of a regular ten-sided pyramid; given the angle $V = 18°$, made at the vertex by two adjacent lateral edges.

About the vertex of the pyramid describe a sphere. The surface of the sphere will intersect the lateral surface of the pyramid, forming a regular spherical decagon, of which each side a is equal to 18°, being measured by the plane angle at the centre of the sphere.

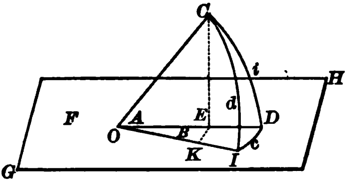
The required angle is an angle A of this decagon.

By Prob. 3, Ex. XXXV,

$$\sin \tfrac{1}{2}A = \sec \tfrac{1}{2}a \cos \frac{180°}{10}$$
$$= \sec 9° \cos 18°.$$
$$\log \cos 18° = 9.97821$$
$$\text{colog} \cos 9° = \underline{0.00538}$$
$$\log \sin \tfrac{1}{2}A = 9.98359$$
$$\tfrac{1}{2}A = 74° \, 21'.$$
$$A = 148° \, 42'.$$

2. Through the foot of a rod which makes the angle A with a plane a straight line is drawn in the plane. This line makes the angle B with the projection of the rod upon the plane. What angle does this line make with the rod?

Let CO be a straight line, making the angle A with the plane GH; OI a straight line passing through the foot of CO, making the angle B with the projection EO of CO upon the plane GH.



It is required to find the angle $COI = x$.

With a radius equal to unity, from O as a centre, construct the spherical triangle DCI.

Then $i = A,$

$c = B,$

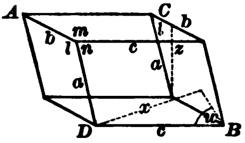
$d = COI = x,$

and $CDI = 90°.$

By [38], $\cos d = \cos i \cos c.$

$$\therefore \cos x = \cos A \cos B.$$

3. **Find the volume V of an oblique parallelopipedon; given the three unequal edges a, b, c, and the three angles l, m, n, which the edges make with one another.**



Let AB be an oblique parallelopipedon, and l, m, and n, the angles which the unequal edges a, b, and c make with one another.

Required the volume, V.

Let w equal the inclination of the edge c to the plane of a and b.

Now, by Geometry, $V =$ area base $\times$ altitude.

By Prob. 12, Ex. XXI, Area base $= ab \sin l$.

$$\text{Altitude} = x = c \sin w.$$

$$\therefore V = abc \sin l \sin w.$$

Suppose a sphere to be described having for its centre the vertex of the trihedral angle whose edges are a, b, and c. The spherical triangle whose vertices are the points where a, b, and c meet the surface has for its sides l, m, n; and w is the perpendicular arc to the side l from the opposite vertex.

Let L, M, N denote the angles of the triangle opposite l, m, n, respectively.

Then, by [39],

$$\sin w = \sin m \sin N$$

By [12], $$= 2 \sin m \sin \tfrac{1}{2} N \cos \tfrac{1}{2} N.$$

Let $$s = \tfrac{1}{2}(l + m + n).$$

By [47], $$\sin \tfrac{1}{2} N = \sqrt{\sin(s-l)\sin(s-m)\csc l \csc m},$$

and $$\cos \tfrac{1}{2} N = \sqrt{\sin s \sin(s-n)\csc l \csc m}.$$

$$\therefore \sin w = 2 \sin m \sqrt{\frac{\sin(s-l)\sin(s-m)}{\sin l \sin m}} \times \sqrt{\frac{\sin s \sin(s-n)}{\sin l \sin m}}$$

$$= \frac{2 \sin m}{\sin l \sin m} \sqrt{\sin s \sin(s-l)\sin(s-m)\sin(s-n)}$$

$$= \frac{2}{\sin l} \sqrt{\sin s \sin(s-l)\sin(s-m)\sin(s-n)}.$$

$$\therefore V = abc \sin l \sin w$$

$$= 2abc \sqrt{\sin s \sin(s-l)\sin(s-m)\sin(s-n)}.$$

4. The continent of Asia has nearly the shape of an equilateral triangle, the vertices being the East Cape, Cape Romania, and the Promontory of Baba. Assuming each side of this triangle to be 4800 geographical miles, and the earth's radius to be 3440 geographical miles, find the area of the triangle: (i) regarded as a plane triangle; (ii) regarded as a spherical triangle.

$$\begin{aligned}\text{(i) Area} &= \tfrac{1}{2}(\text{base} \times \text{altitude}).\\ \text{Altitude} &= \sqrt{4800^2 - 2400^2}\\ &= 2400\sqrt{4-1}\\ &= 2400\sqrt{3}.\end{aligned}$$

$$\begin{aligned}\log 2400 &= 3.38021\\ \log \sqrt{3} &= 0.23856\\ \log 2400 &= \underline{3.38021}\\ \log \text{area} &= 6.99898\end{aligned}$$

$$\text{Area} = 9976500.$$

(ii)

$$F = \frac{E}{180^\circ}\pi R^2.$$

$$\begin{aligned}a = b = c = 4800' &= 80^\circ.\\ 2s &= 240^\circ.\\ s &= 120^\circ.\\ \tfrac{1}{2}s &= 60^\circ.\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}(s-a) &= 20^\circ.\\ \tfrac{1}{2}(s-b) &= 20^\circ.\\ \tfrac{1}{2}(s-c) &= 20^\circ.\end{aligned}$$

$$\begin{aligned}\log\tan\tfrac{1}{2}s &= 10.23856\\ \log\tan\tfrac{1}{2}(s-a) &= 9.56107\\ \log\tan\tfrac{1}{2}(s-b) &= 9.56107\\ \log\tan\tfrac{1}{2}(s-c) &= \underline{9.56107}\\ \log\tan^2\tfrac{1}{4}E &= 8.92177\end{aligned}$$

$$\log\tan\tfrac{1}{4}E = 9.46088$$

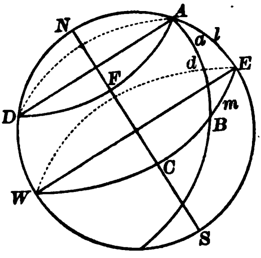
$$\begin{aligned}\tfrac{1}{4}E &= 16^\circ\ 7'\ 7.5''.\\ E &= 64^\circ\,28'\,30''\\ &= 232110''.\end{aligned}$$

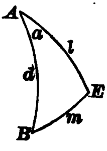
$$\begin{aligned}\log E &= 5.36570\\ \log\frac{\pi}{648000} &= 4.68557\\ \log R^2 &= \underline{7.07312}\\ \log F &= 7.12439\end{aligned}$$

$$F = 13316560.$$

5. A ship sails from a harbor in latitude *l*, and keeps on the arc of a great circle. Her *course* (or angle between the direction in which she sails and the meridian) at starting is a. Find where she will cross the equator, her course at the equator, and the distance she has sailed.

Let $NESW$ be the earth, WCE the equator, N and S the north and south poles. Let A be the point from which the ship starts, AFD the parallel of latitude of A, and AB the great circle of the ship's course.





Then

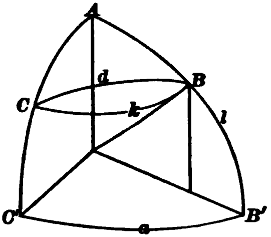
$BAE = a$ = course of ship,
$AE = l$ = latitude of its starting place,
$BE = m$ = place of crossing the equator,
$90° - B$ = course at equator,
$AB = d$ = distance sailed.

By Napier's Rules

$$\sin l = \tan m \cot a,$$
$$\cos B = \cos l \sin a,$$
and
$$\cos a = \tan l \cot d.$$

Whence,
$$\tan m = \sin l \tan a,$$
$$\cos B = \cos l \sin a,$$
and
$$\cot d = \cot l \cos a.$$

6. Two places have the same latitude l, and the distance between the places, measured on an arc of a great circle, is d. How much greater is the arc of the parallel of latitude between the places than the arc of the great circle? Compute the results for $l = 45°$, $d = 90°$.



Let B and C be the two given places of latitude l, d the arc of the great circle passing through B and C, and k the arc BC of the parallel of latitude of B and C.

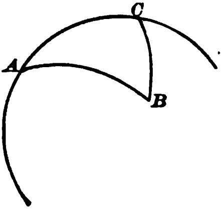
By Prob. 1, Ex. XXXV,

$$\begin{aligned}
\sin \tfrac{1}{2} A &= \sin \tfrac{1}{2} d \csc (90° - l)\\
&= \sin \tfrac{1}{2} d \sec l.\\
\therefore \tfrac{1}{2} A &= \sin^{-1} (\sin \tfrac{1}{2} d \sec l).\\
A &= 2 \sin^{-1} (\sin \tfrac{1}{2} d \sec l).
\end{aligned}$$

Again,
$$\begin{aligned}
A &= \text{arc } a.\\
\therefore \text{arc } k &= a \cos l\\
&= A \cos l\\
&= 2 \cos l \sin^{-1} (\sin \tfrac{1}{2} d \sec l).\\
\therefore k - d &= 2 \cos l \sin^{-1} (\sin \tfrac{1}{2} d \sec l) - d.
\end{aligned}$$

If $l = 45^c$ and $d = 90°$,
$$\begin{aligned}
k - d &= 2 \cos 45° \sin^{-1} (\sin 45° \sec 45°) - 90°\\
&= 2 \times \tfrac{1}{2} \sqrt{2} \sin^{-1} \left(\frac{\sin 45°}{\cos 45°}\right) - 90°\\
&= \sqrt{2} \sin^{-1} (1) - 90°\\
&= \sqrt{2} \times 90° - 90°\\
&= 90° (\sqrt{2} - 1).
\end{aligned}$$

7. The distance d between two places and the latitudes l and l' of the places are known. Find the difference between their longitudes.



Let C represent the north pole, A the position of one place, B the position of the other, and arc $AB = d$.

If the latitudes of A and B are l and l',

$$AC = 90^\circ - l,$$
$$BC = 90^\circ - l'.$$

By [47], $\tan \frac{1}{2} C =$

$$\sqrt{\sec s \sec (s-d) \sin (s-l) \sin (s-l')},$$

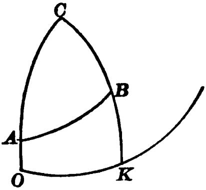
where $2s = l + l' + d$.

8. Given the latitudes and longitudes of three places on the earth's surface, and also the radius of the earth; show how to find the area of the spherical triangle formed by arcs of great circles passing through the three places.

The sides of the triangle are found by Sect. LXV; and the area is found from the sides by Sect. LXIII.

9. The distance between Paris and Berlin (the arc of a great circle) is equal to 472 geographical miles. The latitude of Paris is 48° 50′ 13″; that of Berlin, 52° 30′ 16″. When it is noon at Paris, what time is it at Berlin?

Let AO represent the latitude of Paris, and BK the latitude of Berlin. Then C represents the difference in longitude.



$$\begin{aligned} CA &= b = 41^\circ\ 9'\ 47'' \\ CB &= a = 37^\circ\ 29'\ 44'' \\ AB &= c = \underline{\ \ 7^\circ\ 52'\qquad} \quad (472 \div 60) \\ 2s &= 86^\circ\ 31'\ 31'' \end{aligned}$$

$$\begin{aligned} s &= 43^\circ\ 15'\ 45.5''. \\ s-a &= \ \ 5^\circ\ 46'\ \ 1.5''. \\ s-b &= \ \ 2^\circ\ \ 5'\ 58.5''. \\ s-c &= 35^\circ\ 23'\ 45.5''. \end{aligned}$$

By [47], $\tan^2 \frac{1}{2} C =$

$$\csc s \sin (s-a) \sin (s-b) \csc (s-c).$$

$$\begin{aligned} \log \csc s &= \ \ 0.16409 \\ \log \sin (s-a) &= \ \ 9.00210 \\ \log \sin (s-b) &= \ \ 8.56391 \\ \log \csc (s-c) &= \underline{\ \ 0.23715} \\ \log \tan^2 \tfrac{1}{2} C &= 17.96725 \end{aligned}$$

$$\log \tan \tfrac{1}{2} C = 8.98363.$$

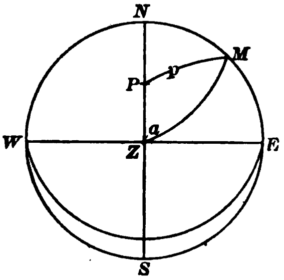
$$\begin{aligned} \tfrac{1}{2} C &= \ \ 5^\circ\ 30'\ 2''. \\ C &= 11^\circ\ \ 0'\ 4''. \end{aligned}$$

$$15\,)\,\underline{11^\circ\ \ 0'\qquad 4''}$$
$$44 \text{ min. } \tfrac{4}{15} \text{ sec.}$$

Therefore, the time at Berlin is 12 hr. 44 min. P.M.

10. Given the altitude of the pole 45°, and the azimuth of a star on the horizon 45°; find the polar distance of the star.

Let Z be the zenith, P the pole, and M the position of the star. In the spherical triangle ZMP,



$$ZP = 90^\circ - l = 45^\circ,$$
$$ZM = z = 90^\circ,$$
$$Z = a = 45^\circ.$$

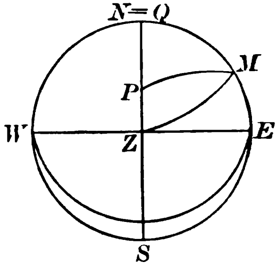
Required p.

By [45],

$$\begin{aligned}\cos p &= \cos(90^\circ - l)\cos z \\ &\quad + \sin(90^\circ - l)\sin z \sin a \\ &= \cos 45^\circ \cos 90^\circ \\ &\quad + \sin 45^\circ \sin 90^\circ \sin 45^\circ \\ &= \sin^2 45^\circ \\ &= \tfrac{1}{2}. \\ \therefore p &= 60^\circ.\end{aligned}$$

11. Given the latitude l of the observer, and the declination d of the sun; find the local time (apparent solar time) of sunrise and sunset, and also the azimuth of the sun at these times (refraction being neglected). When and where does the sun rise on the longest day of the year (at which time $d \doteq + 23^\circ 27'$) in Boston ($l = 42^\circ 21'$), and what is the length of the day from sunrise to sunset? Also, find when and where the sun rises in Boston on the shortest day of the year (when $d = - 23^\circ 27'$), and the length of this day.

(i) To find the hour angle t when the sun is on the horizon.



$$PM = 90^\circ - d,$$
$$ZQ = 90^\circ,$$
$$PQ = l.$$

Then in triangle PMQ, by [40],

$$\cos QPM = \tan PQ \cot PM,$$

or $$\cos t = - \tan l \tan d.$$

Time of sunrise

$$= \left(12 - \frac{t}{15}\right) \text{ o'clock A.M.}$$

Time of sunset

$$= \left(\frac{t}{15}\right) \text{ o'clock P.M.}$$

(ii) To find azimuth $a = MQ$.

By [38],

$$\cos PM = \cos PQ \cos QM,$$

or $$\sin d = \cos l \cos a.$$

$$\therefore \cos a = \sin d \sec l.$$

(iii) At Boston on the longest day

$$\cos t = - \tan d \tan l.$$

$$\log \tan d = 9.63726$$
$$\log \tan l = \underline{9.95977}$$
$$\log \cos t = 9.59703\,(n)$$
$$t = 113^\circ\, 17'\, 26''.$$

$$\frac{t}{15} = 7 \text{ hr. } 33 \text{ min. } 10 \text{ sec.}$$

$$12 - \frac{t}{15} = 4 \text{ hr. } 26 \text{ min. } 50 \text{ sec.}$$

Length of longest day

$$= \frac{2t}{15} = 15 \text{ hr. } 6 \text{ min. } 20 \text{ sec.}$$

$$\cos a = \sin d \sec l.$$

$$\begin{aligned} \log \sin d &= 9.59983 \\ \log \sec l &= \underline{0.13133} \\ \log \cos a &= 9.73116 \\ a &= 57° 25' 15''. \end{aligned}$$

(iv) At Boston on the shortest day

$$\cos t = \tan d \tan l.$$

$$t = 66° 42' 34''.$$

$$\frac{t}{15} = 4 \text{ hr. } 26 \text{ min. } 50 \text{ sec.}$$

$$12 - \frac{t}{15} = 7 \text{ hr. } 33 \text{ min. } 10 \text{ sec.}$$

Length of shortest day

$$= \frac{2t}{15} = 8 \text{ hr. } 53 \text{ min. } 40 \text{ sec.}$$

$$\cos a' = -\sin d \sec l.$$

$$\begin{aligned} \therefore a' &= 180° - a \\ &= 122° 34' 45''. \end{aligned}$$

12. When is the solution of the problem in Example 11 impossible, and for what places is the solution impossible?

The solution is impossible if $\cos t > 1$ or < -1 or if $\cos a > 1$, or < -1, *i.e.*, if (for positive declination)

$$\tan l > \cot d,$$

or $\sin l > \cos d$;

that is, if $l > 90° - d$.

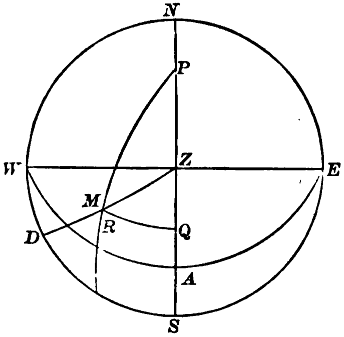
The maximum value of d is 23° 27′; hence the minimum value of l is 66° 33′. The solution is therefore impossible only for places within the Arctic or Antarctic circles. For such places at certain seasons depending on d the sun fails to rise during 24 hours.

13. Given the latitude of a place and the sun's declination; find his altitude and azimuth at 6 o'clock A.M. (neglecting refraction). Compute the results for the longest day of the year at Munich ($l = 48° 9'$).

In the spherical triangle PZM,

$$\begin{aligned} PZM &= a, \\ PZ &= 90° - l, \\ PM &= 90° - d, \\ ZPM &= t = 90°, \\ ZM &= 90° - h. \end{aligned}$$

The declination of the sun on the longest day is 23° 27′.



By [38],

$$\cos MZ = \cos PZ \cos MP.$$

$$\cos(90° - h)$$

$$= \cos(90° - l) \cos(90° - d).$$

$\therefore \sin h = \sin l \sin d.$

By [42],

$\sin PZ = \tan PM \cot PZM.$

$$\begin{aligned}\sin(90^\circ - l) &= \tan(90^\circ - d)\cot a.\\ \cos l &= \cot d \cot a.\\ \therefore \cot a &= \cos l \tan d.\end{aligned}$$

If $l = 48^\circ\ 9'$ and $d = 23^\circ\ 27'$,

$$\begin{aligned}\log\sin l &= 9.87209\\ \log\sin d &= \underline{9.59983}\\ \log\sin h &= 9.47192\end{aligned}$$

$$\text{Altitude} = h = 17^\circ\ 14'\ 35''.$$

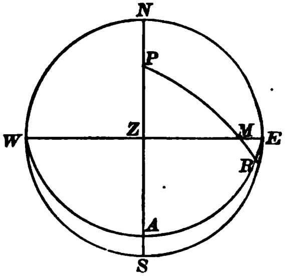
$$\begin{aligned}\log\cos l &= 9.82424\\ \log\tan d &= \underline{9.63726}\\ \log\cot a &= 9.46150\end{aligned}$$

$$\text{Azimuth} = a = 73^\circ\ 51'\ 34''.$$

14. How does the altitude of the sun at 6 A.M. *on a given day* change as we go from the equator to the pole? At what time of the year is it a maximum *at a given place?* (Given $\sin h = \sin l \sin d$.)

The farther the place from the equator, the greater the sun's altitude at 6 A.M. in summer. At the equator it is 0°. At the north pole it is equal to the sun's declination. At a given place, the sun's altitude at 6 A.M. is a maximum on the longest day of the year, and then $\sin h = \sin l \sin e$ (where $e = 23^\circ\ 27'$).

15. Given the latitude of a place north of the equator, and the declination of the sun; find the time of day when the sun bears due east and due west. Compute the results for the longest day at St. Petersburg ($l = 59^\circ\ 56'$).



$$\begin{aligned}PM &= 90^\circ - d.\\ PZ &= 90^\circ - l.\\ PZM &= 90^\circ.\end{aligned}$$

By [40],

$$\begin{aligned}\cos ZPM &= \tan PZ \cot PM.\\ \cos t &= \tan(90^\circ - l)\cot(90^\circ - d).\\ \therefore \cos t &= \cot l \tan d.\end{aligned}$$

The times of bearing due east and due west are

$$12 - \frac{t}{15}\text{ A.M.} \quad\text{and}\quad \frac{t}{15}\text{ P.M.},$$

respectively.

At St. Petersburg on the longest day $l = 59^\circ\ 56'$, $d = 23^\circ\ 27'$.

$$\begin{aligned}\log\cot l &= 9.76261\\ \log\tan d &= \underline{9.63726}\\ \log\cos t &= 9.39987\\ t &= 75^\circ\ 27'\ 24''.\end{aligned}$$

$$\therefore 12 - \frac{t}{15} = 6\text{ hr. }58\text{ min. }10\text{ sec. A.M.},$$

$$\text{and}\quad \frac{t}{15} = 5\text{ hr. }1\text{ min. }50\text{ sec. P.M.}$$

16. Apply the general result in Example 15 ($\cos t = \cot l \tan d$) to the case when the days and nights are equal in length (that is, when $d = 0^\circ$). Why can the sun in summer never be due east before 6 A.M., or due west after 6 P.M.? How does the time of bearing due east and due west change with the declination of the sun? Apply the general result to the cases where $l < d$ and $l = d$. What is it at the north pole?

When the days and nights are equal, $d = 0°$, $\cos t = 0$, and $t = 90°$; that is, the sun is due east at 6 A.M. and due west at 6 P.M. Since l and d must both be less than 90°, $\cos t$ cannot be negative; therefore, t cannot be greater than 90°. As d increases, t decreases; that is, the times of bearing due east and due west both approach noon.

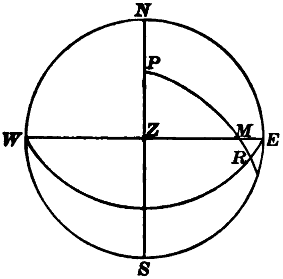
If $l = d$, $\cos t = 1$, $t = 0°$, and the times both coincide with noon.

If $l < d$, then $\cos t > 1$, and the case is impossible.

The explanation of these results is that, if $d = l$, the sun is in the zenith at noon, and north of the prime vertical at every other time. And if $d > l$, the sun is north of the prime vertical the entire day.

If $l > d$, the diurnal circle of the sun and the prime vertical of the place meet in two points, which separate farther and farther as l increases, the distance between them approaching 180° − 23° 27′ as l approaches 90°. At the pole the prime vertical is indeterminate; but near the pole $t = 90°$, and the sun is always east at 6 A.M.

17. Given the sun's declination and his altitude when he bears due east; find the latitude of the observer.



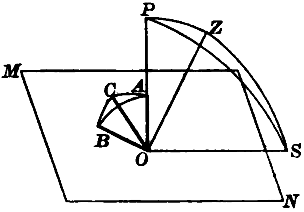
$$ZM = 90° - h.$$
$$PM = 90° - d.$$
$$PZ = 90° - l.$$

Since the sun M bears due east, MZP is a right angle.

By [38],

$$\cos PM = \cos PZ \cos MZ.$$
$$\cos(90° - d) = \cos(90° - l)\cos(90° - h).$$
$$\therefore \sin d = \sin l \sin h.$$
$$\sin l = \sin d \csc h.$$

18. At a point O in a horizontal plane MN a staff OA is fixed so that its angle of inclination AOB with the plane is equal to the latitude of the place, 51° 30′ N., and the direction OB is due north. What angle will OB make with the shadow of OA on the plane, at 1 P.M., when the sun is on the equinoctial?



Given the direction of OB due north, $AOB = 51° 30' = l$, and plane MN horizontal; find BOC.

$SPZ =$ hour angle of sun at 1 P.M.
$= 15°$.

$SPZ = CAB$, being vertical angles.
$\therefore CAB = 15°$.

$ABC = 90°$, since OB is the projection of OA on plane MN.

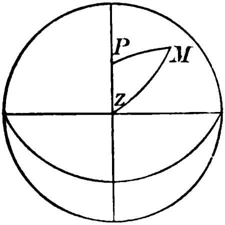
Arc $AB = 51° 30'$, being the measure of plane angle AOB.

Then in right spherical triangle ABC, by [42],

$$\tan BC = \tan BAC \sin AB.$$

$$\log \tan BAC = 9.42805$$
$$\log \sin AB \;\;= \underline{9.89354}$$
$$\log \tan BC \;\;= 9.32159$$
$$\text{Arc } BC \;\;\;= 11° 50' 35''.$$
$$\therefore BOC = 11° 50' 35''.$$

19. What is the direction of a wall in latitude 52° 30° N. which casts no shadow at 6 A.M. on the longest day of the year?



The wall must lie in the plane of ZM in order that it may cast no shadow.

$$PZ = 90° - l,$$
$$PM = 90° - e,$$
$$P = 90°;$$

required $MZP = a$.

By [42],

$$\sin PZ = \tan PM \cot PZM.$$
$$\sin (90° - l) = \tan (90° - e) \cot a.$$
$$\cos l = \cot e \cot a.$$
$$\therefore \cot a = \cos l \tan e.$$

$$\log \cos l = 9.78445$$
$$\log \tan e = \underline{9.63726}$$
$$\log \cot a = 9.42171$$
$$a = 75° 12' 28''.$$

20. Find the latitude of the place at which the sun rises exactly in the northeast on the longest day of the year.

When the sun rises in the northeast on the longest day of the year, $a = 45°$, $d = 23° 27'$.

By Prob. 11, $\cos a = \sin d \sec l$.

$$\therefore \cos l = \sin d \sec a.$$

$$\log \sin d = 9.59983$$
$$\log \sec a = \underline{0.15051}$$
$$\log \cos l = 9.75034$$
$$l = 55° 45' 6''.$$

21. Find the latitude of the place at which the sun sets at 10 o'clock on the longest day.

$$ZPM = 10 \times 15°$$
$$= 150°.$$
$$ZM = 90°.$$
$$MP = 90° - l.$$

By Prob. 15,

$$\cos t = \cot l \tan d.$$
$$\therefore \cot l = \cos t \cot d.$$
$$t = 150°.$$
$$d = 23° 27'.$$

$$\log \cos t = 9.93753$$
$$\log \cot d = \underline{10.36274}$$
$$\log \cot l = 10.30027$$
$$l = 63° 23' 41''.$$

22. To what does the general formula for the hour angle, in Sect. LXX, reduce when (i) $h = 0°$, (ii) $l = 0°$ and $d = 0°$, (iii) l or $d = 90°$?

By Sect. LXX, $\sin \frac{1}{2}t = \pm\left[\cos\frac{1}{2}(l+p+h)\sin\frac{1}{2}(l+p-h)\sec l \csc p\right]^{\frac{1}{2}}$

By [21], $= \pm\left[\frac{1}{2}(\sin\{l+p\} - \sin h)\sec l \csc p\right]^{\frac{1}{2}}$.

(i) If $h = 0°$, $\sin\frac{1}{2}t = \pm\left[\frac{1}{2}\sin(l+p)\sec l \csc p\right]^{\frac{1}{2}}$.

By [13],

$$\begin{aligned}\cos t &= 1 - 2\sin^2\tfrac{1}{2}t\\ &= 1 - \sin(l+p)\sec l \csc p\end{aligned}$$

By [4],

$$\begin{aligned}&= 1 - \frac{\sin l \cos p + \cos l \sin p}{\cos l \sin p}\\ &= -\frac{\sin l \cos p}{\cos l \sin p}\\ &= -\tan l \cot p.\end{aligned}$$

(ii) If

$$\begin{aligned}l &= 0° \text{ and } d = 0°,\\ p &= 90° - d\\ &= 90°.\\ \therefore \sin\tfrac{1}{2}t &= \left[\tfrac{1}{2}(1 - \sin h)\right]^{\frac{1}{2}}.\\ \cos t &= 1 - (1 - \sin h)\\ &= \sin h.\\ \therefore t &= 90° - h\\ &= z.\end{aligned}$$

(iii) If l or $d = 90°$, $\sec l$ or $\csc p = \infty$, and the formula is useless. When $d = 90°$, the star is at the pole and its hour angle is indeterminate; and when $l = 90°$, the place of observation is at the terrestrial pole and the meridian is indeterminate.

23. What does the general formula for the azimuth of a celestial body, in Sect. LXXI, become when $t = 90° = 6$ hours?

From Sect. LXXI,

$$\tan m = \cot d \cos t, \quad (1)$$

and

$$\tan a = \sec(l+m)\tan t \sin m. \quad (2)$$

Multiply (1) by (2),

$$\tan a \tan m = \sec(l+m)\cot d \sin t \sin m.$$

$$\therefore \tan a = \sec(l+m)\cot d \sin t \cos m.$$

Here $t = 90°$; hence

$$\tan m = 0,$$

and

$$m = 0°.$$

$$\therefore \tan a = \sec l \cot d,$$

or

$$\cot a = \cos l \tan d.$$

24. Show that the formulas of Sect. LXXII, if $t = 90°$, lead to the equation $\sin l = \sin h \csc d$; and that if $d = 0°$, they lead to the equation $\cos l = \sin h \sec t$.

From Sect. LXXII,

$$\tan m = \cot d \cos t, \quad (1)$$

and

$$\cos n = \cos m \sin h \csc d. \quad (2)$$

(i) If $t = 90°$, then $m = 0°$ and $n = 90° - l$; hence

$$\cos(90° - l) = \cos 0° \sin h \csc d.$$

$$\sin l = \sin h \csc d.$$

(ii) If $d = 0°$, then $m = 90°$, $n = l$.

Divide (2) by (1),

$$\begin{aligned}\cos n \cot m &= \cos m \sin h \sec d \sec t.\\ \therefore \cos n &= \sin m \sin h \sec d \sec t.\\ \therefore \cos l &= \sin h \sec t.\end{aligned}$$

25. Given the latitude of the place of observation 52° 30′ 16″, the declination of a star 38°, its hour angle 28° 17′ 15″; find the altitude of the star.

By Sect. LXXI,

$$\begin{aligned}\tan m &= \cot d \cos t,\\ \text{and}\quad \sin h &= \sin (l+m) \sin d \sec m.\end{aligned}$$

Here

$$\begin{aligned}d &= 38°,\\ l &= 52° \ 30' \ 16'',\\ t &= 28° \ 17' \ 15''.\end{aligned}$$

$$\begin{aligned}\log \cot d &= 10.10719\\ \log \cos t &= \underline{\ 9.94477}\\ \log \tan m &= 10.05196\\ m &= 48° \ 25' \ 10''.\end{aligned}$$

$$\begin{aligned}\log \sin (l+m) &= 9.99206\\ \log \sin d &= 9.78934\\ \log \sec m &= \underline{0.17804}\\ \log \sin h &= 9.95944\\ h &= 65° \ 37' \ 20''.\end{aligned}$$

26. Given the latitude of the place of observation 51° 19′ 20″, the polar distance of a star 67° 59′ 5″, its hour angle 15° 8′ 12″; find the altitude and the azimuth of the star.

By Sect. LXXI,

$$\begin{aligned}\tan m &= \cot d \cos t,\\ \sin h &= \sin (l+m) \sin d \sec m,\\ \tan a &= \sec (l+m) \tan t \sin m.\end{aligned}$$

$$\begin{aligned}l &= 51° \ 19' \ 20''.\\ d &= 90° - 67° \ 59' \ 5''\\ &= 22° \ \ 0' \ 55''.\\ t &= 15° \ \ 8' \ 12''.\end{aligned}$$

$$\begin{aligned}\log \cot d &= 10.39326\\ \log \cos t &= \underline{\ 9.98466}\\ \log \tan m &= 10.37792\\ m &= 67° \ 16' \ 22''.\end{aligned}$$

$$\begin{aligned}\log \sin (l+m) &= 9.94351\\ \log \sin d &= 9.57386\\ \log \sec m &= \underline{0.41302}\\ \log \sin h &= 9.93039\\ h &= 58° \ 25' \ 8''.\end{aligned}$$

$$\begin{aligned}\log \sec (l+m) &= 0.32001 \ (n)\\ \log \tan t &= 9.43218\\ \log \sin m &= \underline{9.96490}\\ \log \tan a &= 9.71709 \ (n)\\ a &= 152° \ 28'.\end{aligned}$$

27. Given the declination of a star 7° 54′, its altitude 22° 45′ 12″, its azimuth 129° 45′ 37; find the hour angle of the star and the latitude of the observer.

In the spherical triangle ZPM,

$$\frac{\sin ZM}{\sin ZPM} = \frac{\sin PM}{\sin PZM}$$

$$\begin{aligned}&\therefore \sin ZPM\\ &\quad = \sin PZM \sin ZM \csc PM.\\ &\sin t = \sin a \sin (90° - h) \csc (90° - d).\\ &\therefore \sin t\\ &\quad = \sin a \cos h \sec d.\end{aligned}$$

Here

$$\begin{aligned}a &= 129° \ 45' \ 37'',\\ h &= 22° \ 45' \ 12'',\\ d &= 7° \ 54'.\end{aligned}$$

$$\begin{aligned}\log \sin a &= 9.88577\\ \log \cos h &= 9.96482\\ \log \sec d &= \underline{0.00414}\\ \log \sin t &= 9.85473\\ t &= 45° \ 42'.\end{aligned}$$

By Sect. LXXII,

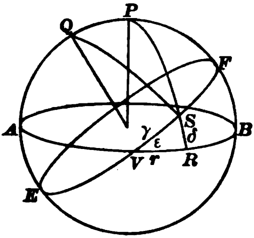
$$\begin{aligned}\tan m &= \cot d \cos t,\\ \cos n &= \cos m \sin h \csc d,\\ l &= 90° - (m \pm n).\end{aligned}$$

$$\begin{aligned}\log\cot d &= 10.85773\\ \log\cos t &= \underline{\ 9.84411}\\ \log\tan m &= 10.70184\\ m &= 78^\circ\ 45'\ 45''.\end{aligned}$$

$$\begin{aligned}\log\cos m &= 9.28976\\ \log\sin h &= 9.58745\\ \log\csc d &= \underline{0.86187}\\ \log\cos n &= 9.73908\\ n &= 56^\circ\ 44'\ 39''.\\ m - n &= 22^\circ\ 1'\ 6''.\\ 90^\circ - (m - n) &= 67^\circ\ 58'\ 54''.\\ \therefore l &= 67^\circ\ 58'\ 54''.\end{aligned}$$

28. Given $e = 23^\circ\ 27'$ and the longitude v of the sun; find the declination d and the right ascension r.

In the figure let P represent the pole of the equinoctial AVB, S the position of the sun, and Q the pole of the ecliptic EVF.



Then $VS = v$,
$VR = r$,
$SR = d$,
$RVS = e$.

Then in the right triangle RVS, by [39],

$$\sin SR = \sin VS \times \sin RVS,$$

or $$\sin d = \sin v \sin e.$$

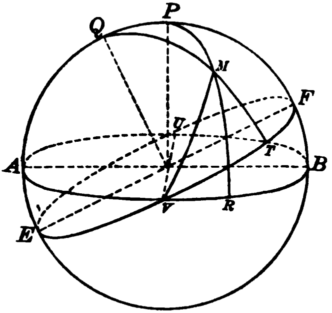
Also, by [40],

$$\cos RVS = \tan RV \cot VS,$$

or $$\cos e = \tan r \cot v.$$

$$\therefore \tan r = \tan v \cos e.$$

29. Given $e = 23^\circ\ 27'$, the latitude of a star 51°, its longitude 315°; find its declination and its right ascension.



Here $VT = 315^\circ$,
$TM = 51^\circ$,
$RVT = 23^\circ\ 27'$;
to find $VR = r$,
and $RM = d$.

In the right triangle VTM,
By [38],

$$\cos VM = \cos VT \cos TM,$$

By [42],

$$\tan MVT = \tan MT \csc VT.$$

$$\begin{aligned}\log\cos VT &= 9.84949\\ \log\cos TM &= \underline{9.79887}\\ \log\cos VM &= 9.64836\\ VM &= 63^\circ\ 34'\ 36''.\end{aligned}$$

$$\begin{aligned}\log\tan TM &= 10.09163\\ \log\csc VT &= \underline{\ 0.15051}\ (n)\\ \log\tan MVT &= 10.24214\ (n)\end{aligned}$$

$$MVT = -(60° \; 12' \; 14.5'').$$

In the right triangle RVM,

$$\begin{aligned} RVM &= RVT + TVM \\ &= 23° \; 27' - (60° \; 12' \; 14.5'') \\ &= -(36° \; 45' \; 14.5''). \end{aligned}$$

By [39],

$$\sin RM = \sin VM \sin RVM.$$

$$\begin{aligned} \log\sin VM &= 9.95208 \\ \log\sin RVM &= \underline{9.77698} \\ \log\sin RM &= 9.72906 \end{aligned}$$

$$RM = d = 32° \; 24' \; 12''.$$

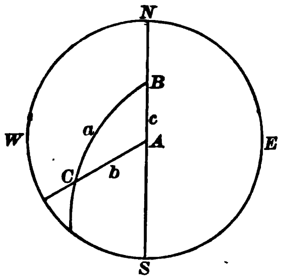
Also, by [42],

$$\sin VR = \tan RM \cot RVM.$$

$$\begin{aligned} \log\tan RM &= 9.80257 \\ \log\cot RVM &= \underline{0.12677} \; (n) \\ \log\sin VR &= 9.92934 \; (n) \end{aligned}$$

$$\begin{aligned} VR &= -(58° \; 11' \; 43''). \\ \therefore \; VR &= 360° - 58° \; 11' \; 43'' \\ &= 301° \; 48' \; 17''. \end{aligned}$$

30. Given the latitude of the observer 44° 50′, the azimuth of a star 138° 58′, its hour angle 20°; find its declination.



$$\begin{aligned} \text{Given} \quad c &= 90° - 44° \; 50' \\ &= 45° \; 10', \\ A &= 138° \; 58', \\ B &= 20°. \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(A - B) &= 59° \; 29'. \\ \tfrac{1}{2}(A + B) &= 79° \; 29'. \\ \tfrac{1}{2}c &= 22° \; 35'. \end{aligned}$$

$$\begin{aligned} \log\cos\tfrac{1}{2}(A - B) &= 9.70568 \\ \log\sec\tfrac{1}{2}(A + B) &= 0.73869 \\ \log\tan\tfrac{1}{2}c &= \underline{9.61901} \\ \log\tan\tfrac{1}{2}(a + b) &= 0.06338 \\ \tfrac{1}{2}(a + b) &= 49° \; 9' \; 57.7''. \end{aligned}$$

$$\begin{aligned} \log\sin\tfrac{1}{2}(A - B) &= 9.93525 \\ \log\csc\tfrac{1}{2}(A + B) &= 0.00736 \\ \log\tan\tfrac{1}{2}c &= \underline{9.61901} \\ \log\tan\tfrac{1}{2}(a - b) &= 9.56162 \\ \tfrac{1}{2}(a - b) &= 20° \; 1' \; 24.6''. \\ \therefore \; a &= 69° \; 11' \; 22''. \end{aligned}$$

$$\begin{aligned} d &= 90° - 69° \; 11' \; 22'' \\ &= 20° \; 48' \; 38''. \end{aligned}$$

31. Given the latitude of the place of observation 51° 31′ 48″, the altitude of the sun west of the meridian 35° 14′ 27″, its declination + 21° 27′; find the local apparent time.

By Sect. LXX,

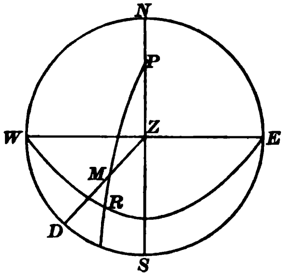
$$\sin\tfrac{1}{2}t = \sqrt{\cos\tfrac{1}{2}(l + p + h)\sin\tfrac{1}{2}(l + p - h)\csc p \sec l}.$$

Here

$$\begin{aligned} l &= 51° \; 31' \; 48'', \\ h &= 35° \; 14' \; 27'', \\ p &= 90° - 21° \; 27' = 68° \; 33'. \end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}(l+p+h) &= 77^\circ\ 39'\ 37.5''.\\
\tfrac{1}{2}(l+p-h) &= 42^\circ\ 25'\ 10.5''.\\
\log\cos\tfrac{1}{2}(l+p+h) &= 9.32982\\
\log\sin\tfrac{1}{2}(l+p-h) &= 9.82901\\
\log\csc p &= 0.03117\\
\log\sec l &= \underline{0.20614}\\
\log\sin^2\tfrac{1}{2}t &= 9.39614\\
\log\sin\tfrac{1}{2}t &= 9.69807\\
\tfrac{1}{2}t &= 29^\circ\ 55'\ 54.5''.\\
t &= 59^\circ\ 51'\ 49''.\\
\frac{t}{15} &= 3\text{ hr. }59\text{ min. }27\tfrac{4}{15}\text{ sec. P.M.}
\end{aligned}$$

32. Given the latitude of place l, the polar distance p of a star, and its altitude h; find its azimuth a.



$$\begin{aligned}
\text{Altitude} &= ZM = 90^\circ - h.\\
\text{Co-latitude} &= PZ = 90^\circ - l.\\
\text{Polar distance} &= PM = 90^\circ - d = p.\\
\text{Azimuth} &= PZM = a.
\end{aligned}$$

By [47], $\cos\tfrac{1}{2}A = \sqrt{\sin s\sin(s-a)\csc b\csc c}.$

$$\begin{aligned}
\text{Let } A &= PZM \text{ or } a,\\
a &= p,\\
b &= 90^\circ - h,\\
c &= 90^\circ - l.
\end{aligned}$$

Then

$$\begin{aligned}
\sin s &= \sin[90^\circ - \tfrac{1}{2}(l+h-p)]\\
&= \cos\tfrac{1}{2}(l+h-p).\\
\sin(s-a) &= \sin[90^\circ - \tfrac{1}{2}(l+h+p)]\\
&= \cos\tfrac{1}{2}(l+h+p).\\
\csc b &= \csc(90^\circ - h) = \sec h.\\
\csc c &= \csc(90^\circ - l) = \sec l.\\
\therefore \cos\tfrac{1}{2}a &= \sqrt{\cos\tfrac{1}{2}(l+h+p)\cos\tfrac{1}{2}(l+h-p)\sec l\sec h}.
\end{aligned}$$

SURVEYING.

EXERCISE II. PAGE 230.

2. From the bearings obtained in Example 1 find the value of each of the interior angles. What is their sum?

Since the given field has 5 sides the sum of the interior angles is (*Geometry*, § 205)

$$3 \times 180^\circ = 540^\circ.$$

4. Range out a line whose bearing is N. 38° 30′ W., and at some point in this line range out another line making a right angle with it. What is the bearing of the second line?

The second line makes an angle of 90° with the first.

$$90^\circ - 38^\circ\,30' = 51^\circ\,30'.$$

Therefore, the bearing of the second line is N. 51° 30′ E.

EXERCISE III. PAGE 235.

2. Lay out the entire angular magnitude about some point into four or more angles, and measure each of them. What should the sum of them equal?

The sum of them should equal 360°.

3. If the constant of a transit, adjusted to one foot 100 feet away, is 3.8 inches, what is the true length of a line when the indication on the rod is 2.35 feet?

The true distance is 235 feet 3.8 inches.

EXERCISE V. PAGE 263.

1. Required the area of a triangular field whose sides are 13 chains, 14 chains, and 15 chains.

$$\begin{aligned}
\text{Area} &= \sqrt{s(s-a)(s-b)(s-c)} \\
&= \sqrt{21 \times 8 \times 7 \times 6}\ \text{sq. ch.} \\
&= \sqrt{3^2 \times 7^2 \times 2^4}\ \text{sq. ch.} \\
&= 84\ \text{sq. ch.} \\
&= 8.4\ \text{A.} \\
&= 8\ \text{A.}\ 64\ \text{P.}
\end{aligned}$$

2. Required the area of a triangular field if it has two angles 48° 30′ and 71° 45′, and the included side 20 chains.

$$F = \frac{a^2 \sin B \sin C}{2 \sin (B + C)}$$

$$= \frac{a^2}{2} \sin B \sin C \csc (B + C)$$

$$= 200 \sin 48° \, 30' \sin 71° \, 45' \csc 120° \, 15'.$$

$$\begin{aligned} \log 200 &= 2.30103 \\ \log \sin 48° \, 30' &= 9.87446 \\ \log \sin 71° \, 45' &= 9.97759 \\ \log \csc 120° \, 15' &= \underline{0.06357} \\ \log F &= 2.21665 \end{aligned}$$

$$F = 164.68.$$

$$\text{Area} = 164.68 \text{ sq. ch.} = 16.468 \text{ A.} = 16 \text{ A. } 74\tfrac{22}{25} \text{ P.}$$

3. Required the area of a triangular field whose base is 12.60 chains, and altitude 6.40 chains.

$$\begin{aligned} \text{Area} &= \tfrac{1}{2}(12.6 \times 6.4) \text{ sq. ch.} \\ &= 40.32 \text{ sq. ch.} \\ &= 4.032 \text{ A.} \\ &= 4 \text{ A. } 5\tfrac{3}{25} \text{ P.} \end{aligned}$$

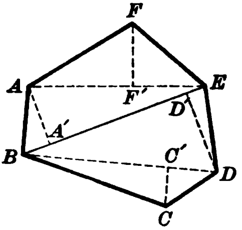
4. Required the area of a triangular field which has two sides 4.50 chains and 3.70 chains, and the included angle 60°.

$$\begin{aligned} \text{Area} &= \tfrac{1}{2} bc \sin A \\ &= \tfrac{1}{2} \times 4.5 \times 3.7 \times 0.866 = 7.20945. \\ \text{Area} &= 7.20945 \text{ sq. ch.} \\ &= 0.720945 \text{ A.} \\ &= 115\tfrac{7}{20} \text{ P.} \end{aligned}$$

5. Required the area of a field in the form of a trapezium, one of whose diagonals is 9 chains, and the two perpendiculars upon this diagonal from the opposite vertices 4.50 chains and 3.25 chains.

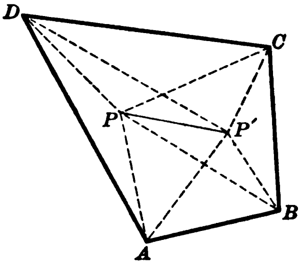
$$\begin{aligned} \text{Area} &= 9 \times \tfrac{1}{2}(4.5 + 3.25) \text{ sq. ch.} \\ &= 34.875 \text{ sq. ch.} \\ &= 3.4875 \text{ A.} \\ &= 3 \text{ A. } 78 \text{ P.} \end{aligned}$$

6. Required the area of the field $ABCDEF$ (Fig. 36), if $AE = 9.25$ chains, $FF' = 6.40$ chains, $BE = 13.75$ chains, $DD' = 7$ chains, $DB = 10$ chains, $CC' = 4$ chains, and $AA' = 4.75$ chains.



$$\begin{array}{lll} 2 \text{ area } AFE & = 6.4 \times 9.25 & = 59.2 \\ 2 \text{ area } BDEA & = 13.75\,(4.75 + 7) & = 161.5625 \\ 2 \text{ area } BDC & = 10 \times 4 & = 40. \\ \hline 2 \text{ area } ABCDEF & & = 260.7625 \\ \quad \text{area } ABCDEF & & = 130.38125 \\ 130.38125 \text{ sq. ch.} & = 13.038125 \text{ A.} & = 13 \text{ A. } 6\tfrac{1}{10} \text{ P.} \end{array}$$

7. Determine the area of the field $ABCD$ from two interior stations, P and P', if $PP' = 1.50$ chains,



$PP'C = 89° 35'$,
$PP'B = 185° 30'$,
$PP'A = 309° 15'$,
$PP'D = 349° 45'$,

$P'PB = 3° 35'$,
$P'PA = 113° 45'$,
$P'PD = 165° 40'$,
$P'PC = 303° 15'$.

Area $= \triangle PAD + \triangle PCD + \triangle PBC + \triangle PAB$.

$\angle PP'D = 10° 15'$,
$\angle PDP' = 4° \; 5'$,
[illegible] 4° 30′,

$\angle PP'A = 50° 45'$,
$\angle PAP' = 15° 30'$,
$\angle PBP' = 1° 55'$,

$\angle P'PC = 56° 45'$,
$\angle PCP' = 33° 40'$.

$$PD = \frac{PP' \sin PP'D}{\sin PDP'}.$$

$$\begin{aligned}
\log PP' &= 0.17609\\
\log \sin PP'D &= 9.25028\\
\text{colog} \sin PDP' &= \underline{1.14748}\\
\log PD &= 0.57385
\end{aligned}$$

$$PA = \frac{PP' \sin PP'A}{\sin PAP'}.$$

$$\begin{aligned}
\log PP' &= 0.17609\\
\log \sin PP'A &= 9.88896\\
\text{colog} \sin PAP' &= \underline{0.57310}\\
\log PA &= 0.63815
\end{aligned}$$

$$PC = \frac{PP' \sin PP'C}{\sin PCP'}.$$

$$\begin{aligned}
\log PP' &= 0.17609\\
\log \sin PP'C &= 9.99999\\
\text{colog} \sin PCP' &= \underline{0.25621}\\
\log PC &= 0.43229
\end{aligned}$$

$$PB = \frac{PP' \sin PP'B}{\sin PBP'}.$$

$$\begin{aligned}
\log PP' &= 0.17609\\
\log \sin PP'B &= 8.98157\\
\text{colog} \sin PBP' &= \underline{1.47566}\\
\log PB &= 0.63332
\end{aligned}$$

$\angle APD = 51° 55'$, $\angle DPC = 137° 35'$, $\angle BPC = 60° 20'$, $\angle APB = 110° 10'$

2 area $PAD = PD \times PA \sin APD$.

$$\begin{aligned}
\log PD &= 0.57385\\
\log PA &= 0.63815\\
\log \sin APD &= \underline{9.89604}\\
\log 2 \text{ area } PAD &= 1.10804\\
2 \text{ area } PAD &= 12.824.
\end{aligned}$$

2 area $PCD = PD \times PC \sin DPC$.

$$\begin{aligned}
\log PD &= 0.57385\\
\log PC &= 0.43229\\
\log \sin DPC &= \underline{9.82899}\\
\log 2 \text{ area } PCD &= 0.83513\\
2 \text{ area } PCD &= 6.8412.
\end{aligned}$$

2 area $PBC = PC \times PB \sin BPC$.

$$\begin{aligned}
\log PC &= 0.43229\\
\log PB &= 0.63332\\
\log \sin BPC &= \underline{9.93898}\\
\log 2 \text{ area } PBC &= 1.00459\\
2 \text{ area } PBC &= 10.106.
\end{aligned}$$

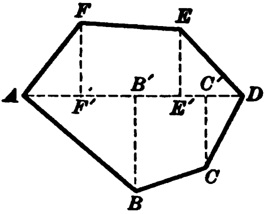
2 area $PAB = PA \times PB \sin APB$.

$$\begin{aligned}
\log PA &= 0.63815\\
\log PB &= 0.63332\\
\log \sin APB &= \underline{9.97252}\\
\log 2 \text{ area } PAB &= 1.24399\\
2 \text{ area } PAB &= 17.538.
\end{aligned}$$

$$\begin{aligned}
2 \triangle PAD &= 12.824\\
2 \triangle PCD &= 6.8412\\
2 \triangle PBC &= 10.106\\
2 \triangle PAB &= \underline{17.538}\\
2\, ABCD &= 47.3092\\
ABCD &= 23.6546.
\end{aligned}$$

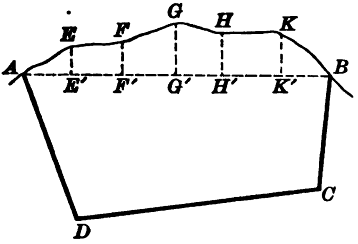
23.6546 sq. ch. = 2.36546 A. = 2 A. 58½ P., nearly.

8. Required the area of the field $ABCDEF$ (Fig. 37), if $AF' = 4$ chains, $FF' = 6$ chains, $EE' = 6.50$ chains, $AE' = 9$ chains, $AD = 14$ chains, $AC' = 10$ chains, $AB' = 6.50$ chains, $BB' = 7$ chains, $CC' = 6.75$ chains.



$$\begin{array}{lll}
2 \text{ area } AFF' & = 4 \times 6 & = 24. \\
2 \text{ area } F'E'EF & = 5(6 + 6.5) & = 62.5 \\
2 \text{ area } EE'D & = 6.5 \times 5 & = 32.5 \\
2 \text{ area } ABB' & = 6.5 \times 7 & = 45.5 \\
2 \text{ area } BCC'B' & = 3.5(7 + 6.75) & = 48.125 \\
2 \text{ area } CDC' & = 6.75 \times 4 & = 27. \\
\hline
2 \text{ area } ABCDEF & & = 239.625 \\
\quad \text{area } ABCDEF & & = 119.8125
\end{array}$$

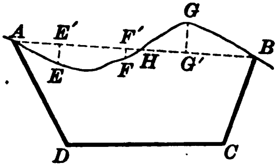
119.8125 sq. ch. = 11.98125 A. = 11 A. 157 P.



9. Required the area of the field $AGBCD$ (Fig. 32, p. 260), if the diagonal $AC = 5$, BB' (the perpendicular from B to AC) = 1, DD' (the perpendicular from D to AC) = 1.60, $EE' = 0.25$, $FF' = 0.25$, $GG' = 0.60$, $HH' = 0.52$, $KK' = 0.54$, $AE' = 0.2$, $E'F' = 0.50$, $F'G' = 0.45$, $G'H' = 0.45$, $H'K' = 0.60$, and $K'B = 0.40$.

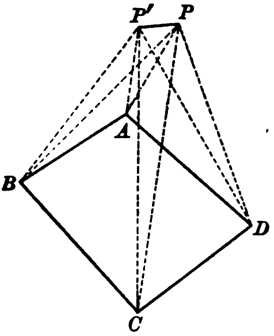
$$\begin{array}{lll}
2 \text{ area } ADCB & = 5(1 + 1.6) & = 13. \\
2 \text{ area } AEE' & = 0.25 \times 0.2 & = 0.05 \\
2 \text{ area } EE'F'F & = 0.5(0.25 + 0.25) & = 0.25 \\
2 \text{ area } FF'G'G & = 0.45(0.25 + 0.6) & = 0.3825 \\
2 \text{ area } GG'H'H & = 0.45(0.6 + 0.52) & = 0.504 \\
2 \text{ area } HH'K'K & = 0.6(0.52 + 0.54) & = 0.636 \\
2 \text{ area } KK'B & = 0.4 \times 0.54 & = 0.216 \\
\hline
2 \text{ area } ADCBKHGFE & & = 15.0385 \\
\quad \text{area } ADCBKHGFE & & = 7.51925.
\end{array}$$

10. Required the area of the field $AGBCD$ (Fig. 33, p. 260), if $AD = 3$, $AC = 5$, $AB = 6$, angle $DAC = 45^\circ$, angle $BAC = 30^\circ$, $AE' = 0.75$, $AF' = 2.25$, $AH = 2.53$, $AG' = 3.15$, $EE' = 0.60$, $FF' = 0.40$, and $GG' = 0.75$.



$$\begin{array}{llll}
2 \text{ area } ADC & = 3 \times 5 \times \sin 45^\circ & = 3 \times 5 \times 0.7071 & = 10.6065 \\
2 \text{ area } ABC & = 5 \times 6 \times \sin 30^\circ & = 5 \times 6 \times 0.5 & = 15. \\
2 \text{ area } HGB & = 0.75 \times (6 - 2.53) & = 0.75 \times 3.47 & = \underline{\ \ 2.6025} \\
\therefore 2 \text{ area } ADCBGHA & & & = 28.209 \\
2 \text{ area } AEE' & = 0.75 \times 0.6 & = \ 0.45 & \\
2 \text{ area } EE'F'F & = 1.5\,(0.6 + 0.4) & = \ 1.5 & \\
2 \text{ area } FF'H & = 0.4 \times 0.28 & = \ \underline{0.112} & \\
\therefore 2 \text{ area } AEFHA & & & = \underline{\ \ 2.062} \\
\therefore 2 \text{ area } AGBCD & & & = 26.147 \\
\text{area } AGBCD & & & = 13.0735
\end{array}$$

11. Determine the area of the field $ABCD$ from two exterior stations P and P', if $PP' = 1.50$ chains,



$P'PB = 41^\circ\ 10'$, $PP'D = 66^\circ\ 45'$,
$P'PA = 55^\circ\ 45'$, $PP'C = 95^\circ\ 40'$,
$P'PC = 77^\circ\ 20'$, $PP'B = 132^\circ\ 15'$,
$P'PD = 104^\circ\ 45'$, $PP'A = 103^\circ\ 0'$.

Area $= (\triangle P'CB + \triangle P'CD) - (\triangle P'AB + \triangle P'AD)$.

$$\angle PBP' = 6^\circ\ 35', \qquad \angle PDP' = 8^\circ\ 30',$$

$$\angle PCP' = 7^\circ\ 0', \qquad \angle PAP' = 21^\circ\ 15'.$$

$$P'B = \frac{PP' \sin P'PB}{\sin PBP'}.$$

$$\begin{aligned} \log PP' &= 0.17609 \\ \log\sin P'PB &= 9.81839 \\ \text{colog}\sin PBP' &= \underline{0.94063} \\ \log P'B &= 0.93511 \end{aligned}$$

$$P'D = \frac{PP' \sin P'PD}{\sin PDP'}.$$

$$\begin{aligned} \log PP' &= 0.17609 \\ \log\sin P'PD &= 9.98545 \\ \text{colog}\sin PDP' &= \underline{0.83030} \\ \log P'D &= 0.99184 \end{aligned}$$

$$P'C = \frac{PP' \sin P'PC}{\sin PCP'}.$$

$$\begin{aligned} \log PP' &= 0.17609 \\ \log\sin P'PC &= 9.98930 \\ \text{colog}\sin PCP' &= \underline{0.91411} \\ \log P'C &= 1.07950 \end{aligned}$$

$$P'A = \frac{PP' \sin P'PA}{\sin PAP'}.$$

$$\begin{aligned} \log PP' &= 0.17609 \\ \log\sin P'PA &= 9.91729 \\ \text{colog}\sin PAP' &= \underline{0.44077} \\ \log P'A &= 0.53415 \end{aligned}$$

$$\angle BP'C = 36^\circ\ 35', \qquad \angle CP'D = 28^\circ\ 55',$$

$$\angle AP'B = 29^\circ\ 15', \qquad \angle AP'D = 36^\circ\ 15'.$$

$$2 \text{ area } P'CB = P'C \times P'B \sin BP'C.$$

$$\begin{aligned} \log P'C &= 1.07950 \\ \log P'B &= 0.93511 \\ \log\sin BP'C &= \underline{9.77524} \\ \log 2 \text{ area } P'CB &= 1.78985 \\ 2 \text{ area } P'CB &= 61.639. \end{aligned}$$

$$2 \text{ area } P'CD = P'C \times P'D \sin CP'D$$

$$\begin{aligned} \log P'C &= 1.07950 \\ \log P'D &= 0.99184 \\ \log\sin CP'D &= \underline{9.68443} \\ \log 2 \text{ area } P'CD &= 1.75577 \\ 2 \text{ area } P'CD &= 56.986. \end{aligned}$$

$$2 \text{ area } P'AB = P'B \times P'A \sin AP'B.$$

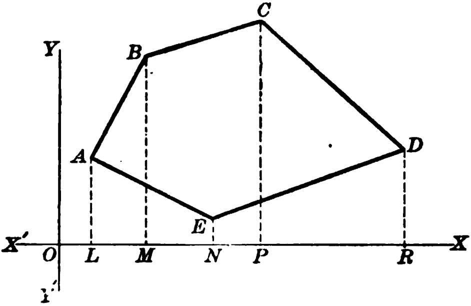
$$\begin{aligned} \log P'B &= 0.93511 \\ \log P'A &= 0.53415 \\ \log\sin AP'B &= \underline{9.68897} \\ \log 2 \text{ area } P'AB &= 1.15823 \\ 2 \text{ area } P'AB &= 14.396. \end{aligned}$$

$$2 \text{ area } P'AD = P'A \times P'D \sin AP'D$$

$$\begin{aligned} \log P'A &= 0.53415 \\ \log P'D &= 0.99184 \\ \log\sin AP'D &= \underline{9.77181} \\ \log 2 \text{ area } P'AD &= 1.29780 \\ 2 \text{ area } P'AD &= 19.852. \end{aligned}$$

$$\begin{aligned} \therefore 2 \text{ area } ABCD &= 61.639 \text{ sq. ch.} + 56.986 \text{ sq. ch.} - (14.396 + 19.852) \text{ sq. ch.} \\ &= 118.625 \text{ sq. ch.} - 34.248 \text{ sq. ch.} \\ &= 84.377 \text{ sq. ch.} \\ \therefore \text{area } ABCD &= 42.1885 \text{ sq. ch.} \\ &= 4.21885 \text{ A.} \\ &= 4 \text{ A. } 35 \text{ P., nearly.} \end{aligned}$$

12. Find the area of the field $ABCDE$ (Fig. 35, p. 262) if the co-ordinates, in chains, of the vertices taken in order are (1.40, 6.75), (4.60, 8.32), (9.00, 9.05), (12.15, 5.58), and (5.27, 1.16).



$$\begin{aligned}
ABCDE &= \tfrac{1}{2}\{x_1(y_5 - y_2) + x_2(y_1 - y_3) + x_3(y_2 - y_4) \\
&\quad + x_4(y_3 - y_5) + x_5(y_4 - y_1)\} \\
&= \tfrac{1}{2}\{1.40(1.16 - 8.32) + 4.60(6.75 - 9.05) \\
&\quad + 9(8.32 - 5.58) + 12.15(9.05 - 1.16) + 5.27(5.58 - 6.75)\} \\
&= \tfrac{1}{2}(-10.0240 - 10.5800 + 24.6600 + 95.8635 - 6.1659) \\
&= \tfrac{1}{2}(120.5235 - 26.7699) \\
&= \tfrac{1}{2} \times 93.7536 \\
&= 46.8768.
\end{aligned}$$

46.8768 sq. ch. = 4.68768 A. = 4 A. 110 P.

13. Find the area of the field $ABCDE$ (Fig. 35, p. 262) by measuring distances as follows: $AL = 400$ feet; $BM = 700$ feet; $CP = 680$ feet; $DR = 380$ feet; $EN = 200$ feet; $LM = 150$ feet; $MN = 250$ feet; $NP = 200$ feet; $PR = 220$ feet.

$$2 \text{ area } ABCDE = 2\,ABML + 2\,BCPM + 2\,CDRP - 2\,AENL - 2\,EDRN.$$

$$\begin{aligned}
2 \text{ area } ABML &= LM(AL + BM) = 150(400 + 700) = 165{,}000. \\
2 \text{ area } BCPM &= MP(BM + CP) = 450(700 + 680) = 621{,}000. \\
2 \text{ area } CDRP &= PR(CP + DR) = 220(680 + 380) = 233{,}200. \\
2 \text{ area } AENL &= LN(AL + EN) = 400(400 + 200) = 240{,}000. \\
2 \text{ area } EDRN &= NR(EN + DR) = 420(200 + 380) = 243{,}600.
\end{aligned}$$

$$\begin{aligned}
\therefore 2 \text{ area } ABCDE &= (165{,}000 + 621{,}000 + 233{,}200 - 240{,}000 - 243{,}600) \text{ sq. ft.} \\
&= (1{,}019{,}200 - 483{,}600) \text{ sq. ft.} \\
&= 535{,}600 \text{ sq. ft.} \\
\therefore \text{ area } ABCDE &= 267{,}800 \text{ sq. ft.} \\
&= \tfrac{267800}{43560} \text{ A.} \\
&= 6.148 \text{ A.} \\
&= 6 \text{ A. } 23\tfrac{17}{25} \text{ P.}
\end{aligned}$$

EXERCISE VI. PAGE 272.

1.

STATION.	BEARING.	DIST.	N.	S.	E.	W.	M.D.	D.M.D.	N.A.	S.A.
1	S. 75° E.	6.00	. . .	1.55	5.79 ~~5.80~~	. . .	5.79	5.79	. . . . .	8.9745
2	S. 15° E.	4.00	. . .	3.86	1.04	. . .	6.83	12.62	. . . . .	48.7132
3	S. 75° W.	6.93	. . .	1.80 ~~1.79~~	. . .	6.70 ~~6.69~~	0.13	6.96	. . . . .	12.5280
4	N. 45° E.	5.00	3.54	. . .	3.54	. . .	3.67	3.80	13.4520	. . . . .
5	N. 45° W.	5.19½	3.67	. . .	. . .	3.67	0.00	3.67	13.4689	. . . . .
			7.21	7.21	10.37	10.37			26.9209	70.2157
										26.9209
	21.647 sq. ch. = 2.1647 A. = 2 A. 26 P., nearly.									43.2948
										21.6474

2.

STATION.	BEARING.	DIST.	N.	S.	E.	W.	M.D.	D.M.D.	N.A.	S.A.
1	N. 45° E.	10.00	7.07	. . .	7.07	. . .	7.07	7.07	49.9849	. . . . .
2	S. 75° E.	11.55	. . .	2.99	11.16	. . .	18.23	25.30	. . . . .	75.6470
3	S. 15° W.	18.21	. . .	17.59	. . .	4.71	13.52	31.75	. . . . .	558.4825
4	N. 45° W.	19.11	13.51	. . .	. . .	13.52 ~~13.51~~	0.00	13.52	182.6552	. . . . .
			20.58	20.58	18.23	18.23			232.6401	634.1295
										232.6401
	200.74 sq. ch. = 20.074 A. = 20 A. 12 P., nearly.									401.4894
										200.7447

3

BEARING.	DIST.	N.	S.	E.	W.
N. 51° 45′ W.	2.39	1.48	. . .	. . .	1.88
S. 85° W.	6.47	. . .	0.56	. . .	6.45
S. 55° 10′ W.	1.62	. . .	0.93	. . .	1.33
			1.49	. . .	9.66
			1.48	. . .	. . .
			0.01		

STATION.	BEARING.	DIST.	N.	S.	E.	W.	M.D.	D.M.D.	N.A.	S.A.
2	S. W.	. . .	. . .	0.03 ~~0.01~~	. . .	9.65 ~~9.66~~	9.65	9.65	. . . . .	0.2895
3	N. 3° 45′ E.	6.39	6.36 ~~6.38~~	. . .	0.43 ~~0.42~~	. . .	9.22	18.87	120.0132	. . . . .
4	S. 66° 45′ E.	1.70	. . .	0.67	1.56	. . .	7.66	16.88	. . . . .	11.3096
5	N. 15° E.	4.98	4.80 ~~4.81~~	. . .	1.29	. . .	6.37	14.03	67.3440	. . . . .
6	S. 82° 45′ E.	6 03	. . .	0.77 ~~0.76~~	5.98	. . .	0.39	6.76	. . . . .	5.2052
1	S. 2° 15′ E.	9.68	. . .	9.69 ~~9.67~~	0.39 ~~0.38~~	. . .	0.00	0.39	. . . . .	3.7791
			11.16	11.16	9.65	9.65			187.3572	20 5834
									20.5834	
	83.39 sq. ch. = 8.339 A. = 8 A. 54 P., nearly.								166.7738	
									83.3869	

4.

Station.	Bearing.	Dist.	N.	S.	E.	W.	M.D.	D.M.D.	N.A.	S.A.
3	S. 3° 00′ E.	5.33	. . .	5.29 ~~5.32~~	0.28	. . .	0.28	0.28	. . . . .	1.4812
4	E.	6.72	0.03 ~~0.00~~	. . .	6.73 ~~6.72~~	. . .	7.01	7.29	0.2187	. . . . .
1	N. 5° 30′ W.	6.08	6.08 ~~6.05~~	. . .	. . .	0.57 ~~0.58~~	6.44	13.45	81.7760	. . . . .
2	S. 82° 30′ W.	6.51	. . .	0.82 ~~0.85~~	. . .	6.44 ~~6.45~~	0.00	6.44	. . . . .	5.2808
			6.11	6.11	7.01	7.01			81.9947	6.7620
									6.7620	
37.61 sq. ch. = 3.761 A. = 3 A. 122 P., nearly.									75.2327	
									37.6163	

5.

Station.	Bearing.	Dist.	N.	S.	E.	W.	M.D.	D.M.D.	N.A.	S.A.
3	S. 5° 00′ E.	5.86	. . .	5.83 ~~5.84~~	0.53 ~~0.51~~	. . .	0.53	0.53	. . . . .	3.0899
4	N. 88° 30′ E.	4.12	0.12 ~~0.11~~	. . .	4.14 ~~4.12~~	. . .	4.67	5.20	0.6240	. . . . .
1	N. 6° 15′ W.	6.31	6.28 ~~6.27~~	. . .	. . .	0.67 ~~0.69~~	4.00	8.67	54.4476	. . . . .
2	S. 81° 50′ W.	4.06	. . .	0.57 ~~0.58~~	. . .	4.00 ~~4.02~~	0.00	4.00	. . . . .	2.2800
			6.40	6.40	4.67	4.67			55.0716	5.3699
									5.3699	
24.85 sq. ch. = 2.485 A. = 2 A. 78 P., nearly.									49.7017	
									24.8508	

6.

STATION.	BEARING.	DIST.	N.	S.	E.	W.	M.D.	D.M.D.	N.A.	S.A.
1	N. 20° 00′ E.	4.62¼	4.35	. . .	1.58	. . .	1.58	1.58	6.8730	. . . . .
2	N. 73° 00′ E.	4.16¼	1.22	. . .	3.98	. . .	5.56	7.14	8.7108	. . . . .
3	S. 45° 15′ E.	6.18¼	. . .	4.35	4.39	. . .	9.95	15.51	. . . . .	67.4685
4	S. 38° 30′ W.	8.00	. . .	6.26	. . .	4.98	4.97	14.92	. . . . .	93.3992
5	Wanting.	. . .	5.04	. . .	. . .	4.97	0 00	4.97	25.0488	. . . . .
			10.61	10.61	9.95	9.95			40.6326	160.8677
										40.6326
										120.2351
										60.1175

60.12 sq. ch = 6.012 A. = 6 A. 2 R., nearly.

7.

BEARING.	DIST.	N.	S.	E.	W.
S. 81° 20′ W.	4.28	. . .	0.65	. . .	4.23
N. 76° 30′ W.	2.67	0.62	. . .	. . .	2.60
			0.65	. . .	6.83
			0.62	. . .	. . .
			0.03		

BEARING.	DIST.	N.	S.	E.	W.
S. 7° E.	1.79	. . .	1.78	0.22	. . .
S. 27° E.	1.94	. . .	1.73	0.88	. . .
S. 10° 30′ E.	5.35	. . .	5.26	0.97	. . .
N. 76° 45′ W.	1.70	0.39	. . .	. . .	1.65
			8.77	2.07	. . .
			0 39	1.65	. . .
			8.38	0.42	

STATION.	BEARING.	DIST.	N.	S.	E.	W.	M.D.	D.M.D.	N.A.	S.A.
1	S. W.	. . .	. . .	0.03	. . .	6.80 6.83	6.80	6.80	. . . . .	0.2040
2	N. 5° E.	8.68	8.65	. . .	0.79 0.76	. . .	6.01	12.81	110.8065	. . . . .
3	S. 87° 30′ E.	5.54	. . .	0.24	5.56 5.54	. . .	0.45	6.46	. . . . .	1.5504
4	S. E.	. . .	. . .	8.38	0.45 0.42	. . .	0.00	0.45	. . . . .	3.7710
			8.65	8.65	6.80	6.80			110.8065	5.5254
									5.5254	
52.64 sq. ch. = 5.264 A. = 5 A. 42 P., nearly.									105.2811	
									52.6405	

8.

STATION.	BEARING.	DIST.	N.	S.	E.	W.	M.D.	D.M.D.	N.A.	S.A.
1	N. 89° 45′ E.	4.94	0.00 0 02	. . .	4.93 4.94	. . .	4.93	4.93	. . . . .	. . . . .
2	S. 7° 00′ W.	2.30	. . .	2.29 2.28	. . .	0.29 0.28	4.64	9.57	. . . . .	21.9153
3	S. 28° 00′ E.	1.52	. . .	1.34	0.71	. . .	5.35	9.99	. . . . .	13.3866
4	S. 0° 45′ E.	2.57	. . .	2.58 2.57	0.02 0.03	. . .	5.37	10.72	. . . . .	27.6576
5	N. 84° 45′ W.	5.11	0.45 0.47	. . .	. . .	5.10 5.09	0.27	5.64	2.5380	. . . . .
6	N. 2° 30′ W.	5.79	5.76 5.78	. . .	. . .	0.27 0.25	0.00	0.27	1 5552	. . . . .
			6.21	6.21	5.66	5.66			4.0932	62.9595
										4.0932
29.43 sq. ch. = 2.943 A. = 2 A. 151 P., nearly.										58.8663
										29.4332

9. An Ohio farm is bounded and described as follows: Beginning at the southwest corner of lot No. 13, thence N. $1\frac{1}{4}$° E. 132 rods and 23 links to a stake in the west boundary line of said lot; thence S. 89° E. 32 rods and $15\frac{4}{10}$ links to a stake; thence N. $1\frac{1}{4}$° E. 29 rods and 15 links to a stake in the north boundary line of said lot; thence S. 89° E. 61 rods and $18\frac{6}{10}$ links to a stake; thence S. $32\frac{1}{2}$° W. 54 rods to a stake; thence S. $35\frac{1}{4}$° E. 22 rods and 4 links to a stake; thence S. 48° E. 33 rods and 2 links to a stake; thence S. $7\frac{1}{2}$° W. 76 rods and 20 links to a stake in the south boundary line of said lot; thence N. 89° W. 96 rods and 10 links to the place of beginning; containing 85.87 acres more or less.

Verify the area given and plot the farm.

STATION.	BEARING.	DIST.	N.	S.	E.	W.	M.D.	D.M.D.	N.A.	S.A.
1	N. 1° 15′ E.	33.23	33.18 33.22	. . .	0.73 0.72	. . .	0.73	0.73	24.2214	. . . . .
2	S. 89° E.	8.154	. . .	0.15 0.14	8.15	. . .	8.88	9.61	. . . . .	1.4415
3	N. 1° 15′ E.	7.40	7.39 7.40	. . .	0.16	. . .	9.04	17.92	132.4288	. . . . .
4	S. 89° E.	15.436	. . .	0.29 0.27	15.44 15.43	. . .	24.48	33.52	. . . . .	9.7208
5	S. 32° 30′ W.	13.50	. . .	11.40 11.39	. . .	7.24 7.25	17.24	41.72	. . . . .	475.6080
6	S. 35° 15′ E.	5.54	. . .	4.53 4.52	3.20	. . .	20.44	37.68	. . . . .	170.6904
7	S. 48° E.	8.27	. . .	5.54 5.53	6.15	. . .	26.59	47.03	. . . . .	260.5462
8	S. 7° 30′ W.	19.20	. . .	19.05 19.03	. . .	2.50 2.51	24.09	50.68	. . . . .	965.4540
9	N. 89° W.	24.10	0.39 0.42	. . .	. . .	24.09 24.10	0.00	24.09	9.3951	. . . . .
			40.96	40.96	33 83	33.83			166.0453	1883.4609
										166.0453
858.7 sq. ch. = 85.87 A.										1717.4156
										858.7078

EXERCISE VII. PAGE 274.

1. Find the area of the field $ABCD$, in which the angle $A = 120°$, $B = 60°$, $C = 150°$, and $D = 30°$; and the side $AB = 4$ chains, $BC = 4$ chains, $CD = 6.928$ chains, and $DA = 8$ chains.

SIDE.	BEARING.	DIST.	N.	S.	E.	W.	M.D.	D.M.D.	N.A.	S.A.
AB	N.	4.000	4.000	. . .	. . .	. . .	0.00	0.00	. . . .	. . . .
BC	S. 60° E.	4.000	. . .	2.000	3.464	. . .	3.464	3.464	. . . .	6.928
CD	S. 30° E.	6.928	. . .	6.000	3.464	. . .	6.928	10.392	. . . .	62.352
DA	N. 60° W.	8.000	4.000	. . .	. . .	6.928	0.00	6.928	27.712	. . . .
			8.000	8.000	6.928	6.928			27.712	69.280
										27.712
20.784 sq. ch. = 2.0784 A. = 2 A. 12½ P., nearly.										41.568
										20.784

2. Find the area of the farm *ABCDE*, in which the angle $A = 106° 19'$, $B = 99° 40'$, $C = 120° 20'$, $D = 86° 8'$, and $E = 127° 33'$; and the side $AB = 79.86$ rods, $BC = 121.13$ rods, $CD = 90$ rods, $DE = 100.65$ rods, and $EA = 100$ rods.

SIDE.	BEARING.	DIST.	N.	S.	E.	W.	M.D.	D.M.D.	N.A.	S.A.
AB	N.	79.86	79.86	. . .	. . .	. . .	0.00	0.00	. . . .	. . . .
BC	N. 80° 20′ E.	121.13	20.34	. . .	119.41	. . .	119.41	119.41	2428.80	. . . .
CD	S. 40° 00′ E.	90.00	. . .	68.94	57.85	. . .	177.26	296.67	. . . .	20452.43
DE	S. 53° 52′ W.	100.65	. . .	59.35	. . .	81.29	95.97	273.23	. . . .	16216.20
EA	N. 73° 41′ W.	100.00	28.09	. . .	. . .	95.97	0.00	95.97	2695.80	. . . .
			128.09	128.09	177.26	177.26			5124 60	36668.63
										5124.60
15772.02 P. = 98 A. 92 P., nearly.										31544.03
										15772.02

EXERCISE VIII. PAGE 277.

1. From the square $ABCD$, containing 6 acres 1 rood 24 perches, part off 3 acres by a line EF parallel to AB.

$$\begin{aligned}&6\text{ A. }1\text{ R. }24\text{ P.} = 64\text{ sq. ch.};\ \sqrt{64}\text{ ch.} = 8\text{ ch.} = AB.\\&3\text{ A.} = 30\text{ sq. ch.}\\&AE = \frac{ABFE}{AB} = \frac{30\text{ sq. ch.}}{8\text{ ch.}} = 3.75\text{ ch.}\end{aligned}$$

2. From the rectangle $ABCD$, containing 8 acres 1 rood 24 perches, part off 2 acres 1 rood 32 perches by a line EF parallel to AD which is equal to 7 chains. Then, from the remainder of the rectangle part off 2 acres 3 roods 25 perches by a line GH parallel to EB.

$$\begin{aligned}&8\text{ A. }1\text{ R. }24\text{ P.} = 84\text{ sq. ch.} = ABCD.\\&2\text{ A. }1\text{ R. }32\text{ P.} = 24.5\text{ sq. ch.} = AEFD.\\&2\text{ A. }3\text{ R. }25\text{ P.} = 29.0625\text{ sq. ch.} = EBHG.\\&AE = \frac{AEFD}{AD} = \frac{24.5\text{ sq. ch.}}{7\text{ ch.}} = 3.5\text{ ch.}\\&AB = \frac{ABCD}{AD} = \frac{84\text{ sq. ch.}}{7\text{ ch.}} = 12\text{ ch.}\\&EB = AB - AE = 12\text{ ch.} - 3.5\text{ ch.} = 8.5\text{ ch.}\\&EG = \frac{EBHG}{EB} = \frac{29.0625\text{ sq. ch.}}{8.5\text{ ch.}} = 3.42\text{ ch., nearly.}\end{aligned}$$

3. Part off 6 acres 3 roods 12 perches from a rectangle $ABCD$, containing 15 acres, by a line EF parallel to AB; AD being 10 chains.

$$\begin{aligned}&6\text{ A. }3\text{ R. }12\text{ P.} = 68.25\text{ sq. ch.} = ABFE.\\&15\text{ A.} = 150\text{ sq. ch.} = ABCD.\\&AB = \frac{ABCD}{AD} = \frac{150\text{ sq. ch.}}{10\text{ ch.}} = 15\text{ ch.}\\&AE = \frac{ABFE}{AB} = \frac{68.25\text{ sq. ch.}}{15\text{ ch.}} = 4.55\text{ ch.}\end{aligned}$$

4. From a square $ABCD$, whose side is 9 chains, part off a triangle which shall contain 2 acres 1 rood 36 perches, by a line BE drawn from B to the side AD.

$$\begin{aligned}&2\text{ A. }1\text{ R. }36\text{ P.} = 24.75\text{ sq. ch.} = ABE.\\&AE = \frac{2\,ABE}{AB} = \frac{2 \times 24.75\text{ sq. ch.}}{9\text{ ch.}} = 5.50\text{ ch.}\end{aligned}$$

5. From $ABCD$ representing the rectangle, whose length is 12.65 chains, and breadth 7.58 chains, part off a trapezoid which shall contain 7 acres 3 roods 24 perches, by a line BE drawn from B to the side DC.

$$7\text{ A. }3\text{ R. }24\text{ P.} = 79\text{ sq. ch.} = ABED.$$

$$ABCD = (12.65 \times 7.58)\text{ sq. ch.} = 95.887\text{ sq. ch.}$$

$$\triangle BCE = 95.887\text{ sq. ch.} - 79\text{ sq. ch.} = 16.887\text{ sq. ch.}$$

$$CE = \frac{2\,BCE}{BC} = \frac{2 \times 16.887\text{ sq. ch.}}{7.58\text{ ch.}} = 4.456\text{ ch., nearly.}$$

6. In the triangle ABC, $AB = 12$ chains, $AC = 10$ chains, and $BC = 8$ chains; part off a trapezoid of 1 acre 2 roods 16 perches, by the line DE parallel to AB.

$$1\text{ A. }2\text{ R. }16\text{ P.} = 16\text{ sq. ch.} = ABED.$$

$$CAB = \sqrt{15 \times 3 \times 5 \times 7}\text{ sq. ch.} = 39.6863\text{ sq. ch.}$$

$$CDE = CAB - ABED = 39.6863\text{ sq. ch.} - 16\text{ sq. ch.} = 23.6863\text{ sq. ch.}$$

$$CAB : CDE = \overline{CA}^2 : \overline{CD}^2 = \overline{CB}^2 : \overline{CE}^2.$$

$$39.6863 : 23.6863 = 10^2 : \overline{CD}^2 = 8^2 : \overline{CE}^2.$$

$$\therefore CD = 7.725\text{ ch. and } CE = 6.18\text{ ch.}$$

$$AD = CA - CD = 10\text{ ch.} - 7.725\text{ ch.} = 2.275\text{ ch.}$$

$$BE = CB - CE = 8\text{ ch.} - 6.18\text{ ch.} = 1.82\text{ ch.}$$

7. In the triangle ABC, $AB = 26$ chains, $AC = 20$ chains, and $BC = 16$ chains; part off a trapezoid of 6 acres 1 rood 24 perches, by the line DE parallel to AB.

$$6\text{ A. }1\text{ R. }24\text{ P.} = 64\text{ sq. ch.} = ABED.$$

$$CAB = \sqrt{31 \times 5 \times 11 \times 15}\text{ sq. ch.} = 159.9218\text{ sq. ch.}$$

$$CDE = CAB - ABED = 159.9218\text{ sq. ch.} - 64\text{ sq. ch.} = 95.9218\text{ sq. ch.}$$

$$CAB : CDE = \overline{CA}^2 : \overline{CD}^2 = \overline{CB}^2 : \overline{CE}^2.$$

$$159.9218 : 95.9218 = 20^2 : \overline{CD}^2 = 16^2 : \overline{CE}^2.$$

$$\therefore CD = 15.49\text{ ch., and } CE = 12.39\text{ ch.}$$

$$AD = CA - CD = 20\text{ ch.} - 15.49\text{ ch.} = 4.51\text{ ch., nearly.}$$

$$BE = CB - CE = 16\text{ ch.} - 12.39\text{ ch.} = 3.61\text{ ch., nearly.}$$

8. It is required to divide the triangular field ABC among three persons whose claims are as the numbers 2, 3, and 5, so that they may all have the use of a watering place at C; $AB = 10$ chains, $AC = 6.85$ chains, and $CB = 6.10$ chains.

Since the triangles have the same altitude, they are to each other as their bases. Hence, it is necessary only to divide the base 10 into the three parts, 2 chains, 3 chains, 5 chains.

9. Divide the five-sided field $ABCHE$ among three persons, X, Y, and Z, in proportion to their claims, X paying $500, Y paying $750, and Z paying $1000, so that each may have the use of an interior pond at P, the quality of the land being equal throughout. Given $AB = 8.64$ chains, $BC = 8.27$ chains, $CH = 8.06$ chains, $HE = 6.82$ chains, and $EA = 9.90$ chains. The perpendicular PD upon $AB = 5.60$ chains, PD' upon $BC = 6.08$ chains, PD'' upon $CH = 4.80$ chains, PD''' upon $HE = 5.44$ chains, and PD'''' upon $EA = 5.40$ chains. Assume PH as the divisional fence between the shares of X and Z; it is required to determine the position of the fences PM and PN between the shares of X and Y, and between the shares of Y and Z.

If P is joined to the vertices, the field is divided into triangles, whose bases are the sides, and the altitudes the given perpendiculars upon the sides from P.

$$\begin{array}{lrl}
APB = 8.64 \times 2.80 = & 24.1920 & \text{sq. ch.} \\
BPC = 8.27 \times 3.04 = & 25.1408 & \\
CPH = 8.06 \times 2.40 = & 19.3440 & \\
HPE = 6.82 \times 2.72 = & 18.5504 & \\
\underline{EPA = 9.90 \times 2.70 =} & \underline{26.7300} & \\
ABCHE \qquad\qquad = & 113.9572 & \text{sq. ch.}
\end{array}$$

The whole area, 113.9572 sq. ch., must be divided as the numbers 500, 750, 1000, or as 2, 3, 4. $2 + 3 + 4 = 9$.

X's share $= \frac{2}{9}$ of 113.9572 sq. ch. $= 25.3238$ sq. ch.
Y's share $= \frac{3}{9}$ of 113.9572 sq. ch. $= 37.9857$ sq. ch.
Z's share $= \frac{4}{9}$ of 113.9572 sq. ch. $= 50.6477$ sq. ch.

PH is assumed as the line between X's and Z's shares. Since the triangle PHE is less than X's share by 25.3238 sq. ch. $-$ 18.5504 sq. ch. $=$ 6.7734 sq. ch.; this difference must be taken from the triangle PEA. The area of PEM is then 6.7734 sq. ch., and the altitude $PD'''' = 5.40$ ch.

$$\therefore EM = \frac{2\,PEM}{PD''''} = \frac{2 \times 6.7734 \text{ sq. ch.}}{5.40 \text{ ch.}} = 2.5087 \text{ ch.}$$

$$\begin{aligned} PMA &= PEA - PEM = 26.7300 \text{ sq. ch.} - 6.7734 \text{ sq. ch.} \\ &= 19.9566 \text{ sq. ch.} \end{aligned}$$

Since Y's share is greater than PMA (19.9566 sq. ch.) and less than $PMA + PAB$ (44.1486 sq. ch.), the point N is on AB.

Y's share diminished by PMA equals PAN; that is,

$$PAN = 37.9857 \text{ sq. ch.} - 19.9566 \text{ sq. ch.} = 18.0291 \text{ sq. ch.}$$

$$AN = \frac{2\,PAN}{PD} = \frac{2 \times 18.0291 \text{ sq. ch.}}{5.60 \text{ ch.}} = 6.4390 \text{ ch.}$$

10. Divide the triangular field ABC, whose sides AB, AC, and BC are 15, 12, and 10 chains, respectively, into three equal parts, by fences EG and DF parallel to BC, without finding the area of the field.

$$\frac{\Delta ABC}{\Delta AEG} = \frac{3}{2} = \frac{\overline{AB}^2}{\overline{AE}^2} = \frac{\overline{AC}^2}{\overline{AG}^2}.$$

$$\therefore \frac{3}{2} = \frac{225}{\overline{AE}^2}. \quad \therefore \overline{AE}^2 = 150. \quad AE = \sqrt{150} \text{ ch.} = 12.247 \text{ ch.}$$

$$\frac{3}{2} = \frac{144}{\overline{AG}^2}. \quad \therefore \overline{AG}^2 = 96. \quad AG = \sqrt{96} \text{ ch.} = 9.798 \text{ ch.}$$

$$\frac{\Delta ABC}{\Delta ADF} = \frac{3}{1} = \frac{\overline{AB}^2}{\overline{AD}^2} = \frac{\overline{AC}^2}{\overline{AF}^2}.$$

$$\therefore \frac{3}{1} = \frac{225}{\overline{AD}^2}. \quad \therefore \overline{AD}^2 = 75. \quad AD = \sqrt{75} \text{ ch.} = 8.660 \text{ ch.}$$

$$\frac{3}{1} = \frac{144}{\overline{AF}^2}. \quad \therefore \overline{AF}^2 = 48. \quad AF = \sqrt{48} \text{ ch.} = 6.928 \text{ ch.}$$

11. Divide the triangular field ABC, whose sides AB, BC, and AC are 22, 17, and 15 chains, respectively, among three persons, A, B, and C, by fences parallel to the base AB, so that A may have 3 acres above the line AB, B 4 acres above A's share, and C the remainder.

$$CAB = \sqrt{27 \times 5 \times 10 \times 12} \text{ sq. ch.} = 127.2792 \text{ sq. ch.}$$

$$CDG = CAB - ABGD = 127.2792 \text{ sq. ch.} - 30 \text{ sq. ch.} = 97.2792 \text{ sq. ch.}$$

$$CEF = CAB - ABFE = 127.2792 \text{ sq. ch.} - 70 \text{ sq. ch.} = 57.2792 \text{ sq. ch.}$$

$$CAB : CDG = \overline{CB}^2 : \overline{CG}^2 = \overline{CA}^2 : \overline{CD}^2.$$

$$127.2792 : 97.2792 = 17^2 : \overline{CG}^2 = 15^2 : \overline{CD}^2.$$

$$\therefore CG = 14.862 \text{ ch. and } CD = 13.113 \text{ ch.}$$

$$CAB : CEF = \overline{CB}^2 : \overline{CF}^2 = \overline{CA}^2 : \overline{CE}^2.$$

$$127.2792 : 57.2792 = 17^2 : \overline{CF}^2 = 15^2 : \overline{CE}^2.$$

$$\therefore CF = 11.404 \text{ ch. and } CE = 10.062 \text{ ch.}$$

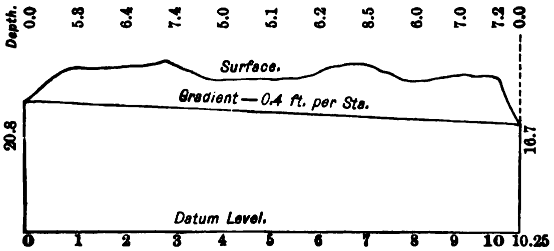
Exercise IX. Page 295.

1. Find the difference of level of two places from the following field notes: backsights, 5.2, 6.8, and 4.0; foresights, 8.1, 9.5, and 7.9.

$$\begin{array}{lr} 8.1 + 9.5 + 7.9 = & 25.5 \\ 5.2 + 6.8 + 4 \quad = & \underline{16.} \\ & 9.5 \end{array}$$

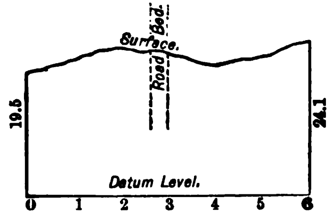
2. Stake 0 of the following notes stands at the lowest point of a pond to be drained into a creek; stake 10 stands at the edge of the bank, and 10.25 at the bottom of the creek. Make a profile, draw the grade line through 0 and 10.25, and fill out the columns H.G. and C., the former to show the height of grade line above the datum, and the latter, the depth of cut at the several stakes necessary to construct the drain.

STATION.	+S.	H.I.	−S.	H.S.	H.G.	C.	REMARKS.
B	6.000	....	....	25	....	....	Bench on rock 30 ft. west of stake 1.
0	....	....	10.2	....	20.8	0.0	
1	....	....	5.3	....	20.4	5.3	
2	....	....	4.6	....	20.0	6.4	
3	....	....	4.0	....	19.6	7.4	
4	....	....	6.8	....	19.2	5.0	
5	4.572	....	7.090	....	18.8	5.1	
6	....	....	3.9	....	18.4	6.2	
7	....	....	2.0	....	18.0	8.5	
8	....	....	4.9	....	17.6	6.0	
9	....	....	4.3	....	17.2	7.0	
10	....	....	4.5	....	16.8	7.2	
10.25	....	....	11.8	....	16.7	0.0	



5. Write the proper numbers in the third and fifth columns of the following table of field notes, and make a profile of the section.

STATION.	+ S.	H.I.	− S.	H.S.	REMARKS.
B	6.944	....	....	20.0	Bench on post 22 ft. north of 0.
0	....	26.944	7.4	19.5	
1	....	....	5.6	21.3	
2	....	....	3.9	23.0	
3	....	....	4.6	22.3	
t. p.	3.855	....	5.513	21.431	
4	....	25.286	4.9	20.4	
5	....	....	3.5	21.8	
6	....	....	1.2	24.1	



EXERCISE X. PAGE 301.

1. The cross-section areas at five stations, 100 feet apart, of a railroad cut are, respectively, 576.8 square feet, 695.1 square feet, 809.5 square feet, 652.0 square feet, and 511.7 square feet. Compute the volume of material in this portion of the cut: (i) on the hypothesis that the cross sections are similar; (ii) on the hypothesis that they are dissimilar, the alternate cross sections being regarded as mid sections.

$$\text{(i)}\quad V = \tfrac{1}{3} H(B + b + \sqrt{Bb}).$$

$$V_1 = \tfrac{1}{3} \times 100\,(576.8 + 695.1 + \sqrt{576.8 \times 695.1})$$

$$= \tfrac{100}{3}(576.8 + 695.1 + 633.2)$$

$$= \tfrac{100}{3} \times 1905.1$$

$$= 63503.3.$$

$$V_2 = \tfrac{1}{3} \times 100\,(695.1 + 809.5 + \sqrt{695.1 \times 809.5})$$

$$= \tfrac{1}{3} \times 100\,(695.1 + 809.5 + 750.1)$$

$$= \tfrac{100}{3} \times 2254.7$$

$$= 75156.7.$$

$$\begin{aligned}
V_3 &= \tfrac{1}{3} \times 100\,(809.5 + 652.0 + \sqrt{809.5 \times 652.0})\\
&= \tfrac{1}{3} \times 100\,(809.5 + 652.0 + 726.5)\\
&= \tfrac{100}{3} \times 2188\\
&= 72933.3.\\
V_4 &= \tfrac{1}{3} \times 100\,(652.0 + 511.7 + \sqrt{652.0 \times 511.7})\\
&= \tfrac{1}{3} \times 100\,(652.0 + 511.7 + 577.6)\\
&= \tfrac{100}{3} \times 1741.3\\
&= 58043.3.\\
V &= (63503.3 + 75156.7 + 72933.3 + 58043.3)\text{ cu. ft.}\\
&= 269636.6\text{ cu. ft.}\\
&= \frac{269636.6}{27}\text{ cu. yd.}\\
&= 9986.5\text{ cu. yd.}
\end{aligned}$$

(ii) $V = \tfrac{1}{6}H(B + b + 4\,M).$

$$\begin{aligned}
V_1 &= \tfrac{1}{6} \times 200\,(576.8 + 809.5 + 4 \times 695.1)\\
&= \tfrac{100}{3}(576.8 + 809.5 + 2780.4)\\
&= \tfrac{100}{3} \times 4166.7\\
&= 138890.\\
V_2 &= \tfrac{1}{6} \times 200\,(809.5 + 511.7 + 4 \times 652)\\
&= \tfrac{100}{3}(809.5 + 511.7 + 2608)\\
&= \tfrac{100}{3} \times 3929.2\\
&= 130973.3.\\
V &= (138890 + 130973.3)\text{ cu. ft.} = 269863.3\text{ cu. ft.} = 9994.9\text{ cu. yd.}
\end{aligned}$$

2. Find the radius of a curve of 1°, of 2°, of 3°, of 4°, of 5°.

$$r = 50 \csc \tfrac{1}{2} D.$$

(i) $r = 50 \csc 0° \, 30'.$

$$\begin{aligned}
\log \csc 0° \, 30' &= 2.05916\\
\log 50 &= \underline{1.69897}\\
\log r &= 3.75813\\
r &= 5729.7\text{ ft.}
\end{aligned}$$

(ii) $r = 50 \csc 1°.$

$$\begin{aligned}
\log \csc 1° &= 1.75814\\
\log 50 &= \underline{1.69897}\\
\log r &= 3.45711\\
r &= 2864.9\text{ ft.}
\end{aligned}$$

(iii) $r = 50 \csc 1° \, 30'.$

$$\begin{aligned}
\log \csc 1° \, 30' &= 1.58208\\
\log 50 &= \underline{1.69897}\\
\log r &= 3.28105\\
r &= 1910.1\text{ ft.}
\end{aligned}$$

(iv) $r = 50 \csc 2°.$

$$\begin{aligned}
\log \csc 2° &= 1.45718\\
\log 50 &= \underline{1.69897}\\
\log r &= 3.15615\\
r &= 1432.7\text{ ft.}
\end{aligned}$$

(v) $r = 50 \csc 2° \, 30'.$

$$\begin{aligned}
\log \csc 2° \, 30' &= 1.36032\\
\log 50 &= \underline{1.69897}\\
\log r &= 3.05929\\
r &= 1146.3\text{ ft.}
\end{aligned}$$

3. Two adjacent straight sections of a railroad form an angle of 148° $16'$. They are joined by a curve touching each of them at the distance of 388 feet from the vertical point. Find the radius and the degree of the curve.

$$\begin{aligned} r &= t \tan \tfrac{1}{2} A. \\ \text{Here} \qquad A &= 148^\circ\ 16'. \\ \therefore \tfrac{1}{2} A &= 74^\circ\ 8'. \\ t &= 388 \text{ ft.} \\ \log 388 &= 2.58883 \\ \log \tan 74^\circ\ 8' &= \underline{10.54633} \\ \log r &= 3.13516 \\ r &= 1365.1 \text{ ft.} \end{aligned}$$

$$\begin{aligned} \sin \tfrac{1}{2} D &= \frac{50}{t} \cot \tfrac{1}{2} A. \\ \log 50 &= 1.69897 \\ \operatorname{colog} 388 &= 7.41117 - 10 \\ \log \cot 74^\circ\ 8' &= \underline{9.45367 \qquad\;\;} \\ \log \sin \tfrac{1}{2} D &= 8.56381 \\ \tfrac{1}{2} D &= 2^\circ\ 5'\ 56.7'' \\ D &= 4^\circ\ 11'\ 53''. \end{aligned}$$

NAVIGATION.

EXERCISE I. PAGE 319.

1. Given compass course S., wind E.S.E., leeway 1¼ points, variation 52° 0′ W., deviation 2° 0′ E.; required true course.

Since the wind is E.S.E., and the compass course is S., the ship is on the port tack; hence, leeway is allowed to the right.

Compass course	0 pts. R. of S.
Leeway	1¼ pts. R.
Compass course corrected for leeway	1¼ pts. R. of S.
	= 14° 3′ 45″ R. of S.
Variation and deviation (52° − 2°) W.	= 50° 0′ 0″ L.
	35° 56′ 15″ L. of S.

True course, S. 35° 56′ E.

2. Given compass course W.N.W., wind N., leeway 3 points, variation 42° 0′ E., deviation 18° 30′ W.; required true course.

Since the wind is N., and the compass course is W.N.W., the ship is on the starboard tack; hence, leeway is allowed to the left.

Compass course	6 pts. L. of N.
Leeway	3 pts. L.
Compass course corrected for leeway	9 pts. L. of N.
	= 7 pts. R. of S.
	= 78° 45′ R. of S.
Variation and deviation (42° − 18° 30′) E.	= 23° 30′ R.
	102° 15′ R. of S.
	77° 45′ L. of N.

True course, N. 77° 45′ W.

3. Given compass course S.S.E. ½ E., wind S.W. ½ S., leeway 3½ points, variation 2½ points E., deviation 1½ points W.; required true course.

Compass course	2½ pts. L. of S.
Leeway (starboard tack)	3½ pts. L.
Variation and deviation	1 pt. R.
	5 pts. L. of S.

True course, S.E. by E.

4. Given true course S. 79° W., wind S. by W., leeway ¾ point, variation 10° 30′ E., deviation 19° 0′ W.; required compass course.

True course	79°	R. of S.
Leeway (port tack)	8° 26′ 15″	L.
Variation and deviation	8° 30′	R.
	79° 3′ 45″	R. of S.

Compass course, S. 79° 4′ W.

5. Given compass course W. ¼ N., wind N.N.W., leeway 1¾ points, variation 8° 30′ E., deviation 15° 35′ E., required true course.

Compass course	7¾ pts.	L. of N.
Leeway (starboard tack)	1¾ pts.	L.
	9½ pts.	L. of N.
	= 6½ pts.	R. of S.
	= 73° 7′ 30″	R. of S.
Variation and deviation	24° 5′	R.
	97° 12′ 30″	R. of S.
	= 82° 47′ 30″	L. of N.

True course, N. 82° 47′ W.

6. Given compass course E. ¼ N., wind N.N.E., leeway 2¼ points, variation 13° 0′ W., deviation 20° 0′ E.; required true course.

Compass course	7¾ pts.	R. of N.
Leeway (port tack)	2¼ pts.	R.
	10 pts.	R. of N.
	= 6 pts.	L. of S.
Variation and deviation	= 67° 30′	L. of S.
	7°	R.
	60° 30′	L. of S.

True course, S. 60° 30′ E.

7. Given true course S. 85° E., wind N. by W., leeway ½ point, variation 14° 0′ E., deviation 19° 0′ E.; required compass course.

True course	85°	L. of S.
Leeway (port tack)	5° 37′ 30″	L.
Variation and deviation	33°	L.
	123° 37′ 30″	L. of S.
	= 56° 22′ 30″	R. of N.

Compass course, N. 56° 22′ E.

8. Given compass course W., wind N.N.W., leeway 1¼ points, variation °° 30′ E., deviation 21° 0′ W.; required true course.

Compass course . . . . . . 8 pts. L. of N.
Leeway (starboard tack) . . . . 1¼ pts. L.
9¼ pts. L. of N.
= 6¾ pts. R. of S.
= 75° 56′ 15″ R. of S.
Variation and deviation . . . 2° 30′ L.
73° 26′ 15″ R. of S.
True course, S. 73° 26′ W.

9. Given compass course E. ½ S., wind N.N.E. ½ E., leeway 2½ points, variation 21° 0′ E., deviation 4° 0′ W.; required true course.

Compass course . . . . . . 7½ pts. L. of S.
Leeway (port tack) . . . . . 2½ pts. R.
5 pts. L. of S.
= 56° 15′ L. of S.
Variation and deviation . . . . 17° R.
39° 15′ L. of S.
True course, S. 39° 15′ E.

10. Given true course, E. by S. ½ S., wind N. by W., leeway 2¾ points, variation 2 points W., deviation 3¾ points E.; required compass course.

True course . . . . 6½ pts. L. of S.
Leeway (port tack) . . . . . 2¾ pts. L.
Variation and deviation . . . . 1¾ pts. L.
11 pts. L. of S.
= 5 pts. R. of N.
Compass course, N.E. by E.

11. Given true course N. by W., wind N.E., leeway 3½ points, variation 2¾ points E., deviation 1¾ points E.; required compass course.

True course . . . . 1 pt. L. of N.
Leeway (starboard tack) . . . . 3½ pts. R.
Variation and deviation . . . . 4½ pts. L.
2 pts. L. of N.
Compass course, N.N.W.

12. Given true course N.N.W., wind S.S.W., leeway 2½ points, variation 2¾ points E., deviation ¾ point E.; required compass course.

True course . . . . 2 pts. L. of N.
Leeway (port tack) . . . . . 2½ pts. L.
Variation and deviation . . . . 3½ pts. L.
8 pts. L. of N.
Compass course, W.

13. Given true course S. 64° E., leeway 0, variation 7° 0′ W., deviation 15° 0′ W.; required compass course.

True course	64° L. of S.
Variation and deviation	22° R.
	42° L. of S.

Compass course, S. 42° E.

14. Given true course N. 44 W., leeway 0, variation 6° 0′ E., deviation 20° 0′ W.; required compass course.

True course	44° L. of N.
Variation and deviation	14° R.
	30° L. of N.

Compass course, N. 30° W.

15. Given compass course N. 65° W., leeway 0, variation 10° 0′ E., deviation 3° 0′ E.; required true course.

Compass course	65° L. of N.
Variation and deviation	13° R.
	52° L. of N.

True course, N. 52° W.

16. Given compass course S. 15° W., leeway 0, variation 6° 0′ W., deviation 18° 0′ E.; required true course.

Compass course	15° R. of S.
Variation and deviation	12° R.
	27° R. of S.

True course, S. 27° W.

17. Given compass course S. 18° E., leeway 0, variation 25° 0′ E., deviation 10° 0′ E., required true course.

Compass course	18° L. of S.
Variation and deviation	35° R.
	17° R. of S.

True course, S. 17° W.

18. Given compass course N. 30° E., wind S. by W., leeway 1¼ points, variation 12° 0′ E., deviation 10° 0′ W.; required true course.

Compass course	30°	R. of N.
Leeway (starboard tack)	14° 3′ 45″	L.
Variation and deviation	2°	R.
	17° 56′ 15″	R. of N.

True course, N. 17° 56′ E.

EXERCISE II. PAGE 332.

1. Given L' 49° 57′ N., C S.W. by W., D 488.0; required L'' and p.

$D = 488.0$
$C = 56° 15'$

$p = D \sin C.$
$\log D = 2.68842$
$\log \sin C = 9.91985$
$\log p = 2.60826$
$p = 405.8.$

$L_d = D \cos C.$
$\log D = 2.68842$
$\log \cos C = 9.74474$
$\log L_d = 2.43316$
$L_d = 271'$
$= 4° 31'$ S.
$L' = 49° 57'$ N.
$L'' = 45° 26'$ N.

2. Given L' 1° 45′ N., C S.E. by E., D 487.8; required L'' and p.

$D = 487.8$
$C = 56° 15'$

$p = D \sin C.$
$\log D = 2.68824$
$\log \sin C = 9.91985$
$\log p = 2.60809$
$p = 405.6.$

$L_d = D \cos C.$
$\log D = 2.68824$
$\log \cos C = 9.74474$
$\log L_d = 2.43298$
$L_d = 271'$
$= 4° 31'$ S.
$L' = 1° 45'$ N.
$L'' = 2° 46'$ S.

3. Given L' 3° 15′ S., C N.E. by E. ¾ E., D 449.1; required L'' and p.

$D = 449.1$
$C = 64° 41' 15''$

$p = D \sin C.$
$\log D = 2.65234$
$\log \sin C = 9.95616$
$\log p = 2.60850$
$p = 406.$

$L_d = D \cos C.$
$\log D = 2.65234$
$\log \cos C = 9.63099$
$\log L_d = 2.28333$
$L_d = 192'$
$= 3° 12'$ N.
$L' = 3° 15'$ S.
$L'' = 0° \ 3'$ S.

4. Given L' 2° 10′ S., C N. by E., D 267.0; required L'' and p.

$D = 267.0$
$C = 11° 15'$

$p = D \sin C.$
$\log D = 2.42651$
$\log \sin C = 9.29024$
$\log p = 1.71675$
$p = 52.1$

$L_d = D \cos C.$
$\log D = 2.42651$
$\log \cos C = 9.99157$
$\log L_d = 2.41808$
$L_d = 262'$
$= 4° 22'$ N.
$L' = 2° 10'$ S.
$L'' = 2° 12'$ N.

5. Given L' 41° 30′ N., C S.S.W., D 295.5; required L'' and p.

$$\begin{aligned} D &= 295.5 \\ C &= 22^\circ\ 30' \end{aligned}$$

$$\begin{aligned} p &= D \sin C. \\ \log D &= 2.47056 \\ \log \sin C &= \underline{9.58284} \\ \log p &= 2.05340 \\ p &= 113.1. \end{aligned}$$

$$\begin{aligned} L_d &= D \cos C. \\ \log D &= 2.47056 \\ \log \cos C &= \underline{9.96562} \\ \log L_d &= 2.43618 \\ L_d &= 273' \\ &= 4^\circ 33'\ \text{S.} \\ L' &= \underline{41^\circ 30'\ \text{N.}} \\ L'' &= 36^\circ 57'\ \text{N.} \end{aligned}$$

6. Given L' 21° 59′ S., L'' 24° 49′ S., D 360; required C and p.

$$\begin{aligned} D &= 360 \\ L_d &= 170 \end{aligned}$$

$$\begin{aligned} \cos C &= \frac{L_d}{D}. \\ \log L_d &= 2.23045 \\ \log D &= \underline{2.55630} \\ \log \cos C &= 9.67415 \\ C &= 61^\circ\ 49' \\ &= 5\tfrac{1}{2}\ \text{pts., nearly.} \end{aligned}$$

$$\begin{aligned} p^2 &= (D - L_d)(D + L_d) \\ &= 190 \times 530. \\ \log 190 &= 2.27875 \\ \log 530 &= \underline{2.72428} \\ \log p^2 &= 5.00303 \\ \log p &= 2.50151 \\ p &= 317.3. \end{aligned}$$

7. Given L' 2° 9′ S., L'' 3° 11′ N., D 354; required C and p.

$$\begin{aligned} D &= 354 \\ L_d &= 320 \end{aligned}$$

$$\begin{aligned} \cos C &= \frac{L_d}{D}. \\ \log L_d &= 2.50515 \\ \log D &= \underline{2.54900} \\ \log \cos C &= 9.95615 \\ C &= 25^\circ\ 19' \\ &= 2\tfrac{1}{4}\ \text{pts.} \end{aligned}$$

$$\begin{aligned} p^2 &= (D - L_d)(D + L_d) \\ &= 34 \times 674. \\ \log\ 34 &= 1.53148 \\ \log 674 &= \underline{2.82866} \\ \log p^2 &= 4.36014 \\ \log p &= 2.18007 \\ p &= 151.4. \end{aligned}$$

8. Given L' 1° 30′ N., L'' 0° 26′ S., C S. by W.; required D and p.

$$\begin{aligned} L_d &= 116 \\ C &= 11^\circ\ 15' \end{aligned}$$

$$\begin{aligned} D &= L_d \sec C. \\ \log L_d &= 2.06446 \\ \log \sec C &= \underline{0.00843} \\ \log D &= 2.07289 \\ D &= 118.3. \end{aligned}$$

$$\begin{aligned} p &= L_d \tan C. \\ \log L_d &= 2.06466 \\ \log \tan C &= \underline{9.29866} \\ \log p &= 1.36312 \\ p &= 23.1. \end{aligned}$$

9. Given L' 40° 17′ N., L″ 37° 6′ N., C S. by W. ½ W.; required D and p.

$$\begin{aligned} L_d &= 191 \\ C &= 16^\circ 52' 30'' \end{aligned}$$

$$\begin{aligned} D &= L_d \sec C. \\ \log L_d &= 2.28103 \\ \log \sec C &= \underline{0.01911} \\ \log D &= 2.30014 \\ D &= 199.6. \end{aligned}$$

$$\begin{aligned} p &= L_d \tan C. \\ \log L_d &= 2.28103 \\ \log \tan C &= \underline{9.48194} \\ \log p &= 1.76297 \\ p &= 57.9. \end{aligned}$$

10. Given L' 38° 0′ N., C S.W. by W., p 48.2; required L'' and D.

$$\begin{aligned} p &= 48.2 \\ C &= 56^\circ\ 15' \end{aligned}$$

$$\begin{aligned} D &= p \csc C. \\ \log p &= 1.68305 \\ \log \csc C &= 0.08015 \\ \hline \log D &= 1.76320 \\ D &= 58.0. \end{aligned}$$

$$\begin{aligned} L_d &= p \cot C. \\ \log p &= 1.68305 \\ \log \cot C &= 9.82489 \\ \hline \log L_d &= 1.50794 \\ L_d &= 32'\ \text{S.} \\ \hline L' &= 38^\circ\ \ 0'\ \text{N.} \\ L'' &= 37^\circ 28'\ \text{N.} \end{aligned}$$

11. Given L' 18° 25′ N., C S.W. by W. ¾ W., p 65.1; required L'' and D.

$$\begin{aligned} p &= 65.1 \\ C &= 64^\circ\ 41'\ 15'' \end{aligned}$$

$$\begin{aligned} D &= p \csc C. \\ \log p &= 1.81358 \\ \log \csc C &= 0.04384 \\ \hline \log D &= 1.85742 \\ D &= 72.02 \end{aligned}$$

$$\begin{aligned} L_d &= p \cot C. \\ \log p &= 1.81358 \\ \log \cot C &= 9.67483 \\ \hline \log L_d &= 1.48841 \\ L_d &= 31'\ S. \\ L' &= 18^\circ 25'\ \text{N.} \\ \hline L'' &= 17^\circ 54'\ \text{N.} \end{aligned}$$

12. Given L' 50° 18′ N., L'' 54° 48′ N., D 299.0; required C and p.

$$\begin{aligned} D &= 299.0 \\ L_d &= 270 \end{aligned}$$

$$\begin{aligned} \cos C &= \frac{L_d}{D}. \\ \log L_d &= 2.43136 \\ \log D &= 2.47567 \\ \hline \log \cos C &= 9.95569 \\ C &= 25^\circ\ 26'\ 30'' \\ &= 2\tfrac{1}{4}\ \text{pts., nearly.} \end{aligned}$$

$$\begin{aligned} p^2 &= (D - L_d)(D + L_d) \\ &= 29 \times 569. \\ \log 29 &= 1.46240 \\ \log 569 &= 2.75511 \\ \hline \log p^2 &= 4.21751 \\ \log p &= 2.10875 \\ p &= 128.5. \end{aligned}$$

13. Given L' 32° 30′ N., L'' 19° 59′ N., D 812.0; required C and p.

$$\begin{aligned} D &= 812.0 \\ L_d &= 751 \end{aligned}$$

$$\begin{aligned} \cos C &= \frac{L_d}{D}. \\ \log L_d &= 2.87564 \\ \log D &= 2.90956 \\ \hline \log \cos C &= 9.96608 \\ C &= 22^\circ\ 21' \\ &= 2\ \text{pts., nearly.} \end{aligned}$$

$$\begin{aligned} p^2 &= (D - L_d)(D + L_d) \\ &= 61 \times 1563. \\ \log 61 &= 1.78533 \\ \log 1563 &= 3.19396 \\ \hline \log p^2 &= 4.97929 \\ \log p &= 2.48964 \\ p &= 308.8. \end{aligned}$$

14. Given L' 2° 8′ S., C N. 11° E., D 500; required L'' and p.

$$\begin{aligned} D &= 500 \\ C &= 11^\circ \end{aligned}$$

$$\begin{aligned} p &= D \sin C. \\ \log D &= 2.69897 \\ \log \sin C &= \underline{9.28060} \\ \log p &= 1.97957 \\ p &= 95.4. \end{aligned}$$

$$\begin{aligned} L_d &= D \cos C. \\ \log D &= 2.69897 \\ \log \cos C &= \underline{9.99195} \\ \log L_d &= 2.69092 \\ L_d &\doteq 491' \\ &= 8^\circ\ 11'\ \text{N.} \\ L' &= \underline{2^\circ\ \ 8'\ \text{S.}} \\ L'' &= 6^\circ\ \ 3'\ \text{N.} \end{aligned}$$

15. Given L' 20° 21′ S., C N. 20° E., D 402.0; required L'' and p.

$$\begin{aligned} D &= 402.0 \\ C &= 20^\circ \end{aligned}$$

$$\begin{aligned} p &= D \sin C. \\ \log D &= 2.60423 \\ \log \sin C &= \underline{9.53405} \\ \log p &= 2.13828 \\ p &= 137.5. \end{aligned}$$

$$\begin{aligned} L_d &= D \cos C. \\ \log D &= 2.60423 \\ \log \cos C &= \underline{9.97299} \\ \log L_d &= 2.57722 \\ L_d &= 378' \\ &= \ 6^\circ\ 18'\ \text{N.} \\ L' &= \underline{20^\circ\ 21'\ \text{S.}} \\ L'' &= 14^\circ\ \ 3'\ \text{S.} \end{aligned}$$

16. Given L' 40° 25′ S., C N. 87° E., D 240.0; required L'' and p.

$$\begin{aligned} D &= 240.0 \\ C &= 87^\circ \end{aligned}$$

$$\begin{aligned} p &= D \sin C. \\ \log D &= 2.38021 \\ \log \sin C &= \underline{9.99940} \\ \log p &= 2.37961 \\ p &\doteq 239.7. \end{aligned}$$

$$\begin{aligned} L_d &= D \cos C. \\ \log D &= 2.38021 \\ \log \cos C &= \underline{8.71880} \\ \log L_d &= 1.09901 \\ L_d &= \qquad 13'\ \text{N.} \\ L' &= \underline{40^\circ\ 25'\ \text{S.}} \\ L'' &= 40^\circ\ 12'\ \text{S.} \end{aligned}$$

17. Given L' 20° 48′ N., L″ 17° 13′ N., p 289.2 W.; required C and D.

$$\begin{aligned} p &= 289.2 \\ L_d &= 215 \end{aligned}$$

$$\begin{aligned} \tan C &= \frac{p}{L_d}. \\ \log p &= \ \ 2.46120 \\ \log L_d &= \ \ \underline{2.33244} \\ \log \tan C &= 10.12876 \\ C &= \text{S. } 53^\circ\ 22'\ 18''\ \text{W.} \\ &= \text{S. } 53^\circ\ 22'\ \text{W.} \end{aligned}$$

$$\begin{aligned} D &= p \csc C. \\ \log p &= 2.46120 \\ \log \csc C &= \underline{0.09554} \\ \log D &= 2.55674 \\ D &= 360.4. \end{aligned}$$

18. Given L' 51° 45′ N., L'' 53° 11′ N., p 128.0 E.; required C and D.

$p = 128.0$
$L_d = 86$

$$\tan C = \frac{p}{L_d}.$$

$$\begin{aligned} \log p &= 2.10721 \\ \log L_d &= \underline{1.93450} \\ \log \tan C &= 10.17271 \\ C &= \text{N. } 56^\circ\, 6'\, 13''\text{ E.} \\ &= \text{N.E. by E. nearly.} \end{aligned}$$

$$D = p \csc C.$$

$$\begin{aligned} \log p &= 2.10721 \\ \log \csc C &= \underline{0.08090} \\ \log D &= 2.18811 \\ D &= 154.2. \end{aligned}$$

19. Given L' 0° 20′ S., L'' 0° 18′ N., p 142.7 E.; required C and D.

$p = 142.7$
$L_d = 38$

$$\tan C = \frac{p}{L_d}.$$

$$\begin{aligned} \log p &= 2.15442 \\ \log L_d &= \underline{1.57978} \\ \log \tan C &= 10.57464 \\ C &= \text{N. } 75^\circ\, 5'\, 19''\text{ E.} \\ &= \text{N. } 75^\circ\, 5'\text{ E.} \end{aligned}$$

$$D = p \csc C.$$

$$\begin{aligned} \log p &= 2.15442 \\ \log \csc C &= 0.01488 \\ \log D &= \underline{2.16930} \\ D &= 147.7. \end{aligned}$$

20. Given L' 40° 20′ N., L'' 41° 37′ N., p 52.6 W.; required C and D.

$p = 52.6$
$L_d = 77$

$$\tan C = \frac{p}{L_d}.$$

$$\begin{aligned} \log p &= 1.72099 \\ \log L_d &= \underline{1.88649} \\ \log \tan C &= 9.83450 \\ C &= \text{N. } 34^\circ\, 20'\, 17''\text{ W.} \\ &= \text{N. } 34^\circ\, 20'\text{ W.} \end{aligned}$$

$$D = p \csc C.$$

$$\begin{aligned} \log p &= 1.72099 \\ \log \csc C &= \underline{0.24867} \\ \log D &= 1.96966 \\ D &= 93.3. \end{aligned}$$

EXERCISE III. PAGE 335.

1. Given L 55° 55′, λ' 2° 10′ W., λ'' 12° 52′ E.; required p.

$$p = 15\tfrac{1}{30} \times 33.62 = 505.4\text{ E.}$$

2. Given L 52° 0′, λ' 0° 59′ W., λ'' 2° 24′ E.; required p.

$$p = 3\tfrac{23}{60} \times 36.94 = 125.0\text{ E.}$$

3. Given L 61° 25′, λ' 179° 20′ W., λ'' 176° 52′ E.; required p.

$$p = 3\tfrac{48}{60} \times 28.71 = 109.1\text{ W.}$$

4. Given L 56° 0′, λ' 3° 12′ W., λ'' 4° 8′ E.; required p.

$$p = 7\tfrac{1}{3} \times 33.55 = 246.0\text{ E.}$$

5. Given L 80° 0′, λ′ 10° 0′ W., λ″ 17° 41′ W.; required p.

$$p = 7\tfrac{41}{60} \times 10.42 = 80.1 \text{ W.}$$

6. Given L 60° 0′, p 204.0 E., λ′ 160° 2′ E.; required λ″.

$$\lambda_d = 204.0 \div \tfrac{1}{2} = 408' = 6^\circ\ 48' \text{ E.};$$
$$\lambda'' = 160^\circ\ 2' + 6^\circ\ 48' = 166^\circ\ 50' \text{ E.}$$

7. Given L 51° 28′, p 70.9 E., λ′ 32° 7′ W.; required λ″.

$$\lambda_d = 70.9 \div 37.38 = 1.90^\circ = 1^\circ\ 54' \text{ E.};$$
$$\lambda'' = 32^\circ\ 7' - 1^\circ\ 54' = 30^\circ\ 13' \text{ W.}$$

8. Given L 64° 16′, p. 265.7 W., λ′ 170° 0′ W.; required λ″.

$$\lambda_d = 265.7 \div 26.05 = 10.20 = 10^\circ\ 12' \text{ W.};$$
$$\lambda'' = 170^\circ\ 0' + 10^\circ\ 12' = 180^\circ\ 12' \text{ W.} = 179^\circ\ 48' \text{ E.}$$

9. Given L 46° 37′, p 352.0 E., λ′ 163° 42′ E.; required λ″.

$$\lambda_d = 352.0 \div 41.21 = 8.54^\circ = 8^\circ\ 33' \text{ E.};$$
$$\lambda'' = 163^\circ\ 42' + 8^\circ\ 33' = 172^\circ\ 15' \text{ E.}$$

10. Given L 39° 57′, p 398.0 W., λ′ 4° 8′ W.; required λ″.

$$\lambda_d = 398.0 \div 45.93 = 8.67^\circ = 8^\circ\ 39' \text{ W.};$$
$$\lambda'' = 4^\circ\ 8' + 8^\circ\ 39' = 12^\circ\ 47' \text{ W.}$$

11. From latitude 32° 3′ S., longitude 179° 45′ W., a ship makes 54 miles west (true). Required the longitude in.

$$\lambda_d = 54 \div 50.85 = 1.06^\circ = 1^\circ\ 4' \text{ W.};$$
$$\lambda'' = 179^\circ\ 45' + 1^\circ\ 4' = 180^\circ\ 49' \text{ W.} = 179^\circ\ 11' \text{ E.}$$

12. From latitude 35° 30′ S., longitude 27° 28′ W., a ship sails east (true) 301 miles. Required the longitude in and the compass course; variation 1¾ points E., leeway ¼ point to the left, deviation 8° 50′ E.

$$\lambda_d = 301 \div 48.85 = 6.16^\circ = 6^\circ\ 10';$$
$$\lambda'' = 27^\circ\ 28' - 6^\circ\ 10' = 21^\circ\ 18' \text{ W.}$$

True course	8 pts. R. of N.
Variation and leeway	2 pts. L.
	6 pts. R. of N.
	= 67° 30′ R. of N.
Deviation	8° 50′ L.
Compass course	N. 58° 40′ E.

EXERCISE IV. PAGE 340.

1. Given L' 25° 35′ N., L'' 27° 28′ N., λ' 60° 0′ W., λ'' 54° 55′ W.; required C and D.

$$\begin{aligned} L_d &= 1^\circ\,53' \\ &= 113' \\ L_m &= 26^\circ\,32' \\ \lambda_d &= 5^\circ\,5' \\ &= 305' \end{aligned}$$

$$\begin{aligned} \tan C &= \frac{\lambda_d \cos L_m}{L_d}. \\ \log \lambda_d &= 2.48430 \\ \log \cos L_m &= 9.95167 \\ \operatorname{colog} L_d &= \underline{7.94692} \\ \log \tan C &= 10.38289 \\ C &= \text{N. } 67^\circ\,30' \text{ E.} = \text{E.N.E.} \end{aligned}$$

$$\begin{aligned} D &= L_d \sec C. \\ \log L_d &= 2.05308 \\ \log \sec C &= \underline{10.41716} \\ \log D &= 2.47024 \\ D &= 295.3. \end{aligned}$$

2. Given L' 32° 30′ N., L'' 34° 10′ N., λ' 25° 24′ W., λ'' 29° 8′ W.; required C and D.

$$\begin{aligned} L_d &= 1^\circ\,40' \\ &= 100' \\ L_m &= 33^\circ\,20' \\ \lambda_d &= 3^\circ\,44' \\ &= 224' \end{aligned}$$

$$\begin{aligned} \tan C &= \frac{\lambda_d \cos L_m}{L_d}. \\ \log \lambda_d &= 2.35025 \\ \log \cos L_m &= 9.92194 \\ \operatorname{colog} L_d &= \underline{8.00000} \\ \log \tan C &= 10.27219 \\ \therefore C &= \text{N. } 61^\circ\,53' \text{ W.} \end{aligned}$$

$$\begin{aligned} D &= L_d \sec C. \\ \log L_d &= 2.00000 \\ \log \sec C &= \underline{10.32673} \\ \log D &= 2.32673 \\ D &= 212.2. \end{aligned}$$

3. Given L' 39° 30′ S., L'' 41° 0′ S., λ' 74° 20′ E., λ'' 70° 12′ E.; required C and D.

$$\begin{aligned} L_d &= 1^\circ\,30' \\ &= 90' \\ L_m &= 40^\circ\,15' \\ \lambda_d &= 4^\circ\,8' \\ &= 248' \end{aligned}$$

$$\begin{aligned} \tan &= \frac{\lambda_d \cos L_m}{L_d}. \\ \log \lambda_d &= 2.39445 \\ \log \cos L_m &= 9.88266 \\ \operatorname{colog} L_d &= \underline{8.04576} \\ \log \tan C &= 10.32287 \\ \therefore C &= \text{S. } 64^\circ\,34' \text{ W.} \end{aligned}$$

$$\begin{aligned} D &= L_d \sec C. \\ \log L_d &= 1.95424 \\ \log \sec C &= \underline{10.36708} \\ \log D &= 2.32132 \\ D &= 209.6. \end{aligned}$$

4. Given L' 46° 24′ S., λ' 178° 28′ E., C S.E. ¾ E., D 278.0; required L'' and λ''.

$$\begin{aligned} C &= 53^\circ\,26' \\ D &= 278.0 \end{aligned}$$

$$\begin{aligned} L_d &= D \cos C. \\ \log D &= 2.44404 \\ \log \cos C &= \underline{9.77507} \\ \log L_d &= 2.21911 \\ L_d &= 166' \\ &= 2^\circ\,46' \\ L' &= \underline{46^\circ\,24'} \\ L'' &= 49^\circ\,10' \text{ S.} \\ L_m &= 47^\circ\,47'. \end{aligned}$$

$$\begin{aligned} \lambda_d &= D \sin C \sec L_m. \\ \log D &= 2.44404 \\ \log \sin C &= 9.90480 \\ \log \sec L_m &= \underline{10.17267} \\ \log \lambda_d &= 2.52151 \\ \lambda_d &= 332' \\ &= 5^\circ\,32' \\ \lambda' &= \underline{178^\circ\,28'} \\ \lambda'' &= 184^\circ\ \ 0' \text{ E.} \\ &= 176^\circ\ \ 0' \text{ W.} \end{aligned}$$

5. Given L' 20° 29′ N., λ' 179° 10′ W., C W. by S. ½ S., D 333.0; required L'' and λ''.

$$\begin{aligned} C &= 73^\circ\ 7' \\ D &= 333 \end{aligned}$$

$$\begin{aligned} L_d &= D \cos C. \\ \log D &= 2.52244 \\ \log\cos C &= \underline{9.46303} \\ \log L_d &= 1.98548 \\ L_d &= 97' \\ &= 1^\circ\ 37' \\ L' &= \underline{20^\circ\ 29'} \\ L'' &= 18^\circ\ 52'\ \text{N.} \\ L_m &= 19^\circ\ 40'. \end{aligned}$$

$$\begin{aligned} \lambda_d &= D \sin C \sec L_m. \\ \log D &= 2.52244 \\ \log\sin C &= 9.98087 \\ \log\sec L_m &= \underline{10.02610} \\ \log\lambda_d &= 2.52941 \\ \lambda_d &= 338' \\ &= 5^\circ\ 38' \\ \lambda' &= \underline{179^\circ\ 10'} \\ \lambda'' &= 184^\circ\ 48'\ \text{W.} \\ &= 175^\circ\ 12'\ \text{E.} \end{aligned}$$

6. Given L' 0° 56′ N., λ' 29° 50′ W., C S. 47° E., D 168.0; required L'' and λ''.

$$\begin{aligned} C &= 47^\circ \\ D &= 168 \end{aligned}$$

$$\begin{aligned} L_d &= D \cos C. \\ \log D &= 2.22530 \\ \log\cos C &= \underline{9.83378} \\ \log L_d &= 2.05908 \\ L_d &= 115' \\ &= 1^\circ\ 55' \\ L' &= \underline{0^\circ\ 56'} \\ L'' &= 0^\circ\ 59'\ \text{S.} \\ L_m &= 0^\circ\ 1'. \end{aligned}$$

$$\begin{aligned} \lambda_d &= D \sin C \sec L_m. \\ \log D &= 2.22530 \\ \log\sin C &= 9.86413 \\ \log\sec L_m &= \underline{0.00000} \\ \log\lambda_d &= 2.08943 \\ \lambda_d &= 123' \\ &= 2^\circ\ 3' \\ \lambda' &= \underline{29^\circ\ 50'} \\ \lambda'' &= 27^\circ\ 47'\ \text{W.} \end{aligned}$$

7. Given L' 42° 25′ N., λ' 66° 14′ W., C S.E. by E., D 25.0; required L'' and λ''.

$$\begin{aligned} C &= 56^\circ\ 15' \\ D &= 25.0 \end{aligned}$$

$$\begin{aligned} L_d &= D \cos C. \\ \log D &= 1.39794 \\ \log\cos C &= \underline{9.74474} \\ \log L_d &= 1.14268 \\ L_d &= 14' \\ L' &= \underline{42^\circ\ 25'} \\ L'' &= 42^\circ\ 11'\ \text{N.} \\ L_m &= 42^\circ\ 18'. \end{aligned}$$

$$\begin{aligned} \lambda_d &= D \sin C \sec L_m. \\ \log D &= 1.39794 \\ \log\sin C &= 9.91985 \\ \log\sec L_m &= \underline{0.13098} \\ \log\lambda_d &= 1.44877 \\ \lambda_d &= 28' \\ \lambda' &= \underline{66^\circ\ 14'} \\ \lambda'' &= 65^\circ\ 46'\ \text{W.} \end{aligned}$$

8. Given L' 42° 8′ N., λ' 65° 48′ W., C E. ½ S., D 126.0; required L'' and λ''.

$$\begin{aligned} C &= 84^\circ\ 22' \\ D &= 126.0 \end{aligned}$$

$$\begin{aligned} L_d &= D \cos C. \\ \log D &= 2.10037 \\ \log \cos C &= 8.99194 \\ \hline \log L_d &= 1.09231 \\ L_d &= \quad 12' \\ L' &= 42^\circ\ 8' \\ \hline L'' &= 41^\circ\ 56' \text{ N.} \\ L_m &= 42^\circ\ 2'. \end{aligned}$$

$$\begin{aligned} \lambda_d &= D \sin C \sec L_m. \\ \log D &= 2.10037 \\ \log \sin C &= 9.99790 \\ \log \sec L_m &= 0.12915 \\ \hline \log L_d &= 2.22743 \\ \lambda_d &= 168' \\ &= \ 2^\circ\ 48' \\ \lambda' &= 65^\circ\ 48' \\ \hline \lambda'' &= 63^\circ\ 0' \text{ W.} \end{aligned}$$

9. Given L' 41° 52′ N., λ' 62° 47′ W., C E. ½ S., D 161.0; required L'' and λ''.

$$\begin{aligned} C &= 84^\circ\ 22' \\ D &= 161.0 \end{aligned}$$

$$\begin{aligned} L_d &= D \cos C. \\ \log D &= 2.20683 \\ \log \cos C &= 8.99194 \\ \hline \log L_d &= 1.19877 \\ L_d &= \quad 16' \\ L' &= 41^\circ\ 52' \\ \hline L'' &= 41^\circ\ 36' \text{ N.} \\ L_m &= 41^\circ\ 44' \text{ N.} \end{aligned}$$

$$\begin{aligned} \lambda_d &= D \sin C \sec L_m. \\ \log D &= 2.20683 \\ \log \sin C &= 9.99789 \\ \log \sec L_m &= 0.12712 \\ \hline \log \lambda_d &= 2.33184 \\ \lambda_d &= 215' \\ &= \ 3^\circ\ 35' \\ \lambda' &= 62^\circ\ 47' \\ \hline \lambda'' &= 59^\circ\ 12' \text{ W.} \end{aligned}$$

10. Given L' 41° 38′ N., L'' 41° 26′ N., λ' 59° 16′ W., C E. by S.; required λ'' and D.

$$\begin{aligned} L_d &= 12 \\ L_m &= 41^\circ\ 32' \\ C &= 78^\circ\ 45' \end{aligned}$$

$$\begin{aligned} D &= L_d \sec C. \\ \log L_d &= 1.07918 \\ \log \sec C &= 0.70976 \\ \hline \log D &= 1.78894 \\ D &= 61.5. \end{aligned}$$

$$\begin{aligned} \lambda_d &= D \sin C \sec L_m. \\ \log D &= 1.78894 \\ \log \sin C &= 9.99157 \\ \log \sec L_m &= 0.12577 \\ \hline \log \lambda_d &= 1.90629 \\ \lambda_d &= 81' \\ &= \ 1^\circ\ 21' \\ \lambda' &= 59^\circ\ 16' \\ \hline \lambda'' &= 57^\circ\ 55' \text{ W.} \end{aligned}$$

11. Given L' 41° 19′ N., L'' 41° 11′ N., λ' 57° 47′ W., D 167.0; required λ'' and C.

$$\begin{aligned} L_d &= 8 \\ L_m &= 41^\circ\ 15' \\ D &= 167.0 \end{aligned}$$

$$\cos C = \frac{L_d}{D}.$$

$$\begin{aligned} \log L_d &= 0.90309 \\ \log D &= \underline{2.22272} \\ \log \cos C &= 8.68037 \\ C &= 87^\circ\ 15'. \end{aligned}$$

$$\lambda_d = D \sin C \sec L_m.$$

$$\begin{aligned} \log D &= 2.22272 \\ \log \sin C &= 9.99950 \\ \log \sec L_m &= \underline{0.12387} \\ \log \lambda_d &= 2.34609 \\ \lambda_d &= 222' \\ &= 3^\circ\ 42' \\ \lambda' &= \underline{57^\circ\ 47'\ \text{W.}} \\ \lambda'' &= 61^\circ\ 29'\ \text{W.} \\ \text{or } & 54^\circ\ 5'\ \text{W.} \end{aligned}$$

12. Given L' 46° 28′ N., L'' 45° 17′ N., λ' 22° 18′ W., λ'' 19° 39′ W.; required C and D.

$$\begin{aligned} L_d &= 71 \\ L_m &= 45^\circ\ 52' \\ \lambda_d &= 159 \end{aligned}$$

$$\tan C = \frac{\lambda_d \cos L_m}{L_d}.$$

$$\begin{aligned} \log \lambda_d &= 2.20140 \\ \log \cos L_m &= 9.84282 \\ \text{colog}\, L_d &= \underline{8.14874} \\ \log \tan C &= 10.19296 \\ C &= \text{S. } 57^\circ\ 19'\ \text{E.} \end{aligned}$$

$$D = L_d \sec C.$$

$$\begin{aligned} \log L_d &= 1.85126 \\ \log \sec C &= \underline{0.26761} \\ \log D &= 2.11887 \\ D &= 131.5. \end{aligned}$$

13. Given L' 25° 30′ S., L'' 28° 15′ S., λ' 2° 15′ E., λ'' 11° 17′ E.; required C and D.

$$\begin{aligned} L_d &= 165 \\ L_m &= 26^\circ\ 52' \\ \lambda_d &= 542 \end{aligned}$$

$$\tan C = \frac{\lambda_d \cos L_m}{L_d}.$$

$$\begin{aligned} \log \lambda_d &= 2.73399 \\ \log \cos L_m &= 9.95039 \\ \text{colog}\, L_d &= \underline{7.78252} \\ \log \tan C &= 10.46690 \\ C &= \text{S. } 71^\circ\ 9'\ \text{E.} \end{aligned}$$

$$D = L_d \sec C.$$

$$\begin{aligned} \log L_d &= 2.21748 \\ \log \sec C &= \underline{0.49067} \\ \log D &= 2.70815 \\ D &= 510.7. \end{aligned}$$

14. Given L' 33° 40′ N., L'' 30° 49′ N., λ' 13° 20′ E., λ'' 17° 56′ E.; required C and D.

$$\begin{aligned} L_d &= 171 \\ L_m &= 31^\circ\ 44' \\ \lambda_d &= 276 \end{aligned}$$

$$\tan C = \frac{\lambda_d \cos L_m}{L_d}.$$

$$\begin{aligned} \log \lambda_d &= 2.44090 \\ \log \cos L_m &= 9.92968 \\ \text{colog}\, L_d &= \underline{7.76700} \\ \log \tan C &= 10.13758 \\ C &= \text{S. } 53^\circ\ 46'\ \text{E.} \end{aligned}$$

$$D = L_d \sec C.$$

$$\begin{aligned} \log L_d &= 2.23300 \\ \log \sec C &= \underline{0.22836} \\ \log D &= 2.46136 \\ D &= 289.3. \end{aligned}$$

15. Given L' 19° 30′ S., L'' 17° 24′ S., λ' 0° 10′ E., λ'' 1° 28′ W.; required C and D.

$$\begin{aligned} L_d &= 126 \\ L_m &= 18° \, 27' \\ \lambda_d &= 98 \end{aligned}$$

$$\tan C = \frac{\lambda_d \cos L_m}{L_d}.$$

$$\begin{aligned} \log \lambda_d &= 1.99123 \\ \log \cos L_m &= 9.97708 \\ \operatorname{colog} L_d &= \underline{7.89963} \\ \log \tan C &= 9.86794 \\ C &= \text{N.}\, 36° \, 25' \, \text{W.} \end{aligned}$$

$$D = L_d \sec C.$$

$$\begin{aligned} \log L_d &= 2.10037 \\ \log \sec C &= \underline{0.09435} \\ \log D &= 2.19472 \\ D &= 156.6. \end{aligned}$$

16. A ship sails from Boston light-house, in latitude 42° 20′ N., longitude 71° 4′ W., on a N.N.E. course, 184 miles. Find the latitude and longitude in.

$$\begin{aligned} C &= 22° \, 30' \\ D &= 184 \end{aligned}$$

$$L_d = D \cos C.$$

$$\begin{aligned} \log D &= 2.26482 \\ \log \cos C &= \underline{9.96603} \\ \log L_d &= 2.23085 \\ L_d &= 170' \\ &= \;\; 2° \, 50' \\ L' &= \underline{42° \, 20'} \\ L'' &= 45° \, 10' \, \text{N.} \\ L_m &= 43° \, 45'. \end{aligned}$$

$$\lambda_d = D \sin C \sec L_m.$$

$$\begin{aligned} \log D &= 2.26482 \\ \log \sin C &= 9.58284 \\ \log \sec L_m &= \underline{0.14124} \\ \log \lambda_d &= 1.98890 \\ \lambda_d &= 97' \\ &= \;\; 1° \, 37' \\ \lambda' &= \underline{71° \;\; 4'} \\ \lambda'' &= 69° \, 27' \, \text{W.} \end{aligned}$$

17. A ship sails from Cape May, in latitude 38° 56′ N., longitude 74° 57′ W., on a S.S.E. course, 240 miles. Find the latitude and longitude in.

$$\begin{aligned} C &= 22° \, 30' \\ D &= 240 \end{aligned}$$

$$L_d = D \cos C.$$

$$\begin{aligned} \log D &= 2.38021 \\ \log \cos C &= \underline{9.96603} \\ \log L_d &= 2.34624 \\ L_d &= 222' \\ &= \;\; 3° \, 42' \\ L' &= \underline{38° \, 56'} \\ L'' &= 35° \, 14' \, \text{N.} \\ L_m &= 37° \;\; 5'. \end{aligned}$$

$$\lambda_d = D \sin C \sec L_m.$$

$$\begin{aligned} \log D &= 2.38021 \\ \log \sin C &= 9.58284 \\ \log \sec L_m &= \underline{0.09813} \\ \log \lambda_d &= 2.06118 \\ \lambda_d &= 115' \\ &= \;\; 1° \, 55' \\ \lambda' &= \underline{74° \, 57'} \\ \lambda'' &= 73° \;\; 2' \, \text{W.} \end{aligned}$$

18. A ship sails from Cape Cod light, in latitude 42° 2′ N., longitude 70° 3′ W., on an E. by N. compass course, 170 miles; wind S.E. by S., leeway ¼ point, deviation 17¾° E., variation 11¼° W. Find the latitude and longitude in.

Compass course . . . . 7 pts. R. of N.
Leeway . . . . ¼ pt. L.
6¾ pts. R. of N.
= 75° 56′ R. of N.
Variation and deviation . . . . 6° 30′ R.
True course . . . . N. 82° 26′ E.

$C = 82° 26'$
$D = 170$

$$\begin{aligned} L_d &= D \cos C. \\ \log D &= 2.23045 \\ \log \cos C &= \underline{9.11951} \\ \log L_d &= 1.34996 \\ L_d &= \quad 22' \\ L' &= \underline{42° \;\; 2'} \\ L'' &= 42° \, 24' \text{ N.} \\ L_m &= 42° \, 13'. \end{aligned}$$

$$\begin{aligned} \lambda_d &= D \sin C \sec L_m. \\ \log D &= 2.23045 \\ \log \sin C &= 9.99620 \\ \log \sec L_m &= \underline{0.13041} \\ \log \lambda_d &= 2.35706 \\ \lambda_d &= 228' \\ &= \;\; 3° \, 48' \\ \lambda' &= \underline{70° \;\; 3'} \\ \lambda'' &= 66° \, 15' \text{ W.} \end{aligned}$$

19. A ship sails from Cape Cod light on a S.S.E. compass course, 140 miles; deviation 5½° E., variation 11¼° W. Find the latitude and longitude in.

Compass course . . . . 22° 30′ L. of S.
Variation and deviation . . . . . 5° 45′ L.
True course . . . . S. 28° 15′ E.

$C = 28° 15'$
$D = 140$

$$\begin{aligned} L_d &= \text{D} \cos C. \\ \log D &= 2.14613 \\ \log \cos C &= \underline{9.94492} \\ \log L_d &= 2.09105 \\ L_d &= 123' \\ &= \;\; 2° \, 3' \\ L' &= \underline{42° \;\; 2'} \\ L'' &= 39° \, 59' \text{ N.} \\ L_m &= 41° \, 0'. \end{aligned}$$

$$\begin{aligned} \lambda_d &= D \sin C \sec L_m. \\ \log D &= 2.14613 \\ \log \sin C &= 9.67516 \\ \log \sec L_m &= \underline{0.12222} \\ \log \lambda_d &= 1.94351 \\ \lambda_d &= 88' \\ &= \;\; 1° \, 28' \\ \lambda' &= \underline{70° \;\; 3'} \\ \lambda'' &= 68° \, 35' \text{ W.} \end{aligned}$$

20. A ship sails from latitude 55° 1′ N., longitude 1° 25′ W., on a S.W. compass course, 101 miles; wind W.N.W., leeway 1¼ points, deviation 6° W., variation 24° 56′ W. Find the latitude and longitude in.

Compass course . . . 4 pts. R. of S.
Leeway . . . . . . 1¼ pts. L.
2¾ pts. R. of S.
= 30° 56′ R. of S.
Variation and deviation 30° 56′ L.
True course . . . . S.

$$\begin{aligned} L_d &= D & \lambda_d &= 0. \\ &= 101' & \lambda'' &= \lambda' \\ &= \;\; 1° \, 41' & &= 1° \, 25' \text{ W.} \\ L' &= \underline{55° \;\; 1'} \\ L'' &= 53° \, 20' \text{ N.} \end{aligned}$$

21. A ship sails from the Bermudas, in latitude 32° 18′ N., longitude 64° 50′ W., on a W.S.W. compass course, 190 miles; deviation 1 point W., variation 1 point W. Find the latitude and longitude in.

Compass course	6 pts. R. of S.
Variation and deviation	2 pts. L.
True course	4 pts. R. of S.

$C = 45°$
$D = 190$

$$\begin{aligned} L_d &= D \cos C. \\ \log D &= 2.27875 \\ \log \cos C &= 9.84949 \\ \log L_d &= 2.12824 \\ L_d &= 134' \\ &= 2° \, 14' \\ L' &= 32° \, 18' \\ L'' &= 30° \, 4' \text{ N.} \\ L_m &= 31° \, 11'. \end{aligned}$$

$$\begin{aligned} \lambda_d &= D \sin C \sec L_m. \\ \log D &= 2.27875 \\ \log \sin C &= 9.84949 \\ \log \sec L_m &= 0.06777 \\ \log \lambda_d &= 2.19601 \\ \lambda_d &= 157' \\ &= 2° \, 37' \\ \lambda' &= 64° \, 50' \\ \lambda'' &= 67° \, 27' \text{ W.} \end{aligned}$$

22. A ship sails from the Bermudas on a W.N.W. compass course, 90 miles; wind S.W., leeway 1 point, deviation 1 point E., variation 1 point W. Find the latitude and longitude in.

Compass course	6 pts.	L. of N.
Leeway	1 pt.	R.
Variation and deviation	0	
True course	5 pts.	L. of N.
	= 56° 15′	L. of N.

$C = 56° \, 15'$
$D = 90$

$$\begin{aligned} L_d &= D \cos C. \\ \log D &= 1.95124 \\ \log \cos C &= 9.74474 \\ \log L_d &= 1.69898 \\ L_d &= \, 50' \\ L' &= 32° \, 18' \\ L'' &= 33° \, 8' \text{ N.} \\ L_m &= 32° \, 47'. \end{aligned}$$

$$\begin{aligned} \lambda_d &= D \sin C \sec L_m. \\ \log D &= 1.95124 \\ \log \sin C &= 9.91985 \\ \log \sec L_m &= 0.07502 \\ \log \lambda_d &= 1.94911 \\ \lambda_d &= 89' \\ &= 1° \, 29' \\ \lambda' &= 64° \, 50' \\ \lambda'' &= 66° \, 19' \text{ W.} \end{aligned}$$

23. A navigator wishes to sail on a rhumb from the Bermudas to Cape Fear, in latitude 33° 52′ N., longitude 78° W.; variation 10° W., deviation 7° W. Find the compass course and distance.

$$\begin{aligned} L_d &= 94 \\ L_m &= 32^\circ\ 5' \\ \lambda_d &= 790 \end{aligned}$$

$$\tan C = \frac{\lambda_d \cos L_m}{L_d}.$$

$$\begin{aligned} \log \lambda_d &= 2.89763 \\ \log \cos L_m &= 9.92318 \\ \text{colog } L_d &= \underline{8.02687} \\ \log \tan C &= 10.84768 \\ C &= \text{N. } 81^\circ\ 55'\ \text{W.} \\ \text{Var. and dev.} &= \underline{17^\circ \qquad \text{W.}} \\ \text{Compass course } &\text{N. } 64^\circ\ 55'\ \text{W.} \end{aligned}$$

$$D = L_d \sec C.$$

$$\begin{aligned} \log L_d &= 1.97313 \\ \log \sec C &= \underline{0.85197} \\ \log D &= 2.82510 \\ D &= 668.5. \end{aligned}$$

24. A ship from latitude 36° 32′ N. sails between south and west until she has made 480 miles of departure, and 9° 22′ of difference of longitude. Required the latitude in, the course steered, and the distance run. [Take $L_m = \frac{1}{2}(L' + L'') + 13'$.]

$$\begin{aligned} p &= 480 \\ \lambda_d &= 562 \end{aligned}$$

$$\cos L_m = \frac{p}{\lambda_d}.$$

$$\begin{aligned} \log p &= 2.68124 \\ \log \lambda_d &= \underline{2.74974} \\ \log \cos L_m &= 9.93150 \\ L_m &= 31^\circ\ 20'. \\ L'' &= 2\,(L_m - 13') - L' \\ &= 25^\circ\ 42'\ \text{N.} \\ L_d &= 10^\circ\ 50' \\ &= 650. \end{aligned}$$

$$\tan C = \frac{p}{L_d}.$$

$$\begin{aligned} \log p &= 2.68124 \\ \log L_d &= \underline{2.81291} \\ \log \tan C &= 9.86833 \\ C &= \text{S. } 36^\circ\ 27'\ \text{W.} \end{aligned}$$

$$D = p \csc C.$$

$$\begin{aligned} \log p &= 2.68124 \\ \log \csc C &= \underline{0.22613} \\ \log D &= 2.90737 \\ D &= 807.9. \end{aligned}$$

Exercise V. Page 347.

1. Given L' 38° 14′ N., L'' 39° 51′ N., λ' 2° 7′ E., λ'' 4° 18′ E.; required C and D.

$$\begin{aligned} &\ 39^\circ\ 51'\ \text{N.} \\ &\ \underline{38^\circ\ 14'\ \text{N.}} \\ L_d &= \ \ 1^\circ\ 37' = 97' \end{aligned}$$

$$\begin{aligned} \text{Mer. parts} &= 2596.2 \\ &= \underline{2471.8} \\ \text{Mer. } L_d &= 124.4 \end{aligned}$$

$$\begin{aligned} &\ 4^\circ\ 18''\ \text{E.} \\ &\ \underline{2^\circ\ \ 7''\ \text{E.}} \\ \lambda_d &= 2^\circ\ 11'' = 131' \end{aligned}$$

$$\tan C = \frac{\lambda_d}{\text{Mer. } L_d}$$

$$\begin{aligned} \log \lambda_d &= 2.11727 \\ \text{colog Mer. } L_d &= \underline{7.90518} \\ \log \tan C &= 10.02245 \\ \therefore C &= \text{N. } 46^\circ\ 29'\ \text{E.} \end{aligned}$$

$$D = L_d \sec C.$$

$$\begin{aligned} \log L_d &= 1.98677 \\ \log \sec C &= \underline{0.16205} \\ \log D &= 2.14882 \\ \therefore D &= 140.9. \end{aligned}$$

2. Given L' 49° 53′ N., L'' 48° 28′ N., λ' 6° 19′ W., λ'' 5° 3′ W.; required C and D.

$$\begin{array}{lll}
\phantom{L_d = {}} 49^\circ\, 53'\ \text{N.} & \text{Mer. parts} = 3446.0 & \phantom{\lambda_d = {}} 6^\circ\, 19'\ \text{W.} \\
\phantom{L_d = {}} \underline{48^\circ\, 28'\ \text{N.}} & \text{Mer. parts} = \underline{3316.4} & \phantom{\lambda_d = {}} \underline{5^\circ\ \ 3'\ \text{W.}} \\
L_d = 1^\circ\, 25' = 85' & \text{Mer. } L_d = \ \ 129.6 & \lambda_d = 1^\circ\, 16' = 76'
\end{array}$$

$$\tan C = \frac{\lambda_d}{\text{Mer. } L_d}. \qquad\qquad D = L_d \sec C.$$

$$\begin{array}{ll}
\log \lambda_d = 1.88081 & \log L_d = 1.92941 \\
\text{colog Mer. } L_d = \underline{7.88730} & \log \sec C = \underline{0.06416} \\
\log \tan C = 9.76829 & \log D = 1.99357 \\
\therefore C = \text{S. } 30^\circ\, 23'\ \text{E.} & \therefore D = 98.5.
\end{array}$$

3. Given L' 64° 30′ N., L'' 60° 40′ N., λ' 4° 20′ W., λ'' 0° 10′ E.; required C and D.

$$\begin{array}{lll}
\phantom{L_d = {}} 64^\circ\, 30'\ \text{N.} & \text{Mer. parts} = 5087.7 & \phantom{\lambda_d = {}} 4^\circ\, 20'\ \text{W.} \\
\phantom{L_d = {}} \underline{60^\circ\, 40'\ \text{N.}} & \text{Mer. parts} = \underline{4582.2} & \phantom{\lambda_d = {}} \underline{0^\circ\, 10'\ \text{E.}} \\
L_d = 3^\circ\, 50' = 230' & \text{Mer. } \lambda_d = \ \ 505.5 & \lambda_d = 4^\circ\, 30' = 270'
\end{array}$$

$$\tan C = \frac{\lambda_d}{\text{Mer. } L_d}. \qquad\qquad D = L_d \sec C.$$

$$\begin{array}{ll}
\log \lambda_d = 2.43136 & \log L_d = 2.36173 \\
\text{colog Mer. } L_d = \underline{7.29628} & \log \sec C = \underline{0.05569} \\
\log \tan C = 9.72764 & \log D = 2.42742 \\
\therefore C = \text{S. } 28^\circ\, 24'\ \text{E.} & \therefore D = 261.5.
\end{array}$$

4. Given L' 54° 54′ S., L'' 34° 22′ S., λ' 60° 28′ W., λ'' 18° 24′ W.; required C and D.

$$\begin{array}{lll}
\phantom{L_d = {}} 54^\circ\, 54'\ \text{S.} & \text{Mer. parts} = 3938.7 & \phantom{\lambda_d = {}} 60^\circ\, 28'\ \text{W.} \\
\phantom{L_d = {}} \underline{34^\circ\, 22'\ \text{S.}} & \text{Mer. parts} = \underline{2185.1} & \phantom{\lambda_d = {}} \underline{18^\circ\, 24'\ \text{W.}} \\
L_d = 20^\circ\, 32' = 1232' & \text{Mer. } L_d = 1753.6 & \lambda_d = 42^\circ\ \ 4' = 2524'
\end{array}$$

$$\tan C = \frac{\lambda_d}{\text{Mer. } L_d}. \qquad\qquad D = L_d \sec C.$$

$$\begin{array}{ll}
\log \lambda_d = \ \ 3.40209 & \log L_d = 3.09061 \\
\text{colog Mer. } L_d = \underline{\ \ 6.75607} & \log \sec C = \underline{0.24376} \\
\log \tan C = 10.15816 & \log D = 3.33437 \\
\therefore C = \text{N. } 55^\circ\, 13'\ \text{E.} & \therefore D = 2160.
\end{array}$$

5. Given L' 17° 0′ N., L'' 20° 0′ N., λ' 180° 0′ E., λ'' 177° 0′ E.; required C and D.

$$\begin{array}{lll}
\phantom{L_d = {}} 20^\circ\, 0'\ \text{N.} & \text{Mer. parts} = 1217.3 & \phantom{\lambda_d = {}} 180^\circ\, 0'\ \text{E.} \\
\phantom{L_d = {}} \underline{17^\circ\, 0'\ \text{N.}} & \text{Mer. parts} = \underline{1023.6} & \phantom{\lambda_d = {}} \underline{177^\circ\, 0'\ \text{E.}} \\
L_d = 3^\circ\, 0' = 180' & \text{Mer } L_d = \ \ 188.7 & \lambda_d = 30^\circ\, 0' = 180'
\end{array}$$

$$\tan C = \frac{\lambda_d}{\text{Mer. } \lambda_d}.$$

$$\begin{aligned} \log \lambda_d &= 2.25527 \\ \text{colog Mer. } L_d &= \underline{7.72423} \\ \log \tan C &= 9.97950 \\ C &= \text{N. } 43^\circ\ 39'\ \text{W.} \end{aligned}$$

$$D = L_d \sec C.$$

$$\begin{aligned} \log L_d &= 2.25527 \\ \log \sec C &= \underline{0.14052} \\ \log D &= 2.39579 \\ D &= 248.8. \end{aligned}$$

6. Given L' 45° 15′ N., λ' 35° 26′ W., C N. 49° E., D 175; required L'' and λ''.

$$L_d = D \cos C.$$

$$\begin{aligned} \log D &= 2.24301 \\ \log \cos C &= \underline{9.81694} \\ \log L_d &= 2.05995 \\ L_d &= 115' \\ &= \ 1^\circ\ 55' \\ L' &= \underline{45^\circ\ 15'} \\ L'' &= 47^\circ\ 10'\ \text{N.} \\ L_m &= 46^\circ\ 12'. \end{aligned}$$

$$\lambda_d = D \sin C \sec L_m.$$

$$\begin{aligned} \log D &= \ 2.24301 \\ \log \sin C &= \ 9.87778 \\ \log \sec L_m &= \underline{10.15980} \\ \log \lambda_d &= \ 2.27959 \\ \therefore \lambda_d &= 190' \\ &= \ 3^\circ\ 11' \\ \lambda' &= \underline{35^\circ\ 26'} \\ \lambda'' &= 32^\circ\ 15'\ \text{W.} \end{aligned}$$

7. Given L' 55° 1′ N., λ' 1° 25′ E., C N. 10° E., D 246; required L'' and λ''.

$$L_d = D \cos C.$$

$$\begin{aligned} \log D &= 2.39094 \\ \log \cos C &= \underline{9.99313} \\ \log D &= 2.38407 \\ D &= 242' \\ &= \ 4^\circ\ 2' \\ L' &= \underline{55^\circ\ 1'} \\ L'' &= 59^\circ\ 3'\ \text{N.} \\ L_m &= 57^\circ\ 2'. \end{aligned}$$

$$\begin{aligned} 55^\circ\ 1'\ \text{N., Mer. parts} &= 3950.9 \\ 59^\circ\ 3'\ \text{N., Mer. parts} &= \underline{4395.3} \\ \text{Mer. } L_d &= \ 444.4 \end{aligned}$$

$$\lambda_d = \text{Mer. } L_d \times \tan C.$$

$$\begin{aligned} \log \text{Mer. } L_d &= 2.64777 \\ \log \tan C &= \underline{9.24632} \\ \log \lambda_d &= 1.89409 \\ \therefore \lambda_d &= 78' \\ &= 1^\circ\ 18' \\ \lambda' &= \underline{1^\circ\ 25'} \\ \lambda'' &= 2^\circ\ 43'\ \text{E.} \end{aligned}$$

8. Given L' 50° 48′ N., λ' 9° 10′ W., C S. 41° W., D 275; required L'' and λ''.

$$L_d = D \cos C.$$

$$\begin{aligned} \log D &= 2.43993 \\ \log \cos C &= \underline{9.87778} \\ \log L_d &= 2.31771 \\ \therefore L_d &= 208' \\ &= \ 3^\circ\ 28' \\ L' &= \underline{50^\circ\ 48'} \\ L'' &= 47^\circ\ 20'\ \text{N.} \\ L_m &= 49^\circ\ 4'. \end{aligned}$$

$$\begin{aligned} 50^\circ\ 48'\ \text{N., Mer. parts} &= 3532.0 \\ 47^\circ\ 20'\ \text{N., Mer. parts} &= \underline{3215.2} \\ \text{Mer. } L_d &= \ 316.8 \end{aligned}$$

$$\lambda_d = \text{Mer. } L_d \tan C.$$

$$\begin{aligned} \log \text{Mer. } L_d &= 2.50079 \\ \log \tan C &= \underline{9.93916} \\ \log \lambda_d &= 2.43995 \\ \lambda_d &= 275' \\ &= \ 4^\circ\ 35' \\ \lambda' &= \underline{\ 9^\circ\ 10'} \\ \lambda'' &= 13^\circ\ 45'\ \text{W.} \end{aligned}$$

9. Given L' 37° 0′ N., L'' 51° 18′ N., λ' 48° 20′ W., D 1027; required λ'' and C.

$$\begin{array}{rl} & 51^\circ\ 18'\text{ N.}\\ & \underline{37^\circ\ \ 0'\text{ N.}}\\ L_d = & 14^\circ\ 18'\\ = & 858'.\end{array}$$

$$\begin{array}{rl}\text{Mer. parts} = & 3579.6\\ \text{Mer. parts} = & \underline{2378.8}\\ \text{Mer. } L_d = & 1200.8\end{array}$$

$$\cos C = \frac{L_d}{D}.$$

$$\begin{array}{rl}\log L_d = & 2.93349\\ \log D = & \underline{3.01157}\\ \log\cos C = & 9.92192\\ C = & \text{N. } 33^\circ\ 20'\text{ W.}\end{array}$$

$$\lambda_d = \text{Mer. } L_d \tan C.$$

$$\begin{array}{rl}\log \text{Mer. } L_d = & 3.07947\\ \log\tan C = & \underline{9.81804}\\ \log\lambda_d = & 2.89751\\ \lambda_d = & 790'\\ = & 13^\circ\ 10'\\ \lambda' = & \underline{48^\circ\ 20'}\\ \lambda'' = & 61^\circ 30'\text{ W. or } 35^\circ 10'\text{ W.}\end{array}$$

10. Given L' 51° 15′ N., L'' 37° 5′ N., λ' 9° 50′ W., C S.W. by S.; required λ'' and D.

$$\begin{array}{rl} & 51^\circ\ 15'\text{ N.}\\ & \underline{37^\circ\ \ 5'\text{ N.}}\\ L_d = & 14^\circ\ 10'\\ = & 850'.\\ C = & 33^\circ\ 45'.\\ D = & L_d \sec C.\end{array}$$

$$\begin{array}{rl}\log L_d = & 2.92942\\ \log\sec C = & \underline{0.08015}\\ \log D = & 3.00957\\ D = & 1022.\end{array}$$

$$\begin{array}{rl}\text{Mer. parts} = & 3574.8\\ \text{Mer. parts} = & \underline{2385.1}\\ \text{Mer. } L_d = & 1189.7\end{array}$$

$$\lambda_d = \text{Mer. } L_d \tan C.$$

$$\begin{array}{rl}\log \text{Mer. } L_d = & 3.07542\\ \log\tan C = & \underline{9.82489}\\ \log\lambda_d = & 2.90031\\ \lambda_d = & 795'\\ = & 13^\circ\ 15'\\ \lambda' = & \underline{\ \ 9^\circ\ 50'}\\ \lambda'' = & 23^\circ\ \ 5'\text{ W.}\end{array}$$

11. Required the course and distance from Toulon to Valencia, by Mercator's sailing:

$$\text{Toulon}\begin{cases} L = 43^\circ\ \ 8'\text{ N.}\\ \lambda = \ \ 5^\circ\ 56'\text{ E.}\end{cases}\qquad \text{Valencia}\begin{cases} L = 39^\circ\ 27'\text{ N.}\\ \lambda = \ \ 0^\circ\ 19'\text{ W.}\end{cases}$$

$$\begin{array}{rl} & 43^\circ\ \ 8'\text{ N.}\\ & \underline{39^\circ\ 27'\text{ N.}}\\ L_d = & 3^\circ\ 41' = 221'\end{array}$$

$$\begin{array}{rl}\text{Mer. parts} = & 2858.3\\ \text{Mer. parts} = & \underline{2565.2}\\ \text{Mer. } L_d = & 293.1\end{array}$$

$$\begin{array}{rl} & 5^\circ\ 56'\text{ E.}\\ & \underline{0^\circ\ 19'\text{ W.}}\\ \lambda_d = & 6^\circ\ 15' = 375'\end{array}$$

$$D = L_d \sec C.$$

$$\begin{array}{rl}\log L_d = & 2.34439\\ \log\sec C = & \underline{0.21050}\\ \log D = & 2.55489\\ D = & 358.8.\end{array}$$

$$\tan C = \frac{\lambda_d}{\text{Mer. } L_d}.$$

$$\begin{array}{rl}\log\lambda_d = & 2.57403\\ \log \text{Mer. } L_d = & \underline{2.46702}\\ \log\tan C = & 10.10701\\ C = & \text{S. } 51^\circ\ 59'\text{ W.}\end{array}$$

12. Required the compass course and distance from Cape East, New Zealand, to San Francisco; variation 14° 20′ E., and deviation 5° 40′ E.:

$$\text{Cape East}\begin{cases}L = 37^\circ\, 40' \text{ S.}\\ \lambda = 178^\circ\, 36' \text{ E.}\end{cases}\qquad \text{San Francisco}\begin{cases}L = 37^\circ\, 48' \text{ N.}\\ \lambda = 122^\circ\, 24' \text{ W.}\end{cases}$$

$$\begin{array}{rl@{\qquad}rl@{\qquad}rl}
 & 37^\circ\, 48' \text{ N.} & \text{Mer. parts} = & 2439.0 & & 178^\circ\, 36' \text{ E.}\\
 & \underline{37^\circ\, 40' \text{ S.}} & \text{Mer. parts} = & \underline{2428.9} & & \underline{122^\circ\, 24' \text{ W.}}\\
L_d = & 75^\circ\, 28' = 4528' & \text{Mer. } L_d = & 4867.9 & \lambda_d = & 59^\circ\, 0' = 3540'
\end{array}$$

$$\begin{array}{rl@{\qquad\qquad}rl}
 & D = L_d \sec C. & & \tan C = \dfrac{\lambda_d}{\text{Mer. } L_d}.\\
\log L_d = & 3.65591 & \log \lambda_d = & 3.54900\\
\log \sec C = & \underline{0.09222} & \log \text{Mer. } L_d = & \underline{3.68734}\\
\log D = & 3.74813 & \log \tan C = & 9.86166\\
D = & 5599. & C = & \text{N. } 36^\circ\, 2' \text{ E.}\\
 & & \text{Var. and dev.} = & \underline{\phantom{\text{N. }}20^\circ\, 0' \text{ L.}}\\
 & & \text{Compass course} = & \text{N. } 16^\circ\, 2' \text{ E.}
\end{array}$$

13. Required the course and distance from Cape Lopatka to Callao:

$$\text{Cape Lopatka}\begin{cases}L = 51^\circ\, 2' \text{ N.}\\ \lambda = 156^\circ\, 50' \text{ E.}\end{cases}\qquad \text{Callao}\begin{cases}L = 12^\circ\, 4' \text{ S.}\\ \lambda = 77^\circ\, 14' \text{ W.}\end{cases}$$

$$\begin{array}{rl@{\qquad}rl@{\qquad}l}
 & 51^\circ\, 2' \text{ N.} & \text{Mer. parts} = & 3554.1 & 156^\circ\, 50' \text{ E.}\\
 & \underline{12^\circ\, 4' \text{ S.}} & \text{Mer. parts} = & \underline{724.6} & \underline{77^\circ\, 14' \text{ W.}}\\
L_d = & 63^\circ\, 6' = 3786' & \text{Mer. } L_d = & 4278.7 & 234^\circ\, 4'
\end{array}$$

$$360^\circ - 234^\circ\, 4' = 125^\circ\, 56' = 7556 \text{ m.} = \lambda_d.$$

$$\begin{array}{rl@{\qquad\qquad}rl}
 & D = L_d \sec C. & & \tan C = \dfrac{\lambda_d}{\text{Mer. } L_d}.\\
\log L_d = & 3.57818 & \log \lambda_d = & 3.87829\\
\log \sec C = & \underline{0.30744} & \text{colog Mer. } L_d = & \underline{6.36869}\\
\log D = & 3.88562 & \log \tan C = & 10.24698\\
D = & 7685. & C = & \text{S. } 60^\circ\, 29' \text{ E.}
\end{array}$$

14. A ship from latitude 20° 40′ N. sails N.E. by N. until she is in latitude 27° 16′ N. Required the distance and difference of longitude.

$$\begin{array}{rl@{\qquad\qquad}rl}
 & 27^\circ\, 16' \text{ N.} & \text{Mer. parts} = & 1691.0\\
 & \underline{20^\circ\, 40' \text{ N.}} & \text{Mer. parts} = & \underline{1259.7}\\
L_d = & 6^\circ\, 36' = 396'. & \text{Mer. } L_d = & 431.3\\
C = & 33^\circ\, 45'. & & \\
 & D = L_d \sec C. & \lambda_d = & \text{Mer. } L_d \tan C.\\
\log L_d = & 2.59770 & \log \text{Mer. } L_d = & 2.63478\\
\log \sec C = & \underline{0.08015} & \log \tan C = & \underline{9.82489}\\
\log D = & 2.67785 & \log \lambda_d = & 2.45967\\
D = & 476.3. & \lambda_d = & 288' = 4^\circ\, 48'.
\end{array}$$

15. A ship from Cape Clear, in latitude 51° 26′ N. and longitude 9° 29′ W., sails S.W. by S. until the distance run is 1022 miles. Find the latitude and longitude in by Mercator's and Middle Latitude Sailings. Which method is preferable?

$D = 1022$
$C = 33° 45'$

$$L_d = D \cos C.$$

$\log D = 3.00945$
$\log \cos C = 9.91985$
$\log L_d = 2.92930$
$L_d = 850'$
$= 14° 10'$
$L' = 51° 26'$
$L'' = 37° 16' \text{ N.}$
$L_m = 44° 21' \text{ N.}$

By Mercator's Sailing,

Mer. parts of 51° 26′ = 3592.4
Mer. parts of 37° 16′ = 2398.8
Mer. $L_d = 1193.6$

$$\lambda_d = \text{Mer. } L_d \tan C.$$

$\log \text{Mer. } L_d = 3.07686$
$\log \tan C = 9.82489$
$\log \lambda_d = 2.90175$
$\lambda_d = 798'$
$= 13° 18'$
$\lambda' = 9° 29'$
$\lambda'' = 22° 47' \text{ W.}$

By Mid. Lat. Sailing,

$$\lambda_d = D \sin C \sec L_m.$$

$\log D = 3.00945$
$\log \sin C = 9.74474$
$\log \sec L_m = 0.14564$
$\log \lambda_d = 2.89983$
$\lambda_d = 794'$
$= 13° 14'$
$\lambda' = 9° 29'$
$\lambda'' = 22° 43' \text{ W.}$

Mercator's Sailing is preferable,
since $C < 45°$.

Exercise VI. Page 351.

1.

C.		*D*	*N.*	*S.*	*E.*	*W.*
S.S.W.	2 pts.	48		44.3		18.4
S.W. by S.	3 pts.	36		29.9		20.
N.E.	4 pts.	24	17		17	
Hence, $L_d = 57.2$ S.				74.2	17	38.4
$= 0° 57'$ S.				17.		17.
$p = 21.4$ W.				57.2		21.4

2.

C.		*D.*	*N.*	*S.*	*E.*	*W.*
S. ½ E.	½ pt.	18		17.9	1.8	
S.W. ½ S.	3½ pts.	37		28.6		23.5
S.S.W. ¼ W.	2¼ pts.	56		50.6		23.9
Hence, $L_d = 97.1$ S.			0	97.1	1.8	47.4
$= 1° 37'$ S.				0.		1.8
$p = 45.6$ W.				97.1		45.6

3.

C.		*D.*	*N.*	*S.*	*E.*	*W.*
S.S.W. ¼ W.	2¼ pts.	43		38.9		18.4
S.S.W. ½ W.	2½ pts.	39		34.4		18.4
S. by W. ½ W.	1½ pts.	27		25.8		7.8
Hence, $L_d = 99.1$ S.			0	99.1	0	44.6
$= 1° 39'$ S.				0.		0.
$p = 44.6$ W.				99.1		44.6

4.

C.	*D.*	*N.*	*S.*	*E.*	*W.*
N. 25° W.	16.4	14.9		0.1	6.9
N. 8° E.	7.8	7.7		1.	
N. 19° E.	13.7	13.0		4.5	
N. 76° E.	39.6	9.6		38.4	
Hence, $L_d = 45.2$ N.		45.2	0	44.	6.9
$= 0° 45'$ N.		0.		6.9	
$p = 37.1$ E.		45.2		37.1	

5.

C.		*D.*	*N.*	*S.*	*E.*	*W.*
W.N.W. ¼ W.	6¼ pts.	21	7.1			19.8
N.N.E. ¾ E.	2¾ pts.	9	7.7		4.6	
N. by E. ¾ E.	1¾ pts.	9	8.5		3.0	
S.S.W. ¼ W.	2¼ pts.	30		27.1		12.8
Hence, $L_d = 3.8$ S.			23.3	27.1	7.6	32.6
$= 0° 4'$ S.				23.3		7.6
$p = 25.0$ W.				3.8		25.0

6.

C.	*D.*	*N.*	*S.*	*E.*	*W.*
S. 83° W.	23		2.8		22.8
S. 48° E.	25.2		16.9	18.7	
N. 48° W.	27.1	18.1			20.1
N. 36° W.	2.1	17.			12.3
Hence, $L_d = 15.4$ N.		35.1	19.7	18.7	55.2
$= 0° 15'$ N.		19.7			18.7
∴ $p = 36.5$ W.		15.4			36.5

7.

C.	*D.*	*N.*	*S.*	*E.*	*W.*
S. 17° E.	48		45.9	14.0	
S. 45° W.	19		13.4		13.4
N. 36° W.	18	14.6			10.6
N. 41° W.	50	37.7			32.8
E. (90°).	36	.0	.0	36.0	
		52.3	59.3	50.0	56.8
Hence, $L_d = 0° 7'$ S.			52.3		50.
$p = 6.8$ W.			7.		6.8

8.

C.		*D.*	*N.*	*S.*	*E.*	*W.*
N.N.E.	2 pts.	31	28.6		11.9	
E.N.E.	6 pts.	35	13.4		32.3	
E. by S.	7 pts.	36		7	35.3	
S.S.E.	2 pts.	51		47.1	19.5	
S. by E.	1 pt.	60		58.8	11.7	
Hence, $L_d = 70.9$			42.0	112.9	110.7	
$= 1^\circ 11'$ S.				42.0		
$p = 110.7$ E.				70.9		

9.

C.	*D.*	*N.*	*S.*	*E.*	*W.*
S. 44° E.	69		49.6	47.9	
S. 85° E.	68		5.9	67.7	
S. 27° E.	25		22.3	11.3	
N. 37° W.	5	4.0			3.0
N. 20° W.	13	12.2			4.4
Hence, $L_d = 61.6$		16.2	77.8	126.9	7.4
$= 1^\circ 2'$ S.			16.2	7.4	
$p = 119.5$ E.			61.6	119.5	

EXERCISE VII. PAGE 359.

1. First course: N.N.E. = 2 points R. of N. = N. 22° 30′ E., 31.4 m.
Second course: E.N.E. = 6 points R. of N. = N. 67° 30′ E., 35 m.
Third course: E. by S. = 7 points L. of S. = S. 78° 45′ E., 36.1 m.
Fourth course: S.S.E. = 2 points L. of S. = S. 22° 30′ E., 50.9 m.
Tide course: = 1 point L. of S. = S. 11° 15′ E., 60 m.

THE TRAVERSE.

C.		*D.*	*N.*	*S.*	*E.*	*W.*
N.N.E.	2 pts.	31.4	29.		12.	
E.N.E.	6 pts.	35.	13.4		32.3	
E. by S.	7 pts.	36.1		7.	35.4	
S.S.E.	2 pts.	50.9		47.	19.5	
S. by E.	1 pt.	60.		58.8	11.7	
$L_d = 70.4' =$ 1° 10′ S. $L' =$ 46° 28′ N. $L'' =$ 45° 18′ N.			42.4	112.8 42.4 70.4	110.9	$= p.$

$p = 110.9$

$L_m = 45° 53'$

$\lambda_d = p \sec L_m.$

$\log p = 2.04493$

$\log \sec L_m = \underline{0.15731}$

$\log \lambda_d = 2.20224$

$\lambda_d = 159.3'$

$= 2° 39'$ E.

$\lambda' = \underline{22° 18' \text{ W.}}$

$\lambda'' = 19° 39'$ W.

2. ***First course:***

S. by W. = 1 pt. R. of S. = 11° 15′ R. of S.

Variation . . . . . . 12° 20′ L. of S.

True course . . . . . . 1° 5′ L. of S.

Hence, course and distance S. 1° 5′ E., 40 m.

Second course (starboard tack):

S.W. by S. = 3 pts. R. of S.

Leeway = 1 pt. L.

2 pts. R. of S. = 22° 30′ R. of S.

Variation . . . . . . 12° 20′ L.

True course . . . . . . 10° 10′ R. of S.

Hence, course and distance S. 10° 10′ W., 69.6 m.

Third course:

S.W. by W. = 5 pts. R. of S. = 56° 15′ R. of S.

Variation . . . . . . 12° 20′ L.

True course . . . . . . 43° 55′ R. of S.

Hence, course and distance S. 43° 55′ W., 58.5 m.

Current course:

W.S.W. = 6 pts. R. of S. = 67° 30′ R. of S.

Variation . . . . . . 12° 20′ L.

True course . . . . . . 55° 10′ R. of S.

Hence, course and distance S. 55° 10′ W., 36 m.

THE TRAVERSE.

C.	*D.*	*N.*	*S.*	*E.*	*W.*
S. 1° E.	40.		40.	0.7	
S. 10° W.	69.6		68.6		12.1
S. 44° W.	58.5		42.1		40.7
S. 55° W.	36.		20.6		29.5
$L_d = 171.3' = 2° 51'$ S.			171.3	0.7	82.3
$L' = 33° 40'$ N.					0.7
$L'' = 30° 49'$ N.				$p =$	81.6

$p = 81.6$

$L_m = 32° 15'$

$\lambda_d = p \sec L_m.$

$\log p = 1.91169$

$\log \sec L_m = \underline{0.07277}$

$\log \lambda_d = 1.98446$

$\lambda_d = 96'$

$= 1° 36'$ W.

$\lambda' = \underline{16° 20'}$ W.

$\lambda'' = 17° 56'$ W.

3. ***First course*** **(starboard tack):**

N. by E. = 1 pt. R. of N.
Leeway = 1 pt. L.
0 = due north = 0°
Variation . . . . . W. 13° 30′ L.
13° 30′ L. of N.

Hence, course and distance N. 13° 30′ W., 37.7 m.

Second course (starboard tack):

N. = 0° pt.
Leeway = 1 pt. L.
1 pt. L. = 11° 15′ L. of N.
Variation . . . . . W. 13° 30′ L.
24° 45′ L. of N.

Hence, course and distance N. 24° 45′. W., 38.7 m.

Third course (starboard tack):

N.N.W. = 2 pts. L. of N.
Leeway = 1 pt. L.
3 pts. L. of N. = 33° 45′ L. of N.
Variation . . . . . W. 13° 30′ L.
47° 15′ L. of N.

Hence, course and distance N. 47° 15′ W., 76.5 m.

Current course:

W.N.W. = 6 pts. L. of N. = 67° 30′ L. of N.
Variation . . . . . 13° 30′ L.
81° 0′ L. of N.

Hence, course and distance N. 81° W., 12 m.

THE TRAVERSE.

C.	*D.*	*N.*	*S.*	*E.*	*W.*
N. 14° W.	37.7	36.6			9.2
N. 25° W.	38.7	35.			16.4
N. 47° W.	76.5	52.1			56.
N. 81° W.	12.	1.9			11.9
$L_d = 125.6' =$ 2° 6′ N. $L' =$ 19° 30′ S. $L'' =$ 17° 24′ S.		125.6		$p =$	93.5

$p = 93.5$
$L_m = 18° 27'$

$\lambda_d = p \sec L_m.$
$\log p = 1.97081$
$\log \sec L_m = 0.02296$
$\log \lambda_d = 1.99377$

$\lambda_d = 99'$
$= 1° 39'$ W.
$\lambda' = 0° 10'$ E.
$\lambda'' = 1° 29'$ W.

4. *Departure course* (the opposite of W.S.W.):

E.N.E. The ship's head S.E. by E.; the deviation is the same as for the first course.

E.N.E. = 6 pts. R. of N. = 67° 30′ R. of N.
Variation and deviation . . 17° L.
50° 30′ R. of N.

Hence, course and distance N. 50° 30′ E., 18 m.

First course:

S.E. by E. = 5 pts. L. of S. = 56° 15′ L. of S.
Variation and deviation . . . 17° L.
True course . . . . . . 73° 15′ L. of S.
Hence, course and distance S. 73° 15′ E., 52 m.

Second course (port tack):

S.E. = 4 pts. L. of S.
Leeway = ½ pt. R.
3½ pts. L. of S. = 39° 22′ L. of S.
Variation and deviation . . . 19° L.
True course . . . . . . 58° 22′ L. of S.
Hence, course and distance S. 58° 22′ E., 43 m.

Third course (starboard tack):

E. by N. = 7 pts. R. of N.
Leeway = 1 pt. L.
6 pts. R. of N. = 67° 30′ R. of N.
Variation and deviation . . 11° L.
True course . . . . . . 56° 30′ R. of N.
Hence, course and distance N. 56° 30′ E., 36 m.

Fourth course (starboard tack):

E.N.E. = 6 pts. R. of N.
Leeway = 1½ pts. L.
4½ pts. R. of N. = 50° 37′ R. of N.
Variation and deviation . . 13° L.
True course . . . . . . 37° 37′ R. of N.
Hence, course and distance N. 37° 37′ E., 27 m.

Fifth course (port tack):

S.S.E. = 2 pts. L. of S.
Leeway = 2 pts. R.
0 pts. = due south = 0°
Variation and deviation . . 21° L.
True course . . . . . . 21° L. of S.
Hence, course and distance S. 21° E., 24 m.

Sixth course (port tack):

S.E. by S. = 3 pts. L. of S.
Leeway = 1¼ pts. R.
1¾ pts. L. of S. = 19° 41′ L. of S.
Variation and deviation . . . 20° L.
True course . . . . . . 39° 41′ L. of S.
Hence, course and distance S. 39° 41′ E., 29 m.

Current course:

S. by E. = 1 pt. = L. of S. = 11° 15′ L. of S.
Variation . . . . . . 28° L.
39° 15′ L. of S.
Hence, course and distance S. 39° 15′ E., 12 m.

THE TRAVERSE.

C.	*D.*	*N.*	*S.*	*E.*	*W.*
N. 51° E.	18	11.3		14.0	
S. 73° E.	52		15.2	49.7	
S. 58° E.	43		22.8	36.5	
N. 57° E.	36	19.6		30.2	
N. 38° E.	27	21.3		16.6	
S. 21° E.	24		22.4	8.6	
S. 40° E.	29		22.2	18.6	
S. 39° E.	12		9.3	7.6	
$L_d =$ 40′ S. $L' =$ 47° 31′ N. $L'' =$ 46° 51′ N.		52.2	91.9 52.2 39.7	181.8	$= p$.

$p = 181.8$
$L_m = 47° 11'$

$\lambda_d = p \sec L_m$.
$\log p = 2.25959$
$\log \sec L_m = 0.16771$
$\log \lambda_d = 2.42730$

$\lambda_d = 267'$
$= 4° 27'$ E.
$\lambda' = 52° 33'$ W.
$\lambda'' = 48° 6'$ W.

5. *Departure course* (the opposite of W. by S. ¼ S.):

E. by N. ¼ N. = 6¾ pts. R. of N.
= 75° 56′ R. of N.
Variation and deviation . . . 34° R.
109° 56′ R. of N.

Hence, course and distance S. 70° E., 17 m.

First course (port tack):

S.S.E. = 2 pts. L. of S.
Leeway = 2¼ pts. R.
¼ pt. R. of S. = 2° 49′ R. of S.
Variation and deviation . . . 34° R.
36° 49′ R. of S.

Hence, course and distance S. 37° W., 21 m.

Second course (starboard tack):

S.S.W. ¼ W. = 2¼ pts. R. of S.
Leeway = 2½ pts. L.
¼ pt. L. of S. = 2° 49′ L. of S.
Variation and deviation . . . 27° R.
24° 11′ R. of S.

Hence, course and distance S. 24° W., 20 m.

Third course (port tack):

W.S.W. = 6 pts. R. of S.
Leeway = 2½ pts. R.
8½ pts. R. of S. = 7½ pts. L. of N.
= 84° 22′ L. of N.
Variation and deviation . . . 22° R.
True course . . . . . . 62° 22′ L. of N.

Hence, course and distance N. 62° W., 24 m.

Fourth course (starboard tack):

W. ½ N. = 7½ pts. L. of N. = 84° 22′ L. of N.
Variation and deviation . . . 20° R.
True course . . . . . . 64° 22′ L. of N.

Hence, course and distance N. 64° W., 26 m.

Fifth course (starboard tack):

East = 8 pts. R. of N.
Leeway = 2½ pts. L.
5½ pts. R. of N. = 61° 52′ R. of N.
Variation and deviation . . 41° R.
True course . . . . . . 102° 52′ R. of N.

Hence, course and distance S. 77° E., 19 m.

Sixth course (starboard tack):

E.S.E. = 6 pts. L. of S. = 67° 30′ L. of S.
Variation and deviation . . . 40° R.
True course . . . . . . 27° 30′ L. of S.

Hence, course and distance S. 28° E., 18 m.

Current course: N.N.E. = 2 pts. R. of N.
= 22° 30′ R. of N.
Variation . . . . . . 31° R.
True course . . . . . . 53° 30′ R. of N.

Hence, course and distance N. 54° E., 21 m.

THE TRAVERSE.

C.	*D.*	*N.*	*S.*	*E.*	*W.*
S. 70° E.	17		5.8	16.0	
S. 37° W.	21		16.8		12.6
S. 24° W.	20		18.3		8.1
N. 62° W.	24	11.3			21.2
N. 64° W.	26	11.4			23.4
S. 77° E.	19		4.3	18.5	
S. 28° E.	18		15.9	8.5	
N. 54° E.	21	12.3		17.	
$L_d = \quad 26'$ S. $L' = 62^\circ\ 0'$ N. $L'' = 61^\circ\ 34'$ N.		35.0	61.1 35.0	60.	65.3 60.
			26.1		5.3

$p \ = 5.3$ $\qquad$ $\lambda_d = p \sec L_m.$ $\qquad$ $\lambda_d = \quad 11'$ W.

$L_m = 61^\circ\ 47'$ $\qquad$ $\log p = 0.72428$ $\qquad$ $\lambda' = 150^\circ\ 0'$ E.

$\log \sec L_m = \underline{0.32532}$ $\qquad$ $\lambda'' = 149^\circ\ 49'$ E.

$\log \lambda_d = 1.04960$

6. *Departure course* (the opposite of N. ¾ W.):

S. ¾ E. = ¾ of a pt. = 8° 26′ L. of S.
Variation and deviation . . . 8° R.
0° 26′ L. of S.

Hence, course and distance S., 19 m.

First course (port tack):

S.W. ½ W. = 4½ pts. R. of S. = 50° 37′ R. of S.
Variation and deviation . . . 8° R.
True course . . . . . . 58° 37′ R. of S.
Hence, course and distance S. 59° W., 58 m.

Second course (starboard tack):

N. ¾ E. = ¾ pt. R. of N.
Leeway = 3¼ pts. L.
2½ pts. L. of N. = 28° 7′ L. of N.
Variation and deviation . . . 17° R.
True course . . . . . . 11° 7′ L. of N.
Hence, course and distance N. 11° W., 15 m.

Third course (starboard tack):

S.E. ½ E. = 1½ pts. L. of S.
Leeway = 2¾ pts. L.

4¼ pts. L. of S. = 47° 48′ L. of S.
Variation and deviation . . . 20° R.

True course . . . . . . 27° 48′ L. of S.
Hence, course and distance S. 28° E., 9 m.

Fourth course (port tack):

W. by S. = 7 pts. R. of S.
Leeway = ¼ pt. R.

7¼ pts. R. of S. = 81° 33′ R. of S.
Variation and deviation . . . 0°

True course . . . . . . 81° 33′ R. of S.
Hence, course and distance S. 82° W., 50 m.

Fifth course (starboard tack):

E.N.E. = 6 pts. R. of N.
Leeway = 2½ pts. L.

3½ pts. R. of N. = 39° 22′ R. of N.
Variation and deviation . . . 33° R.

True course . . . . . . 72° 22′ R. of N.
Hence, course and distance N. 72° E., 12 m.

Sixth course (port tack):

S.S.W. ½ W. = 2½ pts. R. of S.
Leeway = 1¾ pts. R.

4¼ pts. R. of S. = 47° 48′ R. of S.
Variation and deviation . . . 10° R.

True course . . . . . . 57° 48′ R. of S.
Hence, course and distance S. 58° W., 22 m.

Current course:

S.W. ¼ W. = 4¼ pts. R. of S. = 47° 48′ R. of S.
Variation . . . . . . 14° R.

True course . . . . . . 61° 48′ R. of S.
Hence, course and distance S. 62° W., 42 m.

THE TRAVERSE.

C.	*D.*	*N.*	*S.*	*E.*	*W.*
S.	19		19.0		
S. 59° W.	58		29.9		49.7
N. 11° W.	15	14.7			2.9
S. 28° E.	9		7.9	4.2	
S. 82° W.	50		7.0		49.5
N. 72° E.	12	3.7		11.4	
S. 58° W.	22		11.7		18.7
S. 62° W.	42		19.7		37.1
$L_d = 77' = 1° 17'$ S.		18.4	95.2	15.6	157.9
$L' = 50° 12'$ S.			18.4		15.6
$L'' = 51° 29'$ S.			76.8		142.3

$p = 142.3$
$L_m = 50° 51'$

$\lambda_d = p \sec L_m.$
$\log p = 2.15320$
$\log \sec L_m = 0.19973$
$\log \lambda_d = 2.35293$

$\lambda_d = 225'$
$= 3° 45'$ W.
$\lambda' = 179° 40'$ W.
$\lambda'' = 176° 35'$ E.

7. ***First course:***

S.E. = 4 pts. L. of S.
Variation and deviation . . 1¼ pts. L.
True course . . . . . 5¼ pts. L. of S.
Hence, course and distance S. 5¼ pts. E., 27.8 m.

Second course:

E.S.E. ¼ E. = 6¼ pts. L. of S.
Variation . . . . . 1¾ pts. L.
True course . . . . . 8 pts. L. of S. = due east.
Hence, course and distance E. 75.2 m.

Third course:

E. = 8 pts. R. of N.
Variation and deviation . . 1¼ pts. L.
True course . . . . . 6¾ pts. R. of N.
Hence, course and distance N. 6¾ pts. E., 8.7 m.

THE TRAVERSE.

C.	*D.*	*N.*	*S.*	*E.*	*W.*
S. 5½ pts. E.	27.8		13.1	24.5	
E.	75.2			75.2	
N. 6¾ pts. E.	8.7	2.1		8.4	
$L_d =$ 11′ S. $L' =$ 36° 42′ N. $L'' =$ 36° 31′ N.		2.1	13.1 2.1	108.1	$= p.$
			11.0		

$p = 108.1$
$L_m = 36° 37'$

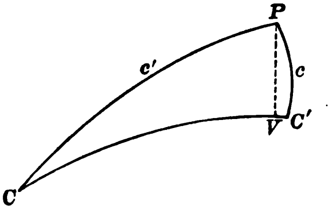
$\lambda_d = p \sec L_m.$

$\log p = 2.03383$
$\log \sec L_m = 0.09548$
$\log \lambda_d = 2.12931$

$\lambda_d = 135'$
$= 2° 15'$ E.
$\lambda' = 4° 25'$ W.
$\lambda'' = 2° 10'$ W.

EXERCISE VIII. PAGE 374.

1. Find the elements (initial courses, distance, and latitude and longitude of the vertex) of the great circle track between the Lizard, in latitude 49° 58′ N., longitude 5° 12′ W., and the Bermuda Islands, in latitude 32° 18′ N., longitude 64° 50′ W.



Referring to the triangle *CPC′*,

Lat. $C' = 49° 58'$ N.
Long. $C' = 5° 12'$ W.
Lat. $C = 32° 18'$ N.
Long. $C = 64° 50'$ W.

$c = 90° - 49° 58' = 40° \ 2'.$
$c' = 90° - 32° 18' = 57° 42'.$
$\lambda_d = 64° 50' - 5° 12' = 59° 38'.$

To find the initial courses:

$$\tan \tfrac{1}{2}(C' + C) = \frac{\cos \frac{1}{2}(c' - c)}{\cos \frac{1}{2}(c' + c)} \cot \tfrac{1}{2}\lambda_d.$$

$$\tan \tfrac{1}{2}(C' - C) = \frac{\sin \frac{1}{2}(c' - c)}{\sin \frac{1}{2}(c' + c)} \cot \tfrac{1}{2}\lambda_d.$$

$$\begin{array}{llll}
\frac{1}{2}(c'-c) = 8^\circ\ 50', & \log\cos = 9.99482 & & \log\sin = 9.18628 \\
\frac{1}{2}(c'+c) = 48^\circ\ 52', & \text{colog}\cos = 0.18190 & & \text{colog}\sin = 0.12310 \\
\frac{1}{2}\lambda_d = 29^\circ\ 48', & \log\cot = \underline{10.24178} & & \log\cot = \underline{10.24178} \\
 & \log\tan\frac{1}{2}(C'+C) = 10.41850 & & \log\tan\frac{1}{2}(C'-C) = 9.55116
\end{array}$$

$$\frac{1}{2}(C'+C) = 69^\circ\ 7' \qquad \frac{1}{2}(C'-C) = 19^\circ\ 35'.$$

$$\frac{1}{2}(C'-C) = \underline{19^\circ\ 35'}$$

$\therefore C' =$ N. 88° 42′ W. = course from Lizard.

$C =$ N. 49° 32′ E. = course from Bermudas.

To find the distance:

$$\cos\frac{1}{2}D = \frac{\cos\frac{1}{2}(c+c')}{\cos\frac{1}{2}(C+C')}\sin\frac{1}{2}\lambda_d.$$

$$\begin{array}{ll}
\frac{1}{2}(c+c') = 48^\circ\ 52', & \log\cos = 9.81810 \\
\frac{1}{2}(C+C') = 69^\circ\ 7', & \text{colog}\cos = 0.44798 \\
\frac{1}{2}\lambda_d = 29^\circ\ 49', & \log\sin = \underline{9.69655} \\
 & \log\cos\frac{1}{2}D = 9.96263
\end{array}$$

$\frac{1}{2}D = 23^\circ\ 26'.$

$D = 46^\circ\ 52' = 2812$ m.

To find L of V.

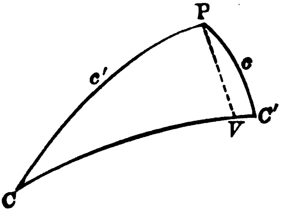
$$\begin{aligned}
\sin PV &= \sin c' \sin C. \\
\log\sin c' &= 9.92699 \\
\log\sin C &= \underline{9.88126} \\
\log\sin PV &= 9.80825 \\
PV &= 40^\circ\ 1'. \\
L \text{ of } V &= 90^\circ - 40^\circ\ 1' \\
&= 49^\circ\ 59' \text{ N.}
\end{aligned}$$

To find λ of V.

$$\begin{aligned}
\cot CPV &= \cos c' \tan C. \\
\log\cos c' &= 9.72783 \\
\log\tan C &= \underline{10.06901} \\
\log\cot CPV &= 9.79684 \\
CPV &= 57^\circ\ 56'. \\
\lambda \text{ of } C &= 64^\circ\ 50' \\
& \quad \underline{57^\circ\ 56'} \\
\lambda \text{ of } V &= 6^\circ\ 54' \text{ W.}
\end{aligned}$$

2. Find the elements of the great circle track between Boston (Minot's Ledge light-house) in latitude 42° 16′ N., longitude 70° 46′ W., and Cape Clear, in latitude 51° 26′ N., longitude 9° 29′ W. [Take $\frac{1}{2}\lambda_d = 30^\circ\ 39'$.]

Lat. C' = 51° 26′ N.
Long. C' = 9° 29′ W.
Lat. C = 42° 16′ N.
Long. C = 70° 46′ W.



$c = 90^\circ - 51^\circ\ 26' = 38^\circ\ 34'.$

$c' = 90^\circ - 42^\circ\ 16' = 47^\circ\ 44'.$

$\lambda_d = 70^\circ\ 46' - 9^\circ\ 26' = 61^\circ\ 17'.$

$$\tan\frac{1}{2}(C'+C) = \frac{\cos\frac{1}{2}(c'-c)}{\cos\frac{1}{2}(c'+c)}\cot\frac{1}{2}\lambda_d.$$

$$\tan\frac{1}{2}(C'-C) = \frac{\sin\frac{1}{2}(c'-c)}{\sin\frac{1}{2}(c'+c)}\cot\frac{1}{2}\lambda_d.$$

$$\begin{array}{lll}
\tfrac{1}{2}(c'-c) = 4^\circ\, 35', & \log\cos = 9.99861 & \log\sin = 8.90260 \\
\tfrac{1}{2}(c'+c) = 43^\circ\, 9', & \text{colog}\cos = 0.13694 & \text{colog}\sin = 0.16500 \\
\tfrac{1}{2}\lambda_d = 30^\circ\, 39', & \log\cot = \underline{10.22726} & \log\cot = \underline{10.22726} \\
 & \log\tan\tfrac{1}{2}(C'+C) = 10.36281 & \log\tan\tfrac{1}{2}(C'-C) = 9.29486 \\
 & \tfrac{1}{2}(C'+C) = 66^\circ\, 33' & \tfrac{1}{2}(C'-C) = 11^\circ\, 9'. \\
 & \tfrac{1}{2}(C'-C) = \underline{11^\circ\, 9'} &
\end{array}$$

$\therefore C' = \text{N. } 77^\circ\, 42' \text{ W.} =$ course from Cape Clear.

$C = \text{N. } 55^\circ\, 24' \text{ E.} =$ course from Boston.

To find the distance:

$$\cos\tfrac{1}{2}D = \frac{\cos\tfrac{1}{2}(c'+c)}{\cos\tfrac{1}{2}(C'+C)}\sin\tfrac{1}{2}\lambda_d.$$

$$\begin{array}{ll}
\tfrac{1}{2}(c'+c) = 43^\circ\, 9', & \log\cos = 9.86306 \\
\tfrac{1}{2}(C'+C) = 66^\circ\, 33', & \text{colog}\cos = 0.40017 \\
\tfrac{1}{2}\lambda_d = 30^\circ\, 39', & \log\sin = \underline{9.70739} \\
 & \log\cos\tfrac{1}{2}D = 9.97062
\end{array}$$

$\therefore \tfrac{1}{2}D = 30^\circ\, 50'.$

$D = 41^\circ\, 40' = 2500$ m.

To find L of V.

$\sin PV = \sin c' \sin C.$

$\log\sin c' = 9.86924$

$\log\sin C = \underline{9.91547}$

$\log\sin PV = 9.78471$

$\therefore PV = 37^\circ\, 32'.$

L of $V = 90^\circ - 37^\circ\, 32'$

$= 52^\circ\, 28'$ N.

To find λ of V.

$\cot CPV = \cos c' \tan C.$

$\log\cos c' = 9.82775$

$\log\tan C = \underline{10.16124}$

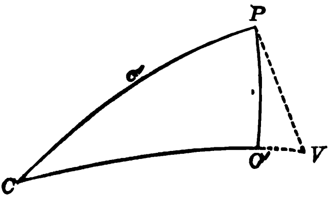
$\log\cot CPV = 9.98899$

$CPV = 45^\circ\, 44'$

λ of $C = \underline{70^\circ\, 46'}$

λ of $V = 25^\circ\, 2'$ W.

3. Find the elements of the great circle track between Vancouver Island, in latitude 50° N., longitude 128° W., and Honolulu, in latitude 21° 18′ N., longitude 157° 52′ W.



Lat. $C' = 50^\circ$ N.

Long. $C' = 128^\circ$ W.

Lat. $C = 21^\circ\, 18'$ N.

Long. $C = 157^\circ\, 52'$ W.

$c = 90^\circ - 50^\circ = 40^\circ.$

$c' = 90^\circ - 21^\circ\, 18' = 68^\circ\, 42'.$

$\lambda_d = 157^\circ\, 52' - 128^\circ = 29^\circ\, 52'.$

$$\tan\tfrac{1}{2}(C'+C) = \frac{\cos\tfrac{1}{2}(c'-c)}{\cos\tfrac{1}{2}(c'+c)}\cot\tfrac{1}{2}\lambda_d.$$

$$\tan\tfrac{1}{2}(C'-C) = \frac{\cos\tfrac{1}{2}(c'-c)}{\sin\tfrac{1}{2}(c'+c)}\cot\tfrac{1}{2}\lambda_d.$$

$$\begin{array}{llll}
\frac{1}{2}(c' - c) = 14^\circ\ 21', & \log\cos = 9.98623 & & \log\sin = 9.39418 \\
\frac{1}{2}(c' + c) = 54^\circ\ 21', & \text{colog}\cos = 0.23446 & & \text{colog}\sin = 0.09013 \\
\frac{1}{2}\lambda_d \quad = 14^\circ\ 56', & \log\cot = \underline{10.57397} & & \log\tan = \underline{10.57397} \\
& \log\tan\frac{1}{2}(C' + C) = 10.79466 & & \log\tan\frac{1}{2}(C' - C) = 10.05828 \\
& \frac{1}{2}(C' + C) = 80^\circ\ 53' & & \frac{1}{2}(C' - C) = 48^\circ\ 50'. \\
& \frac{1}{2}(C' - C) = \underline{48^\circ\ 50'} & & \\
& C' = 129^\circ\ 43' & &
\end{array}$$

$= \text{S. } 50^\circ\ 17' \text{ W.} =$ course from Vancouver.

$C = \text{N. } 32^\circ\ 3' \text{ E.} =$ course from Honolulu.

To find the distance:

$$\cos\frac{1}{2}D = \frac{\cos\frac{1}{2}(c' + c)}{\cos\frac{1}{2}(C' + C)}\sin\frac{1}{2}\lambda_d.$$

$$\begin{array}{llll}
\frac{1}{2}(c' + c) = 54^\circ\ 21', & & \log\cos & = 9.76554 \\
\frac{1}{2}(C' + C) = 80^\circ\ 53', & & \text{colog}\cos & = 0.80012 \\
\frac{1}{2}\lambda_d \quad = 14^\circ\ 56', & & \log\sin & = \underline{9.41110} \\
& & \log\cos\frac{1}{2}D & = 9.97676
\end{array}$$

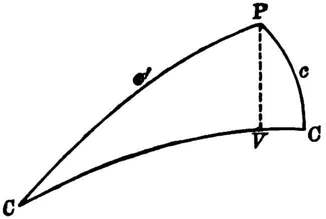
$$\frac{1}{2}D = 18^\circ\ 34'.$$

$$D = 37^\circ\ 8' = 2228 \text{ m.}$$

$$\begin{array}{lll}
\text{To find } L \text{ of } V. & & \text{To find } \lambda \text{ of } V. \\
\sin PV = \sin c' \sin C. & & \cot CPV = \cos c' \tan C. \\
\log\sin c' = 9.96927 & & \log\cos c' = 9.56020 \\
\log\sin C = \underline{9.72482} & & \log\tan C = \underline{9.79663} \\
\log\sin PV = 9.69409 & & \log\cot CPV = 9.35683 \\
PV = 29^\circ\ 38'. & & CPV = 77^\circ\ 11' \\
L \text{ of } V = 90^\circ - 29^\circ\ 38' & & \lambda \text{ of } C = \underline{157^\circ\ 52'} \\
\quad = 60^\circ\ 22' \text{ N.} & & \lambda \text{ of } V = 80^\circ\ 41' \text{ W.}
\end{array}$$

4. Find the elements of the great circle track between Cape Clear, in latitude 51° 26′ N., longitude 9° 29′ W., and Sandy Hook, in latitude 40° N., longitude 74° W.

Lat. $C' = 51^\circ\ 26'$ N.
Long. $C' = 9^\circ\ 29'$ W.
Lat. $C = 40^\circ$ N.
Long. $C = 74^\circ$ W.



$c = 90^\circ - 51^\circ\ 26' = 38^\circ\ 34'.$

$c' = 90^\circ - 40^\circ = 50^\circ.$

$\lambda_d = 74^\circ - 9^\circ\ 29' = 64^\circ\ 31'.$

$$\tan\frac{1}{2}(C + C') = \frac{\cos\frac{1}{2}(c - c')}{\cos\frac{1}{2}(c + c')}\cot\frac{1}{2}\lambda_d.$$

$$\tan\frac{1}{2}(C - C') = \frac{\sin\frac{1}{2}(c - c')}{\sin\frac{1}{2}(c + c')}\cot\frac{1}{2}\lambda_d.$$

$$\begin{array}{lll}
\tfrac{1}{2}(c'-c) = 5^\circ\ 43', & \log\cos = 9.99784 & \log\sin = 8.99839 \\
\tfrac{1}{2}(c+c') = 44^\circ\ 17', & \text{colog}\cos = 0.14515 & \text{colog}\sin = 0.15612 \\
\tfrac{1}{2}\lambda_d = 32^\circ\ 16', & \log\cot = \underline{10.19972} & \log\cot = \underline{10.19972} \\
 & \log\tan\tfrac{1}{2}(C+C'') = 10.34271 & \log\tan\tfrac{1}{2}(C'-C) = 9.354[illegible] \\
 & \tfrac{1}{2}(C+C'') = 65^\circ\ 34' & \tfrac{1}{2}(C'-C) = 12^\circ\ 44. \\
 & \tfrac{1}{2}(C''-C) = \underline{12^\circ\ 44'} &
\end{array}$$

$C'' =$ N. 78° 18′ W. = course from Cape Clear.

$C =$ N. 52° 50′ E. = course from Sandy Hook.

To find the distance:

$$\cos\tfrac{1}{2}D = \frac{\cos\tfrac{1}{2}(c+c')}{\cos\tfrac{1}{2}(C+C')}\sin\tfrac{1}{2}\lambda_d.$$

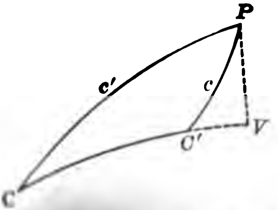
$$\begin{array}{ll}
\tfrac{1}{2}(c+c') = 44^\circ\ 17', & \log\cos = 9.85485 \\
\tfrac{1}{2}(C+C') = 65^\circ\ 34', & \text{colog}\cos = 0.38338 \\
\tfrac{1}{2}\lambda_d = 32^\circ\ 16', & \log\sin = \underline{9.72743} \\
 & \log\cos\tfrac{1}{2}D = 9.96566
\end{array}$$

$\tfrac{1}{2}D = 22^\circ\ 29'.$

$D = 44^\circ\ 58' = 2698$ m.

To find L of V.	To find λ of V.
$\sin PV = \sin c \sin C'.$	$\cot CPV = \cos c' \tan C.$
$\log\sin c = 9.79478$	$\log\cos c' = 9.80807$
$\log\sin C' = \underline{9.99088}$	$\log\tan C = \underline{10.12026}$
$\log\sin PV = 9.78566$	$\log\cot CPV = 9.92833$
$PV = 37^\circ\ 37'.$	$CPV = 49^\circ\ 42'$
L of $V = 90^\circ\ 00' - 37^\circ\ 37'$	λ of $C = \underline{74^\circ\ \ 0'}$
$= 52^\circ\ 23'$ N.	λ of $V = 24^\circ\ 18'$ W.

5. Find the elements of the great circle track between Lizard Light, in latitude 49° 58′ N., longitude 5° 12′ W., and Cape Frio, in latitude 23° S., longitude 42° W.



Lat. $C' = 49^\circ\ 58'$ N.
Long. $C' = 5^\circ\ 12'$ W.
Lat. $C = 23^\circ$ S.
Long. $C = 42^\circ$ W.
$c = 90^\circ - 49^\circ\ 58' = 40^\circ\ 2'.$
$c' = 90^\circ + 23^\circ = 113^\circ.$
$\lambda_d = 42^\circ - 5^\circ\ 12' = 36^\circ\ 48'.$

$$\tan\tfrac{1}{2}(C'+C) = \frac{\cos\tfrac{1}{2}(c'-c)}{\cos\tfrac{1}{2}(c'+c)}\cot\tfrac{1}{2}\lambda_d$$

$$\tan\tfrac{1}{2}(C'-C) = \frac{\sin\tfrac{1}{2}(c'-c)}{\sin\tfrac{1}{2}(c'+c)}\cot\tfrac{1}{2}\lambda_d.$$

$$
\begin{array}{llll}
\tfrac{1}{2}(c'-c) = 36^\circ\,29', & \log\cos = 9.90527 & & \log\sin = 9.77421 \\
\tfrac{1}{2}(c'+c) = 76^\circ\,31', & \text{colog}\cos = 0.63234 & & \text{colog}\sin = 0.01214 \\
\tfrac{1}{2}\lambda_d = 18^\circ\,24', & \log\cot = \underline{10.47801} & & \log\cot = \underline{10.47801} \\
 & \log\tan\tfrac{1}{2}(C'+C) = 11.01562 & & \log\cot\tfrac{1}{2}(C'-C) = 10.26436
\end{array}
$$

$$\tfrac{1}{2}(C'+C) = 84^\circ\,29' \qquad\qquad \tfrac{1}{2}(C'-C) = 61^\circ\,27'.$$

$$\tfrac{1}{2}(C'-C) = \underline{61^\circ\,27'}$$

$C' = \text{S. } 34^\circ\ \ 4'$ W. = course from Lizard Light.

$C\ = \text{N. } 23^\circ\ \ 2'$ E. = course from Cape Frio.

To find the distance:

$$\cos\tfrac{1}{2}D = \frac{\cos\tfrac{1}{2}(c'+c)}{\cos\tfrac{1}{2}(C'+C)}\sin\tfrac{1}{2}\lambda_d.$$

$$
\begin{array}{ll}
\tfrac{1}{2}(c'+c) = 76^\circ\,31', & \log\cos = 9.36766 \\
\tfrac{1}{2}(c'+c) = 84^\circ\,29', & \text{colog}\cos = 1.01712 \\
\tfrac{1}{2}\lambda_d = 18^\circ\,24', & \log\sin = \underline{9.49920} \\
 & \log\cos\tfrac{1}{2}D = 9.88398
\end{array}
$$

$$\therefore \tfrac{1}{2}D = 40^\circ\,3'.$$

$$D = 80^\circ\,6' = 4806 \text{ m.}$$

To find L of V.

$$\sin PV = \sin c' \sin C.$$

$$\sin 180^\circ - 113^\circ = \sin 67^\circ = c'.$$

$$
\begin{array}{l}
\log\sin c' = 9.96403 \\
\log\sin C = \underline{9.59247} \\
\log\sin PV = 9.55650
\end{array}
$$

$$
\begin{array}{r}
PV = 21^\circ\,7' \text{ N.} \\
90^\circ\,00' \text{ N.} \\
\underline{21^\circ\ \ 7' \text{ N.}} \\
L \text{ of } V = 68^\circ\,53' \text{ N.}
\end{array}
$$

To find λ of V.

$$\cot CPV = \cos c' \tan C.$$

$$
\begin{array}{l}
\log\cos c' = 9.59188\ (n) \\
\log\tan C = \underline{9.62855} \\
\log\cot CPV = 9.22043\ (n)
\end{array}
$$

$$
\begin{array}{r}
CPV = 99^\circ\,26' \\
\lambda \text{ of } C = \underline{42^\circ\,00'} \\
\lambda \text{ of } V = 57^\circ\,26' \text{ E.}
\end{array}
$$

6. Find the elements of the great circle track between Cape Frio and Cape Good Hope, in latitude 34° 20′ S., longitude 18° 30′ E. (Reckon from the nearest pole.)

Lat. $C' = 34^\circ\,20'$ S.

Long. $C' = 18^\circ\,30'$ E.

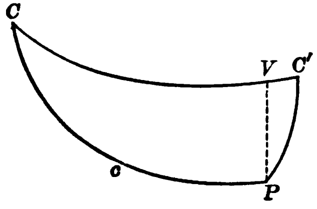
Lat. $C = 23^\circ$ S.

Long. $C = 42^\circ$ W.

$c = 90^\circ - 43^\circ\,20' = 55^\circ\,40'.$

$c' = 90^\circ - 23^\circ = 67^\circ.$

$\lambda_d = 42^\circ + 18^\circ\,30' = 60^\circ\,30'.$



$$\tan\tfrac{1}{2}(C'+C) = \frac{\cos\tfrac{1}{2}(c'-c)}{\cos\tfrac{1}{2}(c'+c)}\cot\tfrac{1}{2}\lambda_d.$$

$$\tan\tfrac{1}{2}(C'-C) = \frac{\sin\tfrac{1}{2}(c'-c)}{\sin\tfrac{1}{2}(c'+c)}\cot\tfrac{1}{2}\lambda_d.$$

$\frac{1}{2}(c' - c) = 5^\circ\ 40'$, log cos $= 9.99787$ log sin $= 8.99450$

$\frac{1}{2}(c' + c) = 61^\circ\ 20'$, colog cos $= 0.31902$ colog sin $= 0.05679$

$\frac{1}{2}\lambda_d = 30^\circ\ 15'$, log cot $= \underline{10.23420}$ log cot $= \underline{10.23420}$

log tan $\frac{1}{2}(C' + C) = 10.55109$ log tan $(C' - C) = 9.28549$

$\frac{1}{2}(C' + C) = 74^\circ\ 18'$ $\frac{1}{2}(C' - C) = 10^\circ\ 55'$.

$\frac{1}{2}(C' - C) = \underline{10^\circ\ 55'}$

$C' =$ S. $85^\circ\ 13'$ W. = course from C. Good Hope.

$C =$ S. $63^\circ\ 23'$ E. = course from Cape Frio.

To find the distance:

$$\cos\tfrac{1}{2}D = \frac{\cos\frac{1}{2}(c' + c)}{\cos\frac{1}{2}(C' + C)}\sin\tfrac{1}{2}\lambda_d.$$

$\frac{1}{2}(c' + c) = 61^\circ\ 20'$, log cos $= 9.68098$

$\frac{1}{2}(C' + C) = 74^\circ\ 18'$, colog cos $= 0.56767$

$\frac{1}{2}\lambda_d = 30^\circ\ 15'$, log sin $= \underline{9.70224}$

log cos $\frac{1}{2}D = 9.95089$

$\frac{1}{2}D = 26^\circ\ 44'$.

$D = 53^\circ\ 28' = 3208$ m.

To find L of V.

$\sin PV = \sin c' \sin C$.

log sin $c' = 9.96403$

log sin $C = \underline{9.95135}$

log sin $PV = 9.91538$

$PV = 55^\circ\ 23'$

L of $V = 90^\circ\ 0' - 55^\circ\ 23'$

$= 34^\circ\ 37'$ S.

To find λ of V.

$\cot CPV = \cos c' \tan C$.

log cos $c' = 9.59188$

log tan $C = \underline{10.30005}$

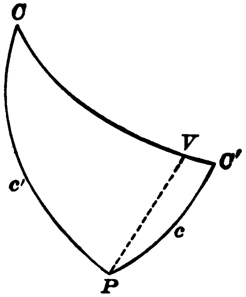
log cot $CPV = 9.89193$

$CPV = 52^\circ\ 3'$

$\underline{42^\circ\ 0'}$

λ of $V = 10^\circ\ 3'$ E.

7. Find the elements of the great circle track between Grand Port, Mauritius, in latitude 20° 24′ S., longitude 57° 47′ E., and Perth, in latitude 32° 3′ S., longitude 115° 45′ E.



Lat. $C' = 32^\circ\ 3'$ S.

Long. $C' = 115^\circ\ 45'$ E.

Lat. $C = 20^\circ\ 24'$ S.

Long. $C = 57^\circ\ 47'$ E.

$c = 90^\circ - 32^\circ\ 3' = 57^\circ\ 57'$.

$c' = 90^\circ - 20^\circ\ 24' = 69^\circ\ 36'$.

$\lambda_d = 115^\circ\ 45' - 57^\circ\ 47' = 57^\circ\ 58'$.

$$\tan\tfrac{1}{2}(C' + C) = \frac{\cos\frac{1}{2}(c' - c)}{\cos\frac{1}{2}(c' + c)}\cot\tfrac{1}{2}\lambda_d.$$

$$\tan\tfrac{1}{2}(C' - C) = \frac{\sin\frac{1}{2}(c' - c)}{\sin\frac{1}{2}(c' + c)}\cot\tfrac{1}{2}\lambda_d.$$

$$\begin{array}{llll}
\tfrac{1}{2}(c'-c) = 5^\circ\ 50, & \log\cos = 9.99776 & & \log\sin = 9.00704 \\
\tfrac{1}{2}(c'+c) = 63^\circ\ 47', & \text{colog}\cos = 0.35481 & & \text{colog}\sin = 0.04714 \\
\tfrac{1}{2}\lambda_d = 28^\circ\ 59', & \log\cot = 10.25655 & & \log\cot = \underline{10.25655} \\
 & \log\tan\tfrac{1}{2}(C'+C) = 10.60912 & & \log\tan\tfrac{1}{2}(C'-C) = 9.31073 \\
 & \tfrac{1}{2}(C'+C) = 76^\circ\ 11' & & \tfrac{1}{2}(C'-C) = 11^\circ\ 34'. \\
 & \tfrac{1}{2}(C'-C) = \underline{11^\circ\ 34'} & & \\
 & C' = \text{S. } 87^\circ\ 45' \text{ E.} = \text{course from Perth.} & & \\
 & C = \text{S. } 64^\circ\ 37' \text{ E.} = \text{course from Mauritius.} & &
\end{array}$$

To find the distance:

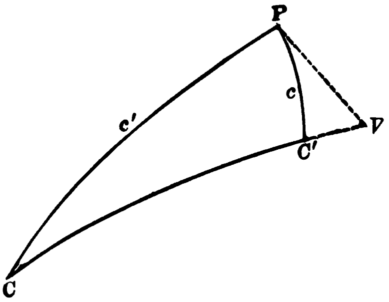
$$\cos\tfrac{1}{2}D = \frac{\cos\tfrac{1}{2}(c'+c)}{\cos\tfrac{1}{2}(C'+C)}\sin\tfrac{1}{2}\lambda_d.$$

$$\begin{array}{ll}
\tfrac{1}{2}(c'+c) = 63^\circ\ 47', & \log\cos = 9.64579 \\
\tfrac{1}{2}(C'+C) = 76^\circ\ 11', & \text{colog}\cos = 0.62194 \\
\tfrac{1}{2}\lambda_d = 28^\circ\ 59', & \log\sin = \underline{9.68534} \\
 & \log\cos\tfrac{1}{2}D = 9.95247
\end{array}$$

$$\tfrac{1}{2}D = 26^\circ\ 19'.$$
$$D = 52^\circ\ 38' = 3158 \text{ m.}$$

To find L of V.	To find λ of V.
$\sin PV = \sin c' \sin C.$	$\cot CPV = \cos c' \tan C.$
$\log\sin c' = 9.97187$	$\log\cos c' = 9.54229$
$\log\sin C = \underline{9.95591}$	$\log\tan C = \underline{10.32378}$
$\log\sin PV = 9.92778$	$\log\cot CPV = 9.86607$
$PV = 57^\circ\ 52'.$	$CPV = 53^\circ\ 42'$
L of $V = 90^\circ\ 0' - 57^\circ\ 52'$	λ of $C = \underline{57^\circ\ 47'}$
$= 32^\circ\ 8'$ S.	λ of $V = 111^\circ\ 29'$ E.

8. Find the elements of the great circle track between A, in latitude 16° 38′ N., longitude 70° 55′ W., and B, in latitude 48° 2′ N., longitude 4° 35′ W.



Lat $C' = 48^\circ\ 2'$ N.

Long. $C' = 4^\circ\ 35'$ W.

Lat. $C = 16^\circ\ 38'$ N.

Long. $C = 70^\circ\ 55'$ W.

$c = 90^\circ - 48^\circ\ 2' = 41^\circ\ 58'.$

$c' = 90^\circ - 16^\circ\ 38' = 73^\circ\ 22'.$

$\lambda_d = 70^\circ\ 55' - 4^\circ\ 35' = 66^\circ\ 20'.$

$$\tan \tfrac{1}{2}(C' + C) = \frac{\cos \tfrac{1}{2}(c' - c)}{\cos \tfrac{1}{2}(c' + c)} \cot \tfrac{1}{2}\lambda_d.$$

$$\tan \tfrac{1}{2}(C' - C) = \frac{\sin \tfrac{1}{2}(c' - c)}{\sin \tfrac{1}{2}(c' + c)} \cot \tfrac{1}{2}\lambda_d.$$

$$\begin{array}{lll}
\tfrac{1}{2}(c' - c) = 15^\circ\, 42', & \log\cos = 9.98349 & \log\sin = 9.43233 \\
\tfrac{1}{2}(c' + c) = 57^\circ\, 40', & \text{colog}\cos = 0.27177 & \text{colog}\sin = 0.07317 \\
\tfrac{1}{2}\lambda_d = 33^\circ\, 10', & \log\cot = \underline{10.18472} & \log\cot = \underline{10.18472} \\
& \log\tan \tfrac{1}{2}(C' + C) = 10.43998 & \log\tan \tfrac{1}{2}(C' - C) = 9.69022 \\
& \tfrac{1}{2}(C' + C) = 70^\circ\ \ 3' & \tfrac{1}{2}(C' - C) = 26^\circ\, 6'. \\
& \tfrac{1}{2}(C' - C) = \underline{26^\circ\ \ 6'} &
\end{array}$$

$C' = \text{N. } 96^\circ\ 9' \text{ W.}$

$= \text{S. } 83^\circ\ 51' \text{ W.} =$ course from B.

$C = \text{N. } 43^\circ\ 57' \text{ E.}$ = course from A.

To find the distance:

$$\cos \tfrac{1}{2}D = \frac{\cos \tfrac{1}{2}(c' + c)}{\cos \tfrac{1}{2}(c' - c)} \sin \tfrac{1}{2}\lambda_d.$$

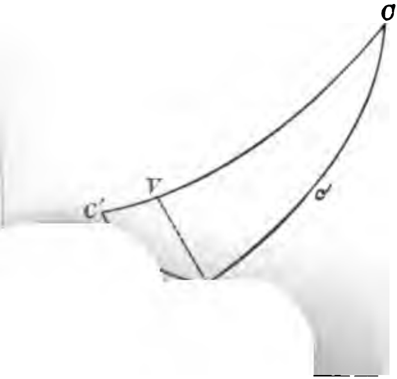
$$\begin{array}{ll}
\tfrac{1}{2}(c' + c) = 57^\circ\, 40', & \log\cos = 9.72823 \\
\tfrac{1}{2}(C' + C) = 70^\circ\ \ 3', & \text{colog}\cos = 0.46699 \\
\tfrac{1}{2}\lambda_d = 33^\circ\, 10', & \log\sin = \underline{9.73805} \\
& \log\cos \tfrac{1}{2}D = 9.93327
\end{array}$$

$\tfrac{1}{2}D = 30^\circ\ 57'.$

$D = 61^\circ\ 54' = 3714 \text{ m.}$

$$\begin{array}{ll}
\text{To find } L \text{ of } V. & \text{To find } \lambda_d \text{ of } V. \\
\sin PV = \sin c' \sin C. & \cot CPV = \cos c' \tan C. \\
\log\sin c = 9.98144 & \log\cos c' = 9.45674 \\
\log\sin C' = \underline{9.84138} & \log\tan C = \underline{9.98408} \\
\log\sin PV = 9.82282 & \log\cot CPV = 9.44082 \\
PV = 41^\circ 41'. & CPV = 74^\circ\ 34' \\
L \text{ of } V = 90^\circ\ \ 0' - 41^\circ 41' & \lambda \text{ of } C = \underline{70^\circ\ 55'} \\
= 48^\circ\ 19' \text{ N.} & \lambda \text{ of } V = \ \ 3^\circ\ 39' \text{ E.}
\end{array}$$

9. A ship sails from A., in latitude 40° S., longitude 148° 30′ E., to B, in latitude 12° 4′ S., longitude 77° 14′ W. Compare the great circle and the rhumb-line between A and B.



$\text{Lat. } C' = 40^\circ \text{ S.}$

$\text{Long. } C' = 148^\circ\ 30' \text{ E.}$

$\text{Lat. } C = 12^\circ\ \ 4' \text{ S.}$

$\text{Long. } C = 77^\circ\ 14' \text{ W.}$

$c = 90^\circ - 40^\circ = 50^\circ.$

$c' = 90^\circ - 12^\circ\ 4' = 77^\circ\ 56'.$

$\lambda_d = 148^\circ\ 30' + 77^\circ\ 14'$

$= 225^\circ\ 44',$ or $134^\circ\ 16'.$

$$\tan\tfrac{1}{2}(C'+C)=\frac{\cos\frac{1}{2}(c'-c)}{\cos\frac{1}{2}(c'+c)}\cot\tfrac{1}{2}\lambda_d.$$

$$\tan\tfrac{1}{2}(C'-C)=\frac{\sin\frac{1}{2}(c'-c)}{\sin\frac{1}{2}(c'+c)}\cot\tfrac{1}{2}\lambda_d.$$

$\frac{1}{2}(c'-c)=13°\ 58'$, $\quad$ log cos $= 9.98697$ $\quad$ log sin $= 9.38266$
$\frac{1}{2}(c'+c)=63°\ 58'$, $\quad$ colog cos $= 0.35764$ $\quad$ colog sin $= 0,04646$
$\frac{1}{2}\lambda_d = 67°\ \ 8'$, $\quad$ log cot $= 9.62504$ $\quad$ log cot $= 9.62504$

log tan $\frac{1}{2}(C'+C) = 9.96965$ $\quad$ log tan $\frac{1}{2}(C'-C) = 9.05416$
$\frac{1}{2}(C'+C) = 43°\ \ 0'$ $\quad$, $\quad$ $\frac{1}{2}(C'-C) = 6°\ 28'$.
$\frac{1}{2}(C'-C) = \ \ 6°\ 28'$

C' = S. 49° 28′ E. = course from A.
C = S. 36° 32′ W. = course from B.

To find the distance:

$$\cos\tfrac{1}{2}D=\frac{\cos\frac{1}{2}(c'+c)}{\cos\frac{1}{2}(c'-c)}\sin\tfrac{1}{2}\lambda_d.$$

$\frac{1}{2}(c'+c) = 63°\ 58'$, $\quad$ log cos $= 9.64236$
$\frac{1}{2}(C'+C) = 43°\ \ 0'$, $\quad$ colog cos $= 0.13587$
$\frac{1}{2}\lambda_d = 67°\ \ 8'$, $\quad$ log sin $= 9.96445$

log cos $\frac{1}{2}D = 9.74268$

$\frac{1}{2}D = \ 56°\ 26'$.
$D = 112°\ 52' = 6772$ m.

To find L of V.	To find λ of V.
$\sin PV = \sin c' \sin C$.	$\cot CPV = \cos c' \tan C$.
$\log \sin c' = 9.99021$	$\log \cos c' = 9.32025$
$\log \sin C = 9.77473$	$\log \tan C = 9.86974$
$\log \sin PV = 9.76494$	$\log \cot CPV = 9.18999$
$PV = 35° 36'$.	$CPV = \ 81°\ 12'$
L of $V = 90°\ \ 0' - 35° 36'$	λ of $C = \ 77°\ 14'$
$= 54° 24'$ S.	λ of $V = 158°\ 26'$ W.

By Rhumb Line:

To find L_d.	To find L_m.	To find λ_d.
$L' = 40°$ S.	$L' = 40°$ S.	$\lambda' = 148°\ 30'$ E.
$L'' = 12°\ \ 4'$ S.	$L'' = 12°\ 4'$ S.	$\lambda'' = \ 77°\ 14'$ W.
$L_d = 27°\ 56'$	$2\overline{)52°\ 4'}$	$\lambda_d = 225°\ 44'$
$= 1676$ m.	$L_m = 26°\ 2'$	$= 360° - 225°\ 44'$
		$= 134°\ 16' = 8056$ m.

To find the course.

$$\tan C = \frac{\lambda_d \cos L_m}{L_d}.$$

$$\begin{array}{lcr}
\log \lambda_d & = & 3.90612 \\
\log \cos L_m & = & 9.95354 \\
\text{colog } L_d & = & \underline{6.77573} \\
\log \tan C & = & 10.63539
\end{array}$$

To find the distance.

$$D = L_d \sec C.$$

$$\begin{array}{lcr}
\log L_d & = & 3.22427 \\
\log \sec C & = & \underline{.64682} \\
\log D & = & 3.87109 \\
D & = & 7432 \text{ m.}
\end{array}$$

$C = 76° 58′$. That is, N. 76° 58′ E. from A, or S. 76° 58′ W. from B.

EXERCISE IX. PAGE 387.

1.

Observed altitude	25° 6′ 10″	Index correction	+ 1′ 15″
Correction	− 4′ 50″	Dip	− 4′ 2″
		Refraction	− 2′ 3.4″
True altitude	25° 1′ 20″		

2.

Observed altitude	15° 20′ 25″	Index correction	− 2′ 20″
Correction	− 9′ 44″	Dip	− 3′ 55″
		Refraction	− 3′ 29.4″
True altitude	15° 10′ 41″		

3.

Observed altitude	18° 17′ 30″	Index correction	+ 0′ 18″
Correction	9′ 41″	Dip	− 4′ 9″
		Refraction	− 2′ 54″
		Semi-diameter	+ 16′ 18″
		Parallax	+ 8″
True altitude	18° 27′ 11″		

4.

Observed altitude	30° 12′ 40″	Semi-diameter	+ 16′ 4″
Correction	10′ 24″	Parallax	+ 8″
		Index correction	− 0′ 0″
		Dip	− 4′ 16″
		Refraction	− 1′ 39″
True altitude	30° 23′ 4″		

5.

Observed altitude	56° 25′ 20″	Semi-diameter	+ 16′ 3″
Correction	10′ 18″	Parallax	+ 5″
		Index correction	− 1′ 20″
		Dip	− 4′ 2″
		Refraction	− 0′ 38″
True altitude	56° 35′ 28″		

6.	Observed altitude	60° 10′ 10″	Semi-diameter	+ 15′ 48″
	Correction	9′ 55″	Parallax	+ 0′ 4″
			Index correction	+ 2′ 15″
			Dip	− 4′ 9″
			Refraction	− 0′ 33″
	True altitude	60° 19′ 5″		

7.	Observed altitude	31° 24′ 35″	Semi-diameter	+ 16′ 14″
	Correction	10′ 38″	Parallax	+ 8″
			Index correction	− 0′ 0″
			Dip	− 4′ 9″
			Refraction	− 1′ 35″
	True altitude	31° 35′ 13″		

8.	Observed altitude	26° 17′ 20″	Semi-diameter	− 16′ 10″
	Correction	19′ 53″	Parallax	+ 8″
			Index correction	+ 2′ 15″
			Dip	− 4′ 9″
			Refraction	− 1′ 57″
	True altitude	25° 57′ 27″		

9.	Observed altitude	20° 35′ 30″	Semi-diameter	− 15′ 46″
	Correction	21′ 49″	Parallax	+ 0′ 8″
			Index correction	+ 0′ 18″
			Dip	− 3′ 55″
			Refraction	− 2′ 34″
	True altitude	20° 13′ 41″		

10.	Observed altitude	36° 12′ 10″	Semi-diameter	− 15′ 47″
	Correction	20′ 57″	Parallax	+ 0′ 8″
			Index correction	+ 0′ 25″
			Dip	− 4′ 23″
			Refraction	− 1′ 20″
	True altitude	35° 51′ 13″		

EXERCISE X. PAGE 389.

	Astronomical Time.	d.	h.	m.	s.		Civil Time.	d.	h.	m.	s.	
1.	July	8	7	6	10	=	July	8	7	6	10	P.M.
2.	Mar.	7	12	25	30	=	Mar.	8	0	25	30	A.M.
3.	Jan.	1	18	10	10	=	Jan.	2	6	10	10	A.M.
4.	Dec.	31	15	0	0	=	Jan.	1	3	0	0	A.M.
5.	Feb.	2	8	4	30	=	Feb.	2	8	4	30	P.M.

	Civil Time.	d.	h.	m.	s.		Astronomical Time.	d.	h.	m.	s.
6.	July	1	11	8	25 A.M.	=	June	30	23	8	25.
7.	Mar.	2	11	56	56 P.M.	=	Mar.	2	11	56	56.
8.	Aug.	3	10	8	20 P.M.	=	Aug.	31	10	8	20.
9.	Sept.	1	0	12	15 A.M.	=	Aug.	31	12	12	15.
10.	Jan.	1	10	41	56 A.M.	=	Dec.	31	22	41	56.

Exercise XI. Page 391.

			d.	h.	m.	s.	
1.	Ship date,	May	4	6	12	15	
	Longitude in time,			+ 11	23	20	15) 170° 50′ 0″
	Greenwich date,	May	4	17	35	35	11 h. 23′ 20″

			d.	h.	m.	s.	
2.	Ship date,	July	30	23	12	30	
	Longitude in time,			+ 2	41	20	15) 40° 20′ 0″
	Greenwich date,	July	31	1	53	50	2 h. 41′ 20″

			d.	h.	m.	s.	
3.	Ship date,	July	31	14	10	15	
	Longitude in time,			5	22	43	15) 80° 40′ 45″
	Greenwich date,	July	31	19	32	58	5 h. 22′ 43″

			d.	h.	m.	s.	
4.	Ship date,	Mar.	2	10	20	0	
	Longitude in time,			3	23	0	15) 50° 45′
	Greenwich date,	Mar.	2	6	57	0	3 h. 23′

			d.	h.	m.	s.	
5.	Ship date,	Mar.	25	11	8	0 P.M.	
	Longitude in time,			6	41	42	15) 100° 25′ 30″
	Greenwich date,	Mar.	25	17	49	42	6 h. 41′ 42″

			d.	h.	m.	s.	
6.	Greenwich date,	Dec.	30	19	47	28	
	Longitude in time,			1	40	28	15) 25° 7′ 0″
		Dec.	30	18	7	0	1 h. 40′ 28″
	Local civil time,	Dec.	31	6	7	0 A.M.	

			d.	h.	m.	s.	
7.	Greenwich date,	July	4	23	51	0	
	Longitude in time,			11	56	0	15) 179° 0′ 0″
	Local civil time,	July	4	11	55	0 P.M.	11 h. 56′

		d.	h.	m.	s.	
8. Greenwich date,	July	3	23	59	0	
Longitude in time,			11	56		
	July	8	35	55	0	
Local civil time,	July	4	11	55	0 P.M.	

		d.	h.	m.	s.	
9. Greenwich date,	May	19	19	40	20	
Longitude in time,				3		15)45′ 0″
	May	19	19	43	20	3′
Local civil time,	May	20	7	43	20 A.M.	

	1880.	d.	h.	m.	s.	
10. Greenwich date,	Dec.	31	15	8	0	
Longitude in time,				8	40	15)2° 10′ 0″
	Dec.	31	15	16	40	8′ 40″
Local civil time,	Jan.	1	3	16	40 A.M.	

EXERCISE XII. PAGE 396.

Find the sun's declination and the equation of time corresponding to the following Greenwich dates:

1. 1895 Jan. 7 d. 3 h. apparent time.

			m.	s.
Jan. 7 d. 0 h.	⊙'s dec. 22° 22′ 25.8″ S.	Eq. of time +	6	29.57
Diff. for 3 h.	− 57.7″		+	3.22
Jan. 7 d. 3 h.	⊙'s dec. 22° 21′ 28.1″ S.	Eq. of time +	6	32.79

2. 1895 Aug. 1 d. 6 h. 12 m. 20 s. apparent time.

			m.	s.
Aug. 1 d. 0 h.	⊙'s dec. 18° 1′ 59.0″ N.	Eq. of time +	6	7.87
Diff. for 6 h. 12 m. 20 s.	− 3′ 54.3″		−	0.92
Aug. 1 d. 6 h. 12 m. 20 s.	⊙'s dec. 17° 58′ 4.7″ N.	Eq. of time +	6	6.95

3. 1895 May 5 d. 10 h. 25 m. apparent time.

			m.	s.
May 5 d. 0 h.	⊙'s dec. 16° 15′ 42.5″ N.	Eq. of time −	3	26.15
Diff. for 10 h. 25 m.	+ 7′ 25.4″		−	2.33
May 5 d. 10 h. 25 m.	⊙'s dec. 16° 23′ 7.9″ N.	Eq. of time −	3	28.48

4. 1895 Aug. 7 d. 15 h. 12 m. apparent time.

			m.	s.
Aug. 8 d. 0 h.	⊙'s dec. 16° 9′ 20.0″ N.	Eq. of time +	5	28.13
Diff. for 8 h. 48 m.	+ 6′ 15.0″	+		2.84
Aug. 8 d. 15 h. 12 m.	⊙'s dec. 16° 15′ 35.0″ N.	Eq. of time +	5	30.97

5. 1895 Dec. 4 d. 6 h. 18 m. apparent time.

			m.	s.
Dec. 4 d. 0 h.	⊙'s dec. 22° 15′ 34.0″ S.	Eq. of time −	9	40.21
Diff. for 6 h. 18 m.	+ 2′ 6.7″	+		6.38
Dec. 4 d. 6 h. 18 m.	⊙'s dec. 22° 17′ 40.7″ S.	Eq. of time −	9	33.83

6. 1895 July 23 d. 20 h. 16 m. 40 s. apparent time.

			m.	s.
July 24 d. 0 h.	⊙'s dec. 19° 53′ 2.9″ N.	Eq. of time +	6	16.42
Diff. for 3 h. 43 m. 20 s.	+ 1′ 57.3″	−		2.12
July 23 d. 20 h. 16 m. 40 s.	⊙'s dec. 19° 55′ 0.2″ N.	Eq. of time +	6	14.30

7. 1895 Nov. 1 d. 3 h. 6 m. apparent time.

			m.	s.
Nov. 1 d. 0 h.	⊙'s dec. 14° 26′ 57.9″ S.	Eq. of time −	16	18.64
Diff. for 3 h. 6 m.	+ 2′ 29.0″	−		0.20
Nov. 1 d. 3 h. 6 m.	⊙'s dec. 14° 29′ 26.9″ S.	Eq. of time −	16	18.84

8. 1895 Oct. 12 d. 5 h. 12 m. apparent time.

			m.	s.
Oct. 12 d. 0 h.	⊙'s dec. 7° 24′ 29.3″ S.	Eq. of time −	13	26.96
Diff. for 5 h. 12 m.	+ 4′ 53.4″	−		3.20
Oct. 12 d. 5 h. 12 m.	⊙'s dec. 7° 30′ 22.7″ S.	Eq. of time −	13	30.16

9. 1895 June 7d. 3 h. 18 m. mean time.

			m.	s.
June 7 d. 0 h.	⊙'s dec. 22° 45′ 50.5″ N.	Eq. of time +	1	26.59
Diff. for 3 h. 18 m.	+ 47.9″	−		1.51
June 7 d. 3 h. 18 m.	⊙'s dec. 22° 46′ 38.4″ N.	Eq. of time +	1	25.08

10. 1895 Feb. 3 d. 9 h. 15 m. mean time.

			m.	s.
Feb. 3 d. 0 h.	⊙'s dec. 16° 30′ 23.8″ S.	Eq. of time −	14	2.48
Diff. for 9 h. 15 m.	− 6′ 48.9″	−		2.40
Feb. 3 d. 9 h. 15 m.	⊙'s dec. 16° 23′ 34.9″ S.	Eq. of time −	14	4.88

EXERCISE XIII. PAGE 404.

1. Given civil date 1895 Jan. 1, longitude 102° 41′ W., observed meridian altitude of $\underline{\odot}$ 59° 59′ 50″ S., index correction + 0′ 50″, eye 15 ft.; find the latitude.

Long. 102° 41′ W. = 6 h. 50 m. 44 s.

$\underline{\odot}$ 59° 59′ 50″	Index cor., + 0′ 50″	$\odot$'s dec. 23° 0′ 34″ S.	12.49
	Semi-diam., + 16′ 18″	1′ 25″	6.84
+ 12′ 50″	Dip, − 3′ 48″	d = 22° 59′ 9″ S.	85.43
	Refraction, − 0′ 34″	z = 29° 47′ 20″ N.	
	Parallax, + 0′ 4″	L = 6° 48′ 11″ N.	
60° 12′ 40″			
90°			
z = 29° 47′ 20″ N.			

2. Given civil date 1895 Feb. 1, longitude 78° 14′ E., observed meridian altitude of $\underline{\odot}$ 78° 4′ 10″ S., index correction + 0′ 55″, eye 12 ft.; find the latitude.

Long. 78° 14′ = 5 h. 12 m. 56 s.

$\underline{\odot}$ 78° 4′ 10″	Index cor., + 0′ 55″	$\odot$'s dec. 17° 5′ 1″ S.	42.76
	Semi-diam., + 16′ 16″	3′ 43″	5.22
+ 13′ 37″	Dip, − 3′ 24″	d = 17° 8′ 44″ S.	223.21
	Refraction, − 0′ 12″	z = 11° 42′ 13″ N.	
	Parallax, + 0′ 2″	L = 5° 26′ 31″ S.	
78° 17′ 47″			
90°			
z = 11° 42′ 13″ N.			

3. Given civil date 1895 Mar. 20, longitude 173° 18′ W., observed meridian altitude of $\underline{\odot}$ 89° 37′ 0″ N., index correction + 4′ 32″, eye 18 ft.; find the latitude.

Long. 173° 18′ = 11 h. 33 m. 12 s.

$\underline{\odot}$ 89° 37′ 0″ N.	Index cor., + 4′ 32″	$\odot$'s dec. 0° 8′ 36″ S.	59.26
	Semi-diam., + 16′ 5″	− 11′ 24″	11.55
+ 16′ 28″	Dip, − 4′ 9″	d = 0° 2′ 48″ N.	684.45
	Refraction, − 0′ 0″	z = 0° 6′ 32″ S.	
	Parallax, + 0′ 0″	L = 0° 3′ 44″ S.	
89° 53′ 28″			
90°			
z = 0° 6′ 32″ S.			

4. Given civil date 1895 April 1, longitude 87° 42′ W., observed meridian altitude of $\underline{\odot}$ 48° 42′ 30″ S., index correction + 1′ 42″, eye 18 ft.; find the latitude.

Long. 87° 42′ = 5 h. 50 m. 48 s.

$\underline{\odot}$ 48° 42′ 30″ S.	Index cor.,	+ 1′ 42″	⊙'s dec.	4° 33′ 23″ N.	57.85
	Semi-diam.,	+ 16′ 2″		+ 5′ 38″	5.85
+ 12′ 50″	Dip,	− 4′ 9″	d =	4° 39′ 1″ N.	338.42
	Refraction,	− 0′ 51″	z =	41° 4′ 40″ N.	
	Parallax,	+ 0′ 6″	L =	45° 43′ 41″ N.	
48° 55′ 20″ S.					
90°					
z=41° 4′ 40″ N.					

5. Given civil date 1895 Sept. 1, longitude 97° 42′ E., observed meridian altitude of $\underline{\odot}$ 51° 4′ 50″ S., index correction − 6′ 0″, eye 23 ft.; find the latitude.

Long. 97° 42′ = 6 h. 30 m. 48 s.

$\underline{\odot}$ 51° 4′ 50″ S.	Index cor.,	− 6′ 0″	⊙'s dec.	8° 17′ 14″ N.	54.43
	Semi-diam.,	+ 15′ 54″		5′ 54″	6.51
+ 4′ 31″	Dip,	− 4′ 42″	d =	8° 23′ 8″ N.	354.34
	Refraction,	− 0′ 47″	z =	38° 50′ 39″ N.	
	Parallax,	+ 0′ 6″	L =	47° 13′ 47″ N.	
51° 9′ 21″ S.					
90°					
z=38° 50′ 39″ N.					

6. Given civil date 1895 Aug. 26, longitude 92° 3′ E., observed meridian altitude of $\underline{\odot}$ 35° 35′ 20″ N., index correction + 2′ 17″, eye 12 ft.; find the latitude.

Long. 92° 3′ = 6 h. 8 m. 12 s.

$\underline{\odot}$ 35° 35′ 20″ N.	Index cor.,	+ 2′ 17″	⊙'s dec.	10° 25′ 18″ N.	52.22
	Semi-diam.,	+ 15′ 52″		5′ 21″	6.14
+ 13′ 31″	Dip,	− 3′ 24″	d =	10° 30′ 39″ N.	320.66
	Refraction,	− 1′ 21″	z =	54° 11′ 9″ S.	
	Parallax,	+ 0′ 7″	L =	43° 40′ 30″ S.	
35° 48′ 51″ N.					
90°					
z=54° 11′ 9″ S.					

7. Given civil date 1895 May 16, longitude 45° 26′ W., observed meridian altitude of $\underline{\odot}$ 86° 34′ 20″ N., index correction + 4′ 16″, eye 15 ft.; the latitude.

Long. 45° 26′ = 3 h. 1 m. 44 s.

$\underline{\odot}$ 86° 34′ 20″ N.	Index cor.,	+ 4′ 16″	⊙'s dec. 19° 6′ 29″ N.	34.63
	Semi-diam.,	+ 15′ 51″	1′ 45″	3.03
+ 16′ 16″	Dip,	− 3′ 48″	d = 19° 8′ 74″ N.	104.93
	Refraction,	− 0′ 4″	z = 3° 9′ 24″ S.	
	Parallax,	+ 0′ 1″	L = 15° 58′ 50″ N.	
86° 50′ 36″ N.				
90°				
z = 3° 9′ 24″ S.				

8. Given civil date 1895 March 20, longitude 174° 0′ W., observed meridian altitude of $\underline{\odot}$ 89° 56′ 10″ N., index correction − 1′ 15″, eye 15 ft.; find the latitude.

Long. 174° 0′ = 11 h. 36 m.

$\underline{\odot}$ 89° 56′ 10″ N.	Index cor.,	− 1′ 15″	⊙'s dec. 0° 8′ 36″ S.	59.26
	Semi-diam.,	+ 16′ 5″	11′ 27″	11.60
+ 10′ 52″	Dip,	− 3′ 48″	d = 0° 2′ 51″ N.	687.42
	Refraction,	− 0′ 0″	z = 0° 7′ 2″ N.	
	Parallax,	+ 0′ 0″	L = 0° 9′ 53″ N.	
90° 7′ 2″ N.				
90°				
z = 0° 7′ 2″ N.				

9. Given civil date 1895 June 1, longitude 44° 40′ E., observed meridian altitude of $\underline{\odot}$ 72° 14′ 10″ N., index correction + 3′ 45″, eye 22 ft.; find the latitude.

Long. 44° 40′ = 2 h. 58 m. 40 s.

$\underline{\odot}$ 72° 14′ 10″ N.	Index cor.,	+ 3′ 45″	⊙'s dec. 22° 3′ 54″ N.	20.39
	Semi-diam.,	+ 15′ 48″	1′ 1″	2.98
+ 14′ 41″	Dip,	− 4′ 36″	d = 22° 2′ 53″ N.	60.76
	Refraction,	− 0′ 19″	z = 17° 31′ 9″ S.	
	Parallax,	+ 0′ 3″	L = 4° 31′ 44″ N.	
72° 28′ 51″ N.				
90°				
z = 17° 31′ 9″ S.				

10. Given civil date 1895 Dec. 1, longitude 67° 56′ E., observed meridian altitude of $\underline{\odot}$ 18° 48′ 10″ S., index correction − 3′ 6″, eye 18 ft.; find the latitude.

Long. 67° 56′ = 4 h. 31 m. 44 s.

$\underline{\odot}$ 18° 48′ 10″ S.	Index cor.,	− 3′ 6″	⊙'s dec. 21° 49′ 31″ S.	23.29
	Semi-diam.,	+ 16′ 16″	1′ 45″	4.53
+ 6′ 20″	Dip,	− 4′ 9″	d = 21° 51′ 16″ S.	105.04
	Refraction,	− 2′ 49″	z = 71° 5′ 30″ N.	
	Parallax,	+ 0′ 8″	L = 49° 14′ 14″ N.	
18° 54′ 30″ S.				
90°				
z = 71° 5′ 30″ N.				

11. Given civil date 1895 Sept. 23, longitude 57° 45′ E., observed meridian altitude of $\underline{\odot}$ 84° 10′ 50″ N., index correction − 1′ 36″, eye 16 ft.; find the latitude.

Long. 57° 45′ = 3 h. 51 m.

$\underline{\odot}$ 84° 10′ 50″ N.	Index cor.,	− 1′ 36″	⊙'s dec. 0° 4′ 35″ S.	58.49
	Semi-diam.,	+ 15′ 59″	3′ 45″	3.85
+ 10′ 23″	Dip,	− 3′ 55″	d = 0° 0′ 50″ S.	225.19
	Refraction,	− 0′ 6″	z = 5° 38′ 47″ S.	
	Parallax,	+ 0′ 1″	L = 5° 39′ 37″ S.	
84° 21′ 13″ N.				
90°				
z = 5° 38′ 47″ S.				

12. Given civil date 1895 Sept. 23, longitude 119° 54′ E., observed meridian altitude of $\underline{\odot}$ 83° 46′ 0″ S., index correction − 5′ 30″, eye 18 ft.; find the latitude.

Long. 119° 54′ = 7 h. 59 m. 36 s.

$\underline{\odot}$ 83° 46′ 0″ S.	Index cor.,	− 5′ 30″	⊙'s dec. 0° 4′ 35″ S.	58.49
	Semi-diam.,	+ 15′ 59″	7′ 48″	7.99
+ 6′ 15″	Dip,	− 4′ 9″	d = 0° 3′ 13″ N.	467.60
	Refraction,	− 0′ 6″	z = 6° 7′ 45″ N.	
	Parallax,	+ 0′ 1″	L = 6° 10′ 58″ N.	
83° 52′ 15″ S.				
90°				
z = 6° 7′ 45″ N.				

13. Given civil date 1895 Nov. 21, longitude 70° 20′ E., observed meridian altitude of $\underline{\odot}$ 80° 20′ 0″ N., index correction − 2′ 50″, eye 20 ft.; find the latitude.

Long. 70° 20′ = 4 h. 41 m. 20 s.

$\underline{\odot}$ 80° 20′ 0″ N.	Index cor.,	− 2′ 50″	⊙'s dec. 19° 56′ 16″ S.	33.11
	Semi-diam.,	+ 16′ 14″	2′ 35″	4.69
+ 8′ 53″	Dip,	− 4′ 23″	d = 19° 53′ 31″ S.	155.29
	Refraction,	− 0′ 10″	z = 9° 31′ 7″ S.	
	Parallax,	+ 0′ 2″	L = 29° 24′ 38″ S.	

80° 28′ 53″ N.
90°
z = 9° 31′ 7″ S.

14. Given civil date 1895 Dec. 31, longitude 123° 45′ W., observed meridian altitude of $\underline{\odot}$ 67° 8′ 10″ S., index correction + 0′ 9″, eye 13 ft.; find the latitude.

Long. 123° 45′ = 8 h. 15 m.

$\underline{\odot}$ 67° 8′ 10″ S.	Index cor.,	+ 0′ 9″	⊙'s dec. 23° 6′ 22″ S.	11.03
	Semi-diam.,	+ 16′ 18″	1′ 31″	8.25
+ 12′ 34″	Dip,	− 3′ 32″	d = 23° 4′ 51″ S.	91.00
	Refraction,	− 0′ 25″	z = 22° 39′ 16″ N.	
	Parallax,	+ 0′ 4″	L = 0° 25′ 35″ S.	

67° 20′ 44″ S.
90°
z = 22° 39′ 16″ N.

15. Given civil date 1895 Oct. 20, longitude 150° 25′ W., observed meridian altitude of $\underline{\odot}$ 49° 58′ 50″ N., index correction + 1′ 10″ eye 19 ft.; find the latitude.

Long. 150° 25′ = 10 h. 1 m. 40 s.

$\underline{\odot}$ 49° 58′ 50″ N.	Index cor.,	+ 1′ 10″	⊙'s dec. 10° 21′ 17″ S.	53.89
	Semi-diam.,	+ 16′ 7″	9′ 11′	10.03
+ 12′ 18″	Dip,	− 4′ 16″	d = 10° 30′ 18″ S.	540.52
	Refraction,	− 0′ 49″	z = 39° 48′ 52″ S.	
	Parallax,	+ 0′ 6″	L = 50° 19′ 10″ S.	

50° 11° 8″ N.
90°
z = 39° 48′ 52″ S.

16. Given civil date 1895 June 1, longitude 96° 17′ E., observed meridian altitude of $\underline{\odot}$ 75° 38′ 15″ N., index correction + 0′ 27″, eye 26 ft.; find the latitude.

Long. 96° 17′ = 6 h. 25 m. 8 s.

$\underline{\odot}$ 75° 38′ 15″ N.	**Index cor.,**	**+ 0′ 27″**	**⊙'s dec.**	**22° 3′ 54″ N.**	**20.39**
	Semi-diam.,	**+ 15′ 48″**		**2′ 11″**	**6.42**
+ 11′ 2″	**Dip,**	**− 5′ 0″**	***d* =**	**22° 1′ 43″ N.**	**130.90**
	Refraction,	**− 0′ 15″**	***z* =**	**14° 10′ 43″ S.**	
	Parallax,	**+ 0′ 2″**	***L* =**	**7° 51′ 0″ N.**	
75° 49′ 17″ N.					
90°					
***z* = 14° 10′ 43″ S.**					

17. Given civil date 1895 June 25, longitude 59° 15′ E., observed meridian altitude of $\overline{\odot}$ 60° 23′ 15″ N., index correction + 2′ 21″, eye 30 ft.; find the latitude.

Long. 59° 15′ = 3 h. 57 m.

$\overline{\odot}$ 60° 23′ 15″ N.	**Index cor.,**	**+ 2′ 21″**	**⊙'s dec.**	**23° 24′ 19″ N.**	**3.93**
	Semi-diam.,	**− 15′ 46″**		**15″**	**3.95**
− 19′ 16″	**Dip,**	**− 5′ 22″**	***d* =**	**23° 24′ 34″ N.**	**15.52**
	Refraction,	**− 0′ 33″**	***z* =**	**29° 56′ 1″ S.**	
	Parallax,	**+ 0′ 4″**	***L* =**	**6° 31′ 27″ S.**	
60° 3′ 59″ N.					
90°					
***z* = 29° 56′ 1″ S.**					

Exercise XIV. Page 405.

1. Given civil date 1895 Jan. 29, observed meridian altitude of Aldebaran 52° 36′ 0″ S., index correction − 0′ 23″, eye 20 ft.; find the latitude.

Obs. alt. = 52° 36′ 0″ S.
− 5′ 31″
True alt. = 52° 30′ 29″ S.
90° N.
Zenith dis. = 37° 29′ 31″ N.
Dec. = 16° 18′ 2″ N.
Lat. = 53° 37′ 33″ N.

Index correction,	− 23″
Dip,	− 4′ 23″
Refraction,	− 45″
	− 5′ 31″

2. Given civil date 1895 Feb. 18, observed meridian altitude of Procyon 77° 18′ 10″ S., index correction + 0′ 19″, eye 16 ft.; find the latitude.

Obs. alt. = 77° 18′ 10″ S.
− 3′ 50″
True alt. = 77° 14′ 20″ S.
90° N.
Zenith dis. = 12° 45′ 40″ N.
Dec. = 5° 29′ 39″ N.
= 18° 15′ 19″ N.

Index correction,	+ 19″
Dip,	− 3′ 55″
Refraction,	− 1′ 13.5″
	− 3′ 49.5″

3. **Given civil date 1895 March 20, observed meridian altitude of Arcturus 36° 10′ 20″ N., index correction + 2′ 42″, eye 20 ft.; find the latitude.**

Obs. alt. = 36° 10′ 20″ N.
− 3′ 1″
True alt. = 36° 7′ 19″ N.
90° S.
Zenith dis. = 53° 52′ 41″ S.
Dec. = 19° 43′ 23″ N.
Lat. = 34° 9′ 18″ S.

Index correction,	+ 2′ 42″
Dip,	− 4′ 23″
Refraction,	− 1′ 20″
	− 3′ 1″

4. **Given civil date 1895 Aug. 17, observed meridian altitude of Altair 66° 51′ 10″ N., index correction + 0′ 58″, eye 13 ft.; find the latitude.**

Obs. alt. = 66° 51′ 10″ N.
− 3′ 0″
True alt. = 66° 48′ 10″ N.
90° S.
Zenith dis. = 23° 11′ 50″ S.
Dec. = 8° 35′ 34″ N.
Lat. = 14° 36′ 16″ S.

Index correction,	+ 58″
Dip,	− 3′ 32″
Refraction,	− 25.5″
	− 3′ 0″

5. **Given civil date 1895 Nov. 4, observed meridian altitude of Fomalhaut 59° 40′ 0″ N., index correction + 1′ 12″, eye 23 ft.; find the latitude.**

Obs. alt. = 59° 40′ 0″ N.
− 4′ 4″
True alt. = 59° 35′ 56″ N.
90° S.
Zenith dis. = 30° 24′ 4″
Dec. = 30° 10′ 32″ S.
Lat. = 60° 34′ 36″ S.

Index correction,	1′ 12″
Dip,	− 4′ 42″
Refraction,	− 34″
	− 4′ 4″

6. **Given civil date 1895 Sept. 6, observed meridian altitude of Arcturus 86° 35′ 50″ N., index correction − 1′ 10″, eye 12 ft.; find the latitude.**

Obs. alt. = 86° 35′ 50″
− 4′ 38″
True alt. = 86° 31′ 12″
= 90° S.
Zenith dis. = 3° 28′ 48″ S.
Dec. = 19° 43′ 37″ N.
Lat. = 16° 14′ 49″ N.

Index correction,	− 1′ 10″
Dip,	− 3′ 24″
Refraction,	− 4″
	− 4′ 38″

7. Given civil date 1895 Oct. 6, observed meridian altitude of Markab 54° 10′ 15″ S., index correction 0, eye 13 ft.; find the latitude.

Obs. alt. = 54° 10′ 15″ S.
− 4′ 14″
True alt. = 54° 6′ 1″ S.
= 90° N.
Zenith dis. = 35° 53′ 59″ N.
Dec. = 14° 38′ 49″ N.
Lat. = 50° 32′ 48″ N.

Index correction,	+ 0′ 0″
Dip,	− 3′ 32″
Refraction,	− 42″
	− 4′ 14″

8. Given civil date 1895 Aug. 17, observed meridian altitude of β Centauri 59° 47′ 13″ S., index correction 0, eye 25 ft.; find the latitude.

Obs. alt. = 59° 47′ 13″ S.
− 5′ 28″
True alt. = 59° 41′ 45″ S.
= 90° N.
Zenith dis. = 30° 18′ 15″ N.
Dec. = 59° 52′ 29″ S.
Lat. = 29° 34′ 14″ S.

Index correction,	+ 0′ 0″
Dip,	− 4′ 54″
Refraction,	− 34″
	− 5′ 28″

9. Given civil date 1895 Dec. 4, observed meridian altitude of α Arietis 60° 29′ 50″ S., index correction − 2′ 10″, eye 18 ft.; find the latitude.

Obs. alt. = 60° 29′ 50″ S.
− 6′ 52″
True alt. = 60° 22′ 58″ S.
90° N.
Zenith dis. = 29° 37′ 2″ N.
Dec. = 22° 58′ 26″ N.
Lat. = 52° 35′ 28″ N.

Index correction,	− 2′ 10″
Dip,	− 4′ 9″
Refraction,	− 33″
	− 6′ 52″

10. Given civil date 1895 Feb. 8, observed meridian altitude of Sirius 37° 50′ 20″ S., index correction + 1′ 4″, eye 19 ft.; find the latitude.

Obs. alt. = 37° 50′ 20″ S.
− 4′ 27″
True alt. = 37° 45′ 53″ S.
90° N.
Zenith dis. = 52° 14′ 7″ N.
Dec. = 16° 34′ 20″ S.
Lat. = 35° 41′ 47″ N.

Index correction,	+ 1′ 4″
Dip,	− 4′ 16″
Refraction,	− 1′ 15″
	− 4′ 27″

11. Given civil date 1895 April 9, observed meridian altitude of Sirius 61° 3′ 50″ N., index correction 0, eye 16 ft.; find the latitude.

Obs. alt.	= 61° 3′ 50″ N.		
	4′ 27″		
True alt.	= 60° 59′ 23″ N.	Index correction,	+ 0′ 0″
	90° S.	Dip,	− 3′ 55″
Zenith dis.	= 29° 0′ 37″ S.	Refraction,	− 32″
Dec.	16° 34′ 24″ S.		− 4′ 27″
Lat.	= 45° 35′ 1″ S.		

12. Given civil date 1895 March 30, observed meridian altitude of Spica 52° 14′ 0″ N., index correction 0, eye 19 ft.; find the latitude.

Obs. alt.	= 52° 14′ 0″ N.		
	5′ 1″		
True alt.	= 52° 8′ 59″ N.	Index correction,	+ 0′ 0″
	90° S.	Dip,	− 4′ 16″
Zenith dis.	= 37° 51′ 1″ S.	Refraction,	− 45″
Dec.	= 10° 37′ 4″ S.		− 5′ 1″
Lat.	= 48° 28′ 5″ S.		

13. Given civil date 1895 July 8, observed meridian altitude of Antares 70° 10′ 30″ N., index correction 0, eye 21 ft.; find the latitude.

Obs. alt.	= 70° 10′ 30″ N.		
	4′ 50″		
True alt.	= 70° 5′ 40″ N.	Index correction,	+ 0′ 0″
	90° S.	Dip,	− 4′ 29″
Zenith dis.	= 19° 54′ 20″ S.	Refraction,	− 21″
Dec.	= 26° 12′ 12″ S.		− 4′ 50″
Lat.	= 46° 6′ 32″ S.		

EXERCISE XV. PAGE 413.

1. 1895, Oct. 19, A.M., at sea, in latitude 33° 27′ S.; the observed altitude $\underline{\odot}$ 28° 22′ 30″; index correction + 30″; height of eye 18 ft.; Greenwich mean time by chronometer Oct. 18 d. 18 h. 28 m. 38 s. Required the longitude.

$\underline{\odot}$ 28° 22′ 30″	Index cor.,	+ 0′ 30″	☉'s dec.	9° 59′ 52.4″ S.	54.27
	Semi-diam.,	+ 16′ 6″		4′ 59.6″	5.52
+ 10′ 47″	Dip,	− 4′ 9″		9° 54′ 52.8″ S.	299.57
	Refraction,	− 1′ 48″		90° S.	
	Parallax,	+ 0′ 8″		p = 80° 5′ 7.2″ S.	
h = 28° 33′ 17″					

$h =$ 28° 33′ 17″					
$L =$ 33° 27′ 0″	log sec =	0.07864	Equation of time.		
$p =$ 80° 5′ 7″	log csc =	0.00654	m. s.		
$2S =$ 142° 5′ 24″			14 56.76		0.447
$S =$ 71° 2′ 42″	log cos =	9.51165	2.47		5.52
$R =$ 42° 29′ 25″	log sin =	9.82960	14 54.29		2.47
		2)19.42643			
	log sin ½t =	9.71321			

½t = 31° 6′ 31″. t = 62° 13′ 2″ = 4 h. 8 m. 52 s.

	d.	h.	m.	s.	
Oct.	18	19	51	8	Local apparent astronomical time.
			14	54	Equation of time.
Oct.	18	19	36	14	Local mean astronomical time.
Oct.	18	18	28	38	Greenwich mean time.
		1	7	36	Difference of time.
		16°	54′	0″	E. Longitude.

2. 1895, Oct. 20 A.M., at sea, in latitude 31° 40′ S.; the observed altitude $\underline{\odot}$ 35° 16′ 10″; index correction + 30″; height of eye 18 ft.; Greenwich mean time by chronometer, Oct. 19 d. 19 h. 11 m. 24 s. Required the longitude.

$\underline{\odot}$ 35° 16′ 10″	Index cor., + 0′ 30″	⊙'s dec. 10° 21′ 30.4″ S.		53.89
	Semi-diam., + 16′ 7″	4′ 19.2″		4.81
+ 11′ 14″	Dip, − 4′ 9″	10° 17′ 11″ S.		259.21
	Refraction, − 1′ 22″	90° S.		
	Parallax, + 0′ 8″	$p =$ 79° 42′ 49″ S.		

$h =$ 35° 27′ 24″					
$L =$ 31° 40′ 0″	log sec =	0.07001	Equation of time.		
$p =$ 79° 42′ 49″	log csc =	0.00704	m. s.		
$2S =$ 146° 50′ 13″			15 7.17		0.421
$S =$ 73° 25′ 6″	log cos =	9.45543	2.03		4.81
$R =$ 37° 57′ 42″	log sin =	9.78897	15 5.14		2.03
		2)19.32145			
	log sin ½t =	9.66072			

½t = 27° 14′ 53″. t = 54° 29′ 46″ = 3 h. 37 m. 59 s.

	d.	h.	m.	s.	
Oct.	19	20	22	1	Local apparent astronomical time.
			15	5	Equation of time.
Oct.	19	20	6	56	Local mean astronomical time.
Oct.	19	19	11	24	Greenwich mean time.
			55	32	Difference in time.
		13°	53′	0″	E. Longitude.

3. 1895, Oct. 20, P.M., at sea, in latitude 30° 55′ S.; the observed altitude $\underline{\odot}$ 21° 42′ 30″; index correction + 29″; height of eye 18 ft.; Greenwich mean time by chronometer Oct. 20 d. 3 h. 35 m. 40 s. Required the longitude.

$\underline{\odot}$ 21° 42′ 30″
+ 10′ 11″ { Index cor., + 0′ 29″; Semi-diam., + 16′ 7″; Dip, − 4′ 9″; Refraction, − 2′ 24″; Parallax, + 0′ 8″
h = 21° 52′ 41″

$\underline{\odot}$ dec. 10° 21′ 30.4″ S. | 53.89
3′ 13.5″ | 3.59
10° 24′ 44″ S. | 193.47
90° S.
p = 79° 35′ 16″ S.

L = 30° 55′ 0″ log sec = 0.06656
p = 79° 35′ 16″ log csc = 0.00721
$2S$ = 132° 22′ 57″
S = 66° 11′ 29″ log cos = 9.60604
R = 44° 18′ 48″ log sin = 9.84421
2)19.52402
log sin $\frac{1}{2}t$ = 9.76201
$\frac{1}{2}t$ = 35° 19′ 3″. t = 70° 38′ 6″ = 4 h. 42 m. 32 s.

Equation of time.
m. s.
15 7.17 | 0.421
1.51 | 3.59
15 8.68 | 1.51

	d.	h.	m.	s.	
Oct.	20	4	42	32	Local apparent astronomical time.
			15	9	Equation of time.
Oct.	20	4	27	23	Local mean astronomical time.
Oct.	20	3	35	40	Greenwich mean time.
			51	43	Difference in time.
		12°	55′	45″	E. Longitude.

4. 1895, Oct. 21, A.M., at sea, in latitude 29° 35′ S.; the observed altitude $\underline{\odot}$ 24° 26′ 42″; index correction + 29″; height of eye 18 ft.; Greenwich mean time by chronometer Oct. 20 d. 18 h. 30 m. 39 s. Required the longitude.

$\underline{\odot}$ 24° 26′ 42″
+ 10′ 27″ { Index cor., + 0′ 29″; Semi-diam., + 16′ 7″; Dip, − 4′ 9″; Refraction, − 2′ 8″; Parallax, + 0′ 8″
h = 24° 37′ 9″

$\odot$'s dec. 10° 42′ 59.3″ S. | 53.50
4′ 53.7″ | 5.49
10° 38′ 6″ S. | 293.71
90° S.
p = 79° 21′ 54″ S.

L = 29° 35′ 0″ log sec = 0.06066
p = 79° 21′ 54″ log csc = 0.00752
$2S$ = 133° 34′ 3″
S = 66° 47′ 1″ log cos = 9.59573
R = 42° 9′ 52″ log sin = 9.82691
2)19.49082
log sin $\frac{1}{2}t$ = 9.74541
$\frac{1}{2}t$ = 33° 48′ 33″. t = 67° 37′ 6″ = 4 h. 30 m. 28 s.

Equation of time.
m. s.
15 16.94 | 0.394
2.16 | 5.49
15 14.78 | 2.16

	d.	h.	m.	s.	
Oct.	20	19	29	32	Local apparent astronomical time.
			15	15	Equation of time.
Oct.	20	19	14	17	Local mean astronomical time.
Oct.	20	18	30	39	Greenwich mean time.
			43	38	Difference in time.
				10° 54′ 30″	E. Longitude.

5. 1895 Jan. 29, P.M., at ship, latitude 42° 26′ N.; observed altitude $\underline{\odot}$ 13° 40′; index error − 1′ 8″; height of eye 16 ft.; time by chronometer 29 d. 6 h. 48 m. 40 s., which was slow 11 m. 22.3 s. for mean noon at Greenwich, Dec. 1, 1894, and on Jan. 1, 1895, was 8 m. 7 s. slow for Greenwich mean noon. Required the longitude.

Chronometer.

			m.	s.
1894	Dec. 1	slow	11	22.3
1895	Jan. 1	slow	8	7.0
		31)	3	15.3
				6.3
				28.28
			2	58.2

	d.	h.	m.	s.
Jan.	29	6	48	40
			+ 8	7
			− 2	58
Jan.	29	6	53	49

$\underline{\odot}$ 13° 40′ 0″	Index cor.,	− 1′ 8″	⊙'s dec. 17° 55′ 7.1″ S.	40.43
	Semi-diam.,	+ 16′ 16″	4′ 39.0″	6.90
+ 7′ 27″	Dip,	− 3′ 55″	17° 50′ 28″ S.	278.97
	Refraction,	− 3′ 55″	90° N.	
	Parallax,	+ 0′ 9″	$p =$ 107° 50′ 28″ N.	

$h =$ 13° 47′ 27″		
$L =$ 42° 26′ 0″	log sec =	0.13191
$p =$ 107° 50′ 28″	log csc =	0.02141
$2S =$ 164° 3′ 55″		
$S =$ 82° 1′ 57″	log cos =	9.14180
$R =$ 68° 14′ 30″	log sin =	9.96790
		2) 19.26302
	log sin $\frac{1}{2}t =$	9.63151

Equation of time.

m.	s.	
13	20.81	0.435
	3.10	6.90
13	23.91	3.10

$\frac{1}{2}t =$ 25° 20′ 42″. $t =$ 50° 41′ 24″ = 3 h. 22 m. 46 s.

	d.	h.	m.	s.	
Jan.	29	3	22	46	Local apparent astronomical time.
			13	24	Equation of time.
Jan.	29	3	36	10	Local mean astronomical time.
Jan.	29	6	53	49	Greenwich mean time.
		3	17	39	Difference in time.
				49° 24′ 45″	W. Longitude.

6. 1895, March 31, A.M., at ship, latitude 26° 9′ N.; observed altitude $\underline{\odot}$ 29° 10′ 20″; height of eye 26 ft.; time by chronometer 31 d. 0 h. 4 m. 50 s., which was 58 m. 58 s. fast for mean noon at Greenwich, Nov. 20, 1894, and on December 31, 1894, was 1 h. 2 m. 55.8 s. fast for mean time at Greenwich. Required the longitude.

Chronometer.

	h.	m.	s.
Nov. 20 fast	0	58	58
Dec. 31 fast	1	2	55.8
41)		3	57.8
			5.8
			90.
		8	42.0

	d.	h.	m.	s.
Mar.	31	0	4	50
		− 1	2	56
		—	8	42
Mar.	30	22	53	12

$$\begin{array}{r} \underline{\odot}\;29^\circ\,10'\,20'' \\ \\ +\;9'\,26'' \\ \\ \\ \end{array} \left\{\begin{array}{lrr} \text{Index cor.,} & +\;0' & 0'' \\ \text{Semi-diam.,} & +\;16' & 2'' \\ \text{Dip,} & -\;5' & 0'' \\ \text{Refraction,} & -\;1' & 44'' \\ \text{Parallax,} & +\;0' & 8'' \end{array}\right.$$

$\odot$'s dec.	4° 10′ 8.3″ N.	58.06
	1′ 5.6″	1.13
	4° 9′ 3″ N.	65.61
	90° N.	
$p =$	85° 50′ 57″ N.	

$h =$	29° 19′ 46″		
$L =$	26° 9′ 0″	log sec =	0.04690
$p =$	85° 50′ 57″	log csc =	0.00114
$2S =$	141° 19′ 43″		
$S =$	70° 39′ 51″	log cos =	9.51996
$R =$	41° 20′ 5″	log sin =	9.81984
		2)	19.38784

$\log\sin\frac{1}{2}t = 9.69392$

$\frac{1}{2}t = 29^\circ\,37'\,5''.$

$t = 59^\circ\,14'\,10'' = 3\text{ h. }56\text{ m. }57\text{ s.}$

Equation of time.

m. s.	
4 15.13	0.758
0.86	1.13
4 15.99	0.86

d.	h.	m.	s.	
Mar. 30	20	3	3	Local apparent astronomical time.
		4	16	Equation of time.
Mar. 30	20	7	19	Local mean astronomical time.
Mar. 30	22	53	12	Greenwich mean time.
	2	45	53	Difference in time.
	41°	28′	15″	W. Longitude.

7. 1895, May 22, A.M., at ship, latitude 43° 25′ N.; observed altitude $\underline{\odot}$ 32° 8′; index correction + 1′ 28″; height of eye 15 ft.; time by chronometer 21 d. 21 h. 6 m. 10 s., which was slow 12.6 s. for mean noon at Greenwich, Feb. 24, and on April 1 was 2 m. 45 s. fast for mean noon at Greenwich. Required the longitude.

Chronometer.

	m.	s.
Feb. 24 slow	0	12.6
Apr. 1 fast	2	45.0
36)	2	57.6
		4.93
		51
	4	11.4

	d.	h.	m.	s.
May	21	21	6	10
			−2	45
			−4	11
May	21	20	59	14

$\underline{\odot}$ 32° 8′ 0″	Index cor.,	+ 1′ 28″	☉'s dec. 20° 23′ 39.9″ N.	29.59
	Semi-diam.,	+ 15′ 50″	1′ 29.1″	3.01
+ 12′ 5″	Dip,	− 3′ 48″	20° 22′ 10.8″ N.	89.07
	Refraction,	− 1′ 33″	90° N.	
	Parallax,	+ 0′ 8″	p = 69° 37′ 49″ N.	

h =	32° 20′ 5″				
L =	43° 25′ 0″	log sec =	0.13884	Equation of time.	
p =	69° 37′ 49″	log csc =	0.02805	m. s.	
2 S =	145° 22′ 54″			3 34.66	0.182
S =	72° 41′ 27″	log cos =	9.47353	0.55	3.01
R =	40° 21′ 22″	log sin =	9.81126	3 35.21	0.55
			2)19.45168		
		log sin ½ t =	9.72584		

½ t = 32° 8′ 6″. t = 64° 16′ 12″ = 4 h. 17 m. 5 s.

	d.	h.	m.	s.	
May	21	19	42	55	Local apparent astronomical time.
			3	35	Equation of time.
May	21	19	39	20	Local mean astronomical time.
May	21	20	59	14	Greenwich mean time.
		1	19	54	Difference in time.
		19°	58′	30″	W. Longitude.

8. 1895, Aug. 24, A.M., at ship, latitude at noon 37° 59′ N.; observed altitude $\underline{\odot}$ 37° 13′ 30″; index correction + 2′ 44″; height of eye 18 ft.; time by chronometer Aug. 23 d. 18 h. 13 m. 24 s., which was 1 m. 5 s. fast for mean noon at Greenwich, August 1, and on August 10 was 0 m. 42 s. slow for mean time at Greenwich; course (true) since observation N.N.W.; distance 22.4 miles. Required the longitude at noon.

Chronometer.

	m.	s.
Aug. 1 fast	1	5
Aug. 10 slow	0	42
9)	1	47
		11.89
		13.76
	2	43.60

	d.	h.	m.	s.
Aug.	23	18	13	24
		+	0	42
		+	2	44
Aug.	23	18	16	50

L' = 37° 59′ 0″ N.
L_d = 20′ 41″
L = 37° 38′ 19″ N.

$\underline{\odot}$ 37° 13′ 30″	Index cor.,	+	2′ 44″
	Semi-diam.,	+	15′ 52″
+ 13′ 17″	Dip,	−	4′ 9″
	Refraction,	−	1′ 17″
	Parallax,	+	0′ 7″
$h =$ 37° 26′ 47″			

☉'s dec.	11° 6′ 46.7″ N.	51.38
	4′ 53.9″	5.72
	11° 11′ 40.6″ N.	293.89
	90° N.	
$p =$	78° 48′ 19″ N.	

$L =$	37° 38′ 19″	log sec =	0.10134
$p =$	78° 48′ 19″	log csc =	0.00834
$2S =$	153° 53′ 25″		
$S =$	76° 56′ 42″	log cos =	9.35389
$R =$	39° 29′ 55″	log sin =	9.80350
			2)19.26707
		log sin $\frac{1}{2}t$ =	9.63353

$\frac{1}{2}t = 25° 28′ 18″$.

$t = 50° 56′ 36″ = 3$ h. 23 m. 46 s.

Equation of time.

m. s.	
2 16.42	0.661
3.78	5.72
2 20.20	3.78

		d. h. m. s.	
Mer. L_d=26.1	log=1.41664	Aug. 23 20 36 14	Local appar. ast. time.
C=22° 30′	log tan=9.61722	2 20	Equation of time.
	log λ_d =1.03386	Aug. 23 20 38 34	Local mean ast. time.
	λ_d=10.811	Aug. 23 18 16 50	Greenwich mean time.
	=10′ 49″ W.	2 21 44	Difference in time.
		35° 26′ 0″	E. Long. at sights.
		10′ 49″	W. Long. since observ.
		35° 15′ 11″	E. Long. at noon.

9. 1895, Jan. 29, P.M., at ship, latitude 28° 45′ N.; observed altitude $\underline{\odot}$ 17° 46′ 30″; index correction − 3′ 18″; height of eye 16 ft., time by chronometer January 28 d. 16 h. 31 m. 30 s., which was 1 m. 16.5 s. fast for Greenwich mean noon, December 17, 1894, and on January 1, 1895, was 1 m. 3 s. slow for Greenwich mean time; course (true) since noon N.W. by W.; distance 20 miles. Required the longitude at the time of observation, and also at noon.

Chronometer.

	m. s.
1894, Dec. 17 fast	1 16.5
1895, Jan. 1 slow	1 3.
	15)2 19.5
	9.30
	28.02
	4 20.59

	d. h. m. s.
Jan.	28 16 31 30
	+ 1 3
	+ 4 21
Jan.	28 16 36 54

$\underline{\odot}$ 17° 46′ 30″	Index cor.,	− 3′ 18″	$\underline{\odot}$'s dec. 17° 55′ 7.1″ S.	40.43
	Semi-diam.,	+ 16′ 16″	4′ 58.4″	7.38
+ 6′ 11″	Dip,	− 3′ 55″	18° 0′ 5.5″ S.	298.37
	Refraction,	− 3′ 0″	90° N.	
	Parallax,	+ 0′ 8″	p = 108° 0′ 5.5″ N.	
h = 17° 52′ 41″				

L = 28° 45′ 0″	log sec = 0.05714	Equation of time.	
p = 108° 0′ 5″	log csc = 0.02179	m. s.	
$2S$ = 154° 37′ 46″		13 20.81	0.435
S = 77° 18′ 53″	log cos = 9.34163	3.21	7.38
R = 59° 26′ 12″	log sin = 9.93504	13 17.60	3.21
	2)19.35560		
	log sin ½t = 9.67780		

½t = 28° 26′ 18″. t = 56° 52′ 36″ = 3 h. 47 m. 30 s.

		d. h. m. s.	
L_d = 11′ 7″	log = 1.10037	Jan. 29 3 47 30	Local appar. ast. time.
Mer. L_d = 12.6	log tan = 10.17511	13 18	Equation of time.
C = 56° 15′	log λ_d = 1.27548	Jan. 29 4 0 48	Local mean ast. time.
	λ_d = 18.857	Jan. 28 16 36 54	Greenwich mean time.
	= 18′ 51″ E.	11 23 54	Difference in time.
		170° 58′ 30″	E. Long. at sights.
		18′ 51″	W. Long. since noon.
		171° 17′ 21″	E. Long. at noon.

10. 1895, Aug. 31, P.M., at ship, latitude 0°; observed altitude $\underline{\odot}$ 45° 5′ 30″; index correction − 2′ 4″; height of eye 15 ft.; time by chronometer Aug. 31 d. 9 h. 11 m. 28 s., which was 5 m. 20 s. fast for mean noon at Greenwich April 15, and on June 16 was fast 2 m. 43 s. on mean time at Greenwich. Required the longitude.

Chronometer.

			d. h. m. s.
April 15 fast	5 20		Aug. 31 9 11 28
June 16 fast	2 43		− 2 43
	62)2 37		+ 3 12
	2.53		Aug. 31 9 11 57
	76		
	3 12.28		

$\underline{\odot}$ 45° 5′ 30″	Index cor.,	− 2′ 4″	$\odot$'s dec. 8° 38′ 56.1″ N.	54.11
	Semi-diam.,	+ 15′ 53″	8′ 17.8″	9.20
+ 9′ 9″	Dip,	− 3′ 48″	8° 30′ 38.3″ N.	497.81
	Refraction,	− 0′ 58″	90° N.	
	Parallax,	+ 0′ 6″	p = 81° 29′ 22″ N.	
h = 45° 14′ 39″				

$h =$ 45° 14′ 39″		
$L =$ 0° 0′ 0″	log sec =	0.00000
$p =$ 81° 29′ 22″	log csc =	0.00481
$2S =$ 126° 44′ 1″		
$S =$ 63° 22′ 0″	log cos =	9.65155
$R =$ 18° 7′ 21″	log sin =	9.49283
		2)19.14919
	log sin $\frac{1}{2}t$ =	9.57459

Equation of time.

m. s.	
0 15.51	0.773
7.11	9.20
0 8.40	7.11

$\frac{1}{2}t = 22° 3′ 15″$. $t = 44° 6′ 30″ = 2$ h. 56 m. 26 s.

	d.	h.	m.	s.	
Aug.	31	2	56	26	Local apparent astronomical time.
				8	Equation of time.
Aug.	31	2	56	34	Local mean astronomical time.
Aug.	31	9	11	57	Greenwich mean time.
		6	15	23	Difference in time.
		93°	50′	45″	W. Longitude.

11. 1895, April 15, A.M., at ship, latitude 48° 52′ N.; observed altitude $\underline{\odot}$ 22° 18′; index correction − 3′ 54″; height of eye 17 ft.; time by chronometer April 14 d. 22 h. 30 m. 42 s., which was 0 m. 4 s. slow for mean noon at Greenwich January 1, and on January 12 was fast 0 m. 2 s. Required the longitude.

Chronometer.

	m. s.
Jan. 1 slow	0 4
Jan. 12 fast	0 2
	11)0 6
	0.545
	93
	50.69

	d.	h.	m.	s.
Apr.	14	22	30	42
			−0	2
			−	51
Apr.	14	22	29	49

$\underline{\odot}$ 22° 18′ 0″	Index cor.,	− 3′ 54″
	Semi-diam.,	+ 15′ 58″
+ 5′ 49″	Dip,	− 4′ 2″
	Refraction,	− 2′ 21″
	Parallax,	+ 0′ 8″
$h =$ 22° 23′ 49″		

⊙'s dec.	9° 46′ 35.5″ N.	53.58
	1′ 20.4″	1.50
	9° 45′ 15.1″ N.	80.37
	90° N.	
$p =$	80° 14′ 45″ N.	

$L =$ 48° 52′ 0″	log sec =	0.18190
$p =$ 80° 14′ 45″	log csc =	0.00632
$2S =$ 151° 30′ 34″		
$S =$ 75° 45′ 17″	log cos =	9.39107
$R =$ 53° 21′ 28″	log sin =	9.90438
		2)19.48367
	log sin $\frac{1}{2}t$ =	9.74183

Equation of time.

m. s.	
0 1.59	0.619
0.93	1.50
0 2.52	0.93

$\frac{1}{2}t = 33° 29′ 41″$. $t = 66° 59′ 22″ = 4$ h. 27 m. 57 s.

	d.	h.	m.	s.	
April	14	19	32	3	Local apparent astronomical time.
				3	Equation of time.
April	14	19	32	6	Local mean astronomical time.
April	14	22	29	49	Greenwich mean time.
		2	57	43	Difference in time.
	44°	25′	45″		W. Longitude.

12. 1895, Aug. 28, P.M., at ship, latitude 5° S.; observed altitude $\underline{\odot}$ 38°; index correction + 5′ 27″; height of eye 21 ft.; time by chronometer Aug. 27 d. 22 h. 20 m. 30 s., which was 10 m. 0 s. slow for mean noon at Greenwich Feb. 19, and on May 30 was 2 m. 20 s. slow on mean noon at Greenwich. Required the longitude.

Chronometer.

	m.	s.
Feb. 19 slow	10	0
May 30 slow	2	20
100)	7	40
		4.6
		90
	6	54

	d.	h.	m.	s.
Aug.	27	22	20	30
			+ 2	20
			− 6	54
Aug.	27	22	15	56

$$\underline{\odot}\ 38^\circ\ 0'\ 0'' \qquad +15'\,44''\left\{\begin{array}{lcr} \text{Index cor.,} & + & 5'\,27'' \\ \text{Semi-diam.,} & + & 15'\,53'' \\ \text{Dip,} & - & 4'\,29'' \\ \text{Refraction,} & - & 1'\,14'' \\ \text{Parallax,} & + & 0'\ \ 7'' \end{array}\right.$$

$$h = 38^\circ\,15'\,44''$$

⊙'s dec.	9° 43′ 13.4″ N.	53.02
	1′ 31.7″	1.73
	9° 44′ 45.1″ N.	91.72
	90° S.	
$p =$	99° 44′ 45″ S.	

$$\begin{array}{rlrl} L = & 5^\circ\ 0'\ 0'' & \log\sec = & 0.00166 \\ p = & 99^\circ\,44'\,45'' & \log\csc = & 0.00632 \\ 2S = & 143^\circ\ 0'\,29'' & & \\ S = & 71^\circ\,30'\,14'' & \log\cos = & 9.50139 \\ R = & 33^\circ\,14'\,30'' & \log\sin = & 9.73891 \\ & & 2) & 19.24828 \\ & & \log\sin\tfrac{1}{2}t = & 9.62414 \end{array}$$

$$\tfrac{1}{2}t = 24^\circ\,53'\,20''.$$

$$t = 49^\circ\,46'\,40'' = 3\text{ h. }19\text{ m. }7\text{ s.}$$

Equation of time.

m.	s.	
1	9.63	0.729
	1.26	1.73
1	10.89	1.26

	d.	h.	m.	s.	
Aug.	28	3	19	7	Local apparent astronomical time.
			1	11	Equation of time.
Aug.	28	3	20	18	Local mean astronomical time.
Aug.	27	22	15	56	Greenwich mean time.
		5	4	22	Difference of time.
		76°	5′	30″	E. Longitude.

13. 1895, Sept. 22, A.M., at ship, on the equator, observed altitude $\overline{\odot}$ 17° 20′ 40″; index correction − 1′ 9″; height of eye 20 ft.; time by chronometer Sept. 22 d. 4 h. 59 m. 16 s., which was 15 s. slow for Greenwich mean noon, April 30, and on June 1 was 10.6 s. fast for mean time at Greenwich. Required the longitude.

Chronometer.

	m.	s.
April 30 slow	0	15
June 1 fast	0	10.6
32)	0	25.6
	0	0.8
		113.2
	1	30.6

	d.	h.	m.	s.
Sept.	22	4	59	16
			− 0	10.6
			− 1	30.6
Sept.	22	4	57	35

$$\begin{array}{r} \overline{\odot}\ 17^\circ 20' 40'' \\ \\ -24' 37'' \\ \\ \\ \end{array} \left\{ \begin{array}{lcr} \text{Index cor.,} & - & 1' 19'' \\ \text{Semi-diam.,} & - & 15' 59'' \\ \text{Dip,} & - & 4' 28'' \\ \text{Refraction,} & - & 3'\ \ 4'' \\ \text{Parallax,} & + & 0'\ \ 8'' \end{array} \right.$$

☉'s dec.	0° 18′ 41.2″ N.	58.48
	4′ 50.1″	4.96
	0° 13′ 51.1″ N.	290.06
	90° N.	
$p =$	89° 46′ 9″ N.	

$h =$	16° 56′ 3″		
$L =$	0° 0′ 0″	log sec =	0.00000
$p =$	89° 46′ 9″	log csc =	0.00000
$2S =$	106° 42′ 12″		
$S =$	53° 21′ 6″	log cos =	9.77590
$R =$	36° 25′ 3″	log sin =	9.77354
			2)19.54944
		log sin $\frac{1}{2}t$ =	9.77472

$\frac{1}{2}t = 36^\circ 31' 57''$.

$t = 72^\circ 3' 54'' =$ 4 h. 48 m. 16 s.

Equation of time.

m.	s.	
7	15.58	0.807
	4.32	4.96
7	19.90	4.32

	d.	h.	m.	s.	
Sept.	21	19	11	44	Local apparent astronomical time.
			7	20	Equation of time.
Sept.	21	19	4	24	Local mean astronomical time.
Sept.	22	4	57	35	Greenwich mean time.
		9	53	11	Difference in time.
		148°	17′	45″	W. Longitude.

14. 1895, Aug. 5, A.M., at ship, latitude at noon 30° 30′ N.; observed altitude $\underline{\odot}$ 35° 6′; height of eye 15 ft.; time by chronometer 5 d. 8 h. 39 m. 22 s., which was fast 29 m. 32.4 s. on Greenwich mean noon, July 8, and on July 20 was fast 30 m. 0 s. on Greenwich mean noon; course (true) till noon W.; distance 48 miles. Required the longitude at noon.

Chronometer.

		m.	s.
July 8	fast	29	32.4
July 20	fast	30	0.0
	12)	0	27.6
		0	2.3
			16.4
		0	37.7

	d.	h.	m.	s.
Aug.	5	8	39	22
			− 30	0
			− 0	38
Aug.	5	8	8	44

$\underline{\odot}$ 35° 6′ 0″	Index cor.,	+ 0′ 0″
	Semi-diam.,	+ 15′ 49″
+ 10′ 45″	Dip,	− 3′ 48″
	Refraction,	− 1′ 23″
	Parallax,	+ 0′ 7″

⊙'s dec. 16° 59′ 19.9″ N.	40.60
5′ 30.9″	8.15
16° 53′ 49.0″ N.	330.89
90° N.	
$p =$ 73° 6′ 11″ N.	

$h =$	35° 16′ 45″		
$L =$	30° 30′ 0″	log sec =	0.06468
$p =$	73° 6′ 11″	log csc =	0.01916
$2S =$	138° 52′ 56″		
$S =$	69° 26′ 28″	log cos =	9.54552
$R =$	34° 9′ 43″	log sin =	9.74938
		2)	19.37874
		log sin $\frac{1}{2}t =$	9.68937

$\frac{1}{2}t =$ 29° 16′ 46″.

$t =$ 58° 33′ 32″ = 3 h. 54 m. 14 s.

Equation of time.

m. s.	
5 48.79	0.250
2.04	8.15
5 46.75	2.04

	d.	h.	m.	s.	
Aug.	4	20	5	46	Local apparent astronomical time.
			5	47	Equation of time.
Aug.	4	20	11	33	Local mean astronomical time.
Aug.	5	8	8	44	Greenwich mean time.
		11	57	11	Difference in time.
		179°	17′	45″	W. Longitude at sight.
48 miles =			55′	43″	W.
		180°	13′	28″	
		= 179°	46′	32″	E. Longitude at noon.

15. 1895, Nov. 12, A.M., at sea, in latitude 7° 10′ N.; four observed altitudes of the $\underline{\odot}$ were taken at the times (by watch) standing opposite, viz.:

		h. m. s.
Obs. alt. $\underline{\odot}$	21° 8′ 40″	2 55 48
	11′ 50″	56 0
	14′ 50″	56 13
	17′ 30″	56 26.5

Index correction + 31″; height of eye 18 ft.; correction of watch by chronometer − 5 h. 12 m. 2.1 s. Required the longitude.

	h.	m.	s.
⊙ 21° 8′ 40″	2	55	48
11′ 50″		56	0
14′ 50″		56	13
17′ 30″		56	26.5
52′ 50″		224	27.5
21° 13′ 12.5″	2	56	6.9
	−5	12	2.1
	21	44	4.8

⊙'s declination.	
17° 43′ 11.1″ S.	40.67
1′ 33.3″	2.27
17° 41′ 37.8″ S.	93.32
90° N.	
$p =$ 107° 41′ 38″ N.	

Index correction,	+ 0′ 31″
Semi-diameter,	+ 16′ 12″
Dip,	− 4 9″
Refraction,	− 2′ 29″
Parallax,	− 0′ 8″
	+ 10′ 13″

+ 10′ 13″

$h =$ 21° 23′ 25″

$L =$ 7° 10′ 0″ log sec = 0.00341

$p =$ 107° 41′ 38″ log csc = 0.02104

$2S =$ 136° 15′ 3″

$S =$ 68° 7′ 31″ log cos = 9.57122

$R =$ 46° 44′ 6″ log sin = 9.86224

2)19.45791

log sin $\frac{1}{2}t =$ 9.72895

$\frac{1}{2}t =$ 32° 23′ 38″.

$t =$ 64° 47′ 16″ = 4 h. 19 m. 9 s.

Equation of time.

m. s.	
15 44.88	0.324
0.74	2.27
15 45.62	0.74

	d.	h.	m.	s.	
Nov.	11	19	40	51	Local apparent astronomical time.
			15	46	Equation of time.
Nov.	11	19	25	5	Local mean astronomical time.
Nov.	11	21	44	5	Greenwich mean time.
		2	19	0	Difference in time.
		34°	45′	0″	W. Longitude.

16. 1895, Nov. 13, A.M., at sea, in latitude 9° 30′ N.; five observed altitudes of the ⊙ were taken at the times (by watch) standing opposite, viz.:

	h.	m.	s.
Obs. alt. ⊙ 18° 58′ 40″	2	59	2
19° 1′ 20″			13
3′ 30″			28
7′ 30″			45
11′ 0″			57.5

Index correction + 32″; height of eye 18 ft.; correction of watch by chronometer — 5 h. 11 m. 58.4 s. Required the longitude.

	h. m. s.
☉ 18° 58′ 40″	2 59 2
19° 1′ 20″	13
3′ 30″	28
7′ 30″	45
11′ 0″	57.5
22′ 0″	145.5
19° 4′ 24″	2 59 29.1
	−5 11 58.4
	21 47 30.7

☉'s declination.

17° 59′ 18.1″ S.	39.90
1′ 28.2″	2.21
17° 57′ 49.9″ S.	88.18
90° N.	
p = 107° 57′ 50″ N.	

Index correction,	+ 0′ 32″
Semi-diameter,	+ 16′ 13″
Dip,	− 4′ 9″
Refraction,	− 2′ 48″
Parallax,	+ 0′ 8″
	+ 9′ 56″

	+ 9′ 56″	
h =	19° 14′ 20″	
L =	9° 30′ 0″	log sec = 0.00600
p =	107° 57′ 50″	log csc = 0.02171
$2S$ =	136° 42′ 10″	
S =	68° 21′ 5″	log cos = 9.56692
R =	49° 6′ 45″	log sin = 9.87852
		2)19.47315
		log sin $\frac{1}{2}t$ = 9.73657

$\frac{1}{2}t$ = 33° 2′ 22″.

t = 66° 4′ 44″ = 4 h. 24 m. 19 s.

Equation of time.

m. s.	
15 36.66	0.361
0.80	2.21
15 37.46	0.80

d. h. m. s.	
Nov. 12 19 35 41	Local apparent astronomical time.
15 37	Equation of time.
Nov. 12 19 20 4	Local mean astronomical time.
Nov. 12 21 47 31	Greenwich mean time.
2 27 27	Difference in time.
36° 51′ 45″	W. Longitude.

17. 1895, Nov. 17, A.M., at sea, in latitude 15° 35′ N.; five observed altitudes of the ☉ were taken at the times (by watch) standing opposite, viz.:

	h. m. s.
Obs. alt. ☉ 23° 56′ 0″	4 12 31
24° 0′ 0″	56
4′ 0″	13 2.5
6′ 10″	14
10′ 0″	28.5

Index correction + 31″; height of eye 18 ft.; correction of watch by chronometer − 5 h. 11 m. 43.6 s. Required the longitude.

	h. m. s.
$\underline{\odot}$ 23° 56′ 0″	4 12 31
24° 0′ 0″	46
4′ 0″	13 2.5
6′ 10″	14
10′ 0″	28.5
16′ 10″	65 2
24° 3′ 14″	4 13 0.4
	−5 11 43.6
	23 1 16.8

⊙'s declination.

19° 0′ 34.2″ S.	36.63
35.9″	0.98
18° 59′ 58.3″ S.	35.90
90° N	
$p =$ 108° 59′ 58″ N.	

Index correction,	+ 0′ 31″
Semi-diameter,	+ 16′ 13″
Dip,	− 4′ 9″
Refraction,	− 2′ 10″
Parallax,	+ 0′ 8″
	+ 10′ 33″

	+ 10′ 33″	
$h =$	24° 13′ 47″	
$L =$	15° 35′ 0″	log sec = 0.01627
$p =$	108° 59′ 58″	log csc = 0.02433
$2S =$	148° 48′ 45″	
$S =$	74° 24′ 22″	log cos = 9.42946
$R =$	50° 10′ 35″	log sin = 9.88537
		2)19.35543
	log sin $\frac{1}{2}t$ =	9.67771

$\frac{1}{2}t = 28° 25′ 55″$.

$t = 56° 51′ 50″ =$ 3 h. 47 m. 27 s.

Equation of time.

m. s.	
14 55.18	0.503
0.49	0.98
14 55.67	0.49

d. h. m. s.	
Nov. 16 20 12 33	Local apparent astronomical time.
14 56	Equation of time.
Nov. 16 19 57 37	Local mean astronomical time.
Nov. 16 23 1 17	Greenwich mean time.
3 3 40	Difference in time.
45° 55′ 0″	W. Longitude.

18. 1895, Nov. 18, A.M., at sea, in latitude 16° 25′ N.; five observed altitudes of the $\underline{\odot}$ were taken at the times (by watch) standing opposite, viz.:

	h. m. s.
Obs. alt. $\underline{\odot}$ 18° 13′ 30″	3 52 42
16′ 10″	53.5
19′ 20″	53 6.5
22′ 30″	23
25′ 30″	38

Index correction + 32″: height of eye 18 ft.; correction of watch by chronometer − 5 h. 11 m. 39.9 s. Required the longitude.

		h. m. s.
$\underline{\odot}$	18° 13′ 30″	3 52 42
	16′ 10″	53.5
	19′ 20″	53 6.5
	22′ 30″	23
	25′ 30″	38
	97′ 0″	265 43
	18° 19′ 24″	3 53 8.6
		−5 11 39.9
		22 41 28.7

☉'s declination.

19° 15′ 3.2″ S.		35.77
46.9″		1.31
19° 14′ 16.3″ S.		46.86
90°	N.	
$p =$ 109° 14′ 16″	N.	

Index correction,	+ 0′ 32″
Semi-diameter,	+ 16′ 14″
Dip,	− 4′ 9″
Refraction,	− 2′ 54″
Parallax,	+ 0′ 8″
	+ 9′ 51″

+ 9′ 51″

$h =$ 18° 29′ 15″

$L =$ 16° 25′ 0″ log sec = 0.01808

$p =$ 109° 14′ 16″ log csc = 0.02493

$2S =$ 144° 8′ 31″

$S =$ 72° 4′ 15″ log cos = 9.48832

$R =$ 53° 35′ 0″ log sin = 9.90565

2)19.43698

log sin $\frac{1}{2}t =$ 9.71849

$\frac{1}{2}t =$ 31° 31′ 57″.

$t =$ 63° 3′ 54″ = 4 h. 12 m. 16 s.

Equation of time.

m. s.	
14 42.70	0.537
0.70	1.31
14 43.40	0.703

	d.	h.	m.	s.	
Nov.	17	19	47	44	Local apparent astronomical time.
			14	43	Equation of time.
Nov.	17	19	33	1	Local mean astronomical time.
Nov.	17	22	41	29	Greenwich mean time.
		3	8	28	Difference in time.
		47°	7′	0″	W. Longitude.

19. 1896, Dec. 4, A.M., at sea, in latitude 36° 10′ N.; five observed altitudes of the $\underline{\odot}$ were taken at the times (by watch) standing opposite, viz.:

		h. m. s.
Obs. alt. $\underline{\odot}$	13° 0′ 30″	6 27 14
	3′ 10″	29.5
	5′ 40″	49
	8′ 50″	28 5
	12′ 0″	23

Index correction + 32″; height of eye 18 ft.; correction of watch by chronometer − 5 h. 10 m. 47.1 s. Required the longitude.

	h. m. s.
$\underline{\odot}$ 13° 0′ 30″	6 27 14
3′ 10″	29.5
5′ 40″	49
8′ 50″	28 5
12′ 0″	23
30′ 10″	139 0.5
13° 6′ 2″	6 27 48.1
	−5 10 47.1
	1 17 1

☉'s declination.

22° 15′ 37.3″ S.	20.10
25.7″	1.28
22° 16′ 3.0″ S.	25.73
90° N.	
$p =$ 112° 16′ 3″ N.	

Index correction,	+ 0′ 32″
Semi-diameter,	+ 16′ 16″
Dip,	− 4′ 9″
Refraction,	− 4′ 5″
Parallax,	+ 0′ 9″
	+ 8′ 43″

	+ 8′ 43″		
$h =$	13° 14′ 45″		
$L =$	36° 10′ 0″	log sec =	0.09296
$p =$	112° 16′ 3″	log csc =	0.03366
$2S =$	161° 40′ 48″		
$S =$	80° 50′ 24″	log cos =	9.20192
$R =$	67° 35′ 39″	log sin =	9.96591
			2)19.29445

Equation of time.

m. s.	
9 40.05	1.013
1.30	1.28
9 38.75	1.30

log sin $\frac{1}{2}t =$ 9.64722

$\frac{1}{2}t =$ 26° 20′ 55″.

$t =$ 52° 41′ 50″ = 3 h. 30 m. 47 s.

d. h. m. s.	
Dec. 3 20 29 13	Local apparent astronomical time.
9 39	Equation of time.
Dec. 3 20 19 34	Local mean astronomical time.
Dec. 4 1 17 1	Greenwich mean time.
4 57 27	Difference in time.
74° 21′ 45″	W. Longitude.

20. 1895, Dec. 4, P.M., at sea, in latitude 36° 36′ N.; four observed altitudes of the $\underline{\odot}$ were taken at the times (by watch) standing opposite, viz.:

		h. m. s.
Obs. alt. $\underline{\odot}$	16° 31′ 10″	1 7 26.5
	30′ 0″	35.5
	28′ 30″	46
	27′ 20″	56.5

Index correction + 30″; height of eye 18 ft.; correction of watch by chronometer − 5 h. 10 m. 46.1 s. Required the longitude.

	h. m. s.
☉ 16° 31′ 10″	1 7 26.5
30′ 0″	35.5
28′ 30″	46
27′ 20″	56.5
117′ 0″	164.5
16° 29′ 15″	1 7 41.1
	−5 10 46.1
	7 56 55

☉'s declination.

22° 15′ 37.3″ S.	20.10
2′ 39.8″	7.95
22° 18′ 17.1″ S.	159.79
90° N.	
p = 112° 18′ 17″ N.	

Index correction,	+ 0′ 30″
Semi-diameter,	+ 16′ 16″
Dip,	− 4′ 9″
Refraction,	− 3′ 14″
Parallax,	+ 0′ 8″
	+ 9′ 31″

+ 9′ 31″

h = 16° 38′ 46″		
L = 36° 36′ 0″	log sec =	0.09538
p = 112° 18′ 17″	log csc =	0.03377
2 S = 165° 33′ 3″		
S = 82° 46′ 31″	log cos =	9.09955
R = 66° 7′ 45″	log sin =	9.96116
		2)19.18986
	log sin ½ t =	9.59493

½ t = 23° 10′ 18″.

t = 46° 20′ 36″ = 3 h. 5 m. 22 s.

Equation of time.

m. s.	
9 40.05	1.013
8.05	7.95
9 32.00	8.05

	d.	h.	m.	s.	
Dec.	4	3	5	22	Local apparent astronomical time.
			9	32	Equation of time.
Dec.	4	2	55	50	Local mean astronomical time.
Dec.	4	7	56	55	Greenwich mean time.
		5	1	5	Difference in time.
		75°	16′	15″	W. Longitude.